Incompressible elasticity

Essentially the same equations as Stokes flow (viscous, incompressible, laminar $Re \to 0$).

Background

Hooke's Law for isotropic solid

$$\sigma_{ij} = \lambda \, \varepsilon_{kk} \, \delta_{ij} + 2\mu \, \varepsilon_{ij}$$

Volumetric and deviatoric components

$$\sigma_{ij}^{dev} = \sigma_{ij} - p\,\delta_{ij} \, , \quad p = \sigma_{kk}/3$$

$$\sigma_{ij}^{vol} = p\,\delta_{ij}$$

$$\boxed{\sigma_{ij} = \sigma_{ij}^{dev} + \sigma_{ij}^{vol}} \qquad \sigma_{ij}^{dev}\,\sigma_{ij}^{vol} = 0$$

$$\varepsilon_{ij}^{dev} = \varepsilon_{ij} - \frac{\theta}{3}\delta_{ij} \, , \quad \theta = \varepsilon_{kk} = u_{k,k} = \nabla \cdot u$$

$$\varepsilon_{ij}^{vol} = \frac{\theta}{3}\delta_{ij}$$

$$\boxed{\varepsilon_{ij} = \varepsilon_{ij}^{vol} + \varepsilon_{ij}^{dev}}$$

Hooke's Law

$$\sigma_{ij}^{dev} = 2\mu\, \varepsilon_{ij}^{dev}$$

$$\sigma_{ij}^{vol} = 3K\, \varepsilon_{ij}^{vol} \quad , \quad K = \text{bulk modulus}$$
$$K = \lambda + \frac{2\mu}{3} = \frac{E}{3(1-2\nu)}$$

$\Longrightarrow$ Isotropic Hooke's law decouples into deviatoric and volumetric parts.

Incompressible limit

$\nu \to 0.5$
$K \to \infty$

For $p = K\theta < \infty$ as $K \to \infty$, $\theta \to 0$, p finite

In limit $\theta = 0$, p not determined from constitutive equations

$$\varepsilon_{kk} = \boxed{\nabla \cdot u = 0}$$

Governing equations:

$$\sigma_{ij,j} + f_i = 0 \quad , \quad \sigma_{ij} = \sigma_{ij}^{dev} + p\,\delta_{ij}$$
$$= 2\mu\, \varepsilon_{ij}^{dev} + p\,\delta_{ij}$$

$$\left(2\mu\,\varepsilon_{ij}^{dev} + p\delta_{ij}\right)_{,j} + f_i = 0$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$u_{k,k} = 0 \longrightarrow \varepsilon_{ij}^{dev} = \varepsilon_{ij}$$

$$\varepsilon_{ij,j}^{dev} = \frac{1}{2}\left(u_{i,jj} + \cancel{u_{j,ij}}^{\,0}\right)$$

$$\boxed{\mu\, u_{i,jj} + p_{,i} + f_i = 0} \quad \text{in } B$$

incompressible elasticity

$$u_i = \bar{u}_i \quad \text{on } S_1$$

$$\left(2\mu\, u_{i,j} + p\delta_{ij}\right) n_j = \bar{t}_i \quad \text{on } S_2$$

$$u_{i,i} = 0$$

Completely different variational structure
(saddle-point problem.

What happens in the discrete case?

$$K_h u_h = f_h \qquad \text{(finite element solutions)}$$

$$K_h = \sum_{e=1}^{E} \sum_{q=1}^{Q} w_q \left(B^{e^T} C \, B\right)_{(\xi_q)}$$

"C" can be decomposed into volumetric and deviatoric parts:

$$C = C^{dev} + C^{vol} \quad \Longrightarrow$$

$$K_h = K_h^{dev} + K_h^{vol} \quad , \quad \text{normalize} \rightarrow$$

$$= 2\mu \, \hat{K}_h^{dev} + 3\underbrace{K}_{\text{bulk modulus}} \hat{K}_h^{vol}$$

$$\Longrightarrow \left(2\mu \, \hat{K}_h^{dev} + 3K \, \hat{K}_h^{vol}\right) u_h = f_h$$

$$\left(\frac{2\mu}{3K} \hat{K}_h^{dev} + \hat{K}_h^{vol}\right) u_h = \frac{f_h}{3K}$$

$$\boxed{K \rightarrow \infty \qquad \hat{K}_h^{vol} u_h = 0}$$

If $\hat{K}_h^{vol}$ is non-singular $\rightarrow u_h \rightarrow 0$ when $K \rightarrow \infty$

K_h^{dev}, K_h^{vol} decomposition

• Deviatoric/volumetric projections

Adopt Voigt's notation:

$$\sigma = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} \in \mathbb{R}^{6\times1} \qquad \varepsilon = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix} \in \mathbb{R}^{6\times1}$$

$$\varepsilon^{dev} = P^{dev}\varepsilon \quad , \quad \varepsilon^{vol} = P^{vol}\varepsilon$$

$$P^{vol}, P^{dev} \in \mathbb{R}^{6\times6}$$

$$P^{vol} = \frac{1}{3}\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \hline & 0 & & & 0 & \end{array}\right)$$

$$P^{dev} = I - P^{vol} = \begin{pmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & 2/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & -1/3 & 2/3 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & \text{sym} & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$$

Properties

1) $P^{vol} + P^{dev} = I$

2) $(P^{dev})^2 = P^{dev}$, $(P^{vol})^2 = P^{vol}$

3) $(P^{dev})^T = P^{dev}$, $(P^{vol})^T = P^{vol}$ (self-adjoint) (symmetric in finite dim)

4) $P^{dev} P^{vol} = P^{vol} P^{dev} = 0$ (orthogonality)

5) Isotropic elasticity

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$C P^{dev} = P^{dev} C \quad , \quad C P^{vol} = P^{vol} C$$

projections commute with C

$\Rightarrow$ Can express $C = C^{dev} + C^{vol}$ such that

$$\sigma^{dev} = C^{dev} \varepsilon^{dev} \quad , \quad \sigma^{vol} = C^{vol} \varepsilon^{vol}$$

where $C^{dev} = 2\mu \, P^{dev}$

$C^{vol} = 3K \, P^{vol}$

$$\boxed{C = 2\mu \, P^{dev} + 3K \, P^{vol}}$$

"B" - matrix: From Principle of virtual displacements

$$\underbrace{\int_\Omega \sigma_{ij} \, \delta\varepsilon_{ij} \, dv}_{\text{Voigt notation}} = \int_V f_i \, \delta u_i \, dv + \int_{S_2} \bar{t}_i \, \delta u_i \, ds$$

$$\int_\Omega \sigma^T \delta\varepsilon \, dv = \int_\Omega (C\varepsilon)^T \delta\varepsilon \, dv = \int_\Omega \varepsilon^T C \, \delta\varepsilon \, dv$$

$\varepsilon \in \mathbb{R}^{6\times1}$, $C \in \mathbb{R}^{6\times6}$

$$\varepsilon^T = \{ u_{1,1} ; u_{2,2} ; u_{3,3} ; 0.5(u_{2,3} + u_{3,2}) ; \ldots \}$$

Finite element interpolation:

$$u_i = \sum_{a=1}^{N} N_a \, u_{ia}$$

Can write: $\underset{6\times 1}{\varepsilon} = \underset{6\times(n\times a)}{B}\ \underset{(n\times a)\times 1}{u_h}$

<u>$d=2,\ n=4$</u>

$$\varepsilon = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} \qquad u_h^T = \{u_{11}\ u_{21}\ u_{12}\ u_{22}\ u_{31}\ u_{32}\ u_{41}\ u_{4\ldots}$$

$$\varepsilon_{11} = u_{1,1} = N_{a,1}\, u_{1a} = \{N_{1,1}\ 0\ N_{2,1}\ 0\ N_{3,1}\ 0\ N_{4,1}\ 0\}\{u\ldots$$

$$\Rightarrow B = \begin{pmatrix} N_{1,1} & 0 & N_{2,1} & 0 & N_{3,1} & 0 & N_{4,1} & 0 \\ 0 & N_{1,2} & 0 & N_{2,2} & 0 & N_{3,2} & 0 & N_{4,2} \\ N_{1,2} & N_{1,1} & N_{2,2} & N_{2,1} & N_{3,2} & N_{3,1} & N_{4,2} & N_{4,1} \end{pmatrix}$$

$$\Longrightarrow \int_\Omega \varepsilon^T C\, \delta\varepsilon\, dV = \int_\Omega u_h^T B^T C B\, \eta_h\, dV$$

$$= \eta_h^T \underbrace{\int_\Omega B^T C B\, dV}_{K_h} u_h$$

$$\Longrightarrow \boxed{K_h = \int_\Omega B^T C B\, dV}$$

<u>Volumetric and deviatoric components of "K_h"</u>

$$K_h = \sum_{e=1}^{E} K^e = \sum_{e=1}^{E} \sum_{q=1}^{Q} w_q^e \, B^{e^T} C B^e$$

$$= 2\mu \sum_{e=1}^{E} \sum_{q=1}^{Q} w_q^e \, B^{e^T} P^{dev} B^e \longrightarrow K_h^{dev}$$

$$+ 3K \sum_{e=1}^{E} \sum_{q=1}^{Q} w_q^e \, B^{e^T} P^{vol} B^e \longrightarrow K_h^{vol}$$

$$K_h u_h = K_h^{dev} u_h + 3K \left(\sum_{e=1}^{E} \sum_{q=1}^{Q} w_p^e \, B^{e^T} P^{vol} B^e \right) u_h$$

$$\varepsilon_h^e = B^e u_h^e \rightarrow \left(\varepsilon_h^e\right)^{dev} = P^{dev} \varepsilon_h^e = \underbrace{P^{dev} B^e}_{\left(B^e\right)^{dev}} u_h^e$$

$$\left(\varepsilon_h^e\right)^{vol} = P^{vol} \varepsilon_h^e = \underbrace{P^{vol} B^e}_{\left(B^e\right)^{vol}} u_h^e$$

$$\Rightarrow \left.\begin{aligned} \left(\varepsilon_h^e\right)^{dev} &= \left(B^e\right)^{dev} u_h \\ \left(\varepsilon_h^e\right)^{vol} &= \left(B^e\right)^{vol} u_h \end{aligned}\right\} \underline{B^e = \left(B^e\right)^{vol} + \left(B^e\right)^{dev}}$$

$$B^{e^T} P^{vol} B^e = B^{e^T} \left(P^{vol}\right)^2 B^e = B^{e^T} P^{vol^T} P^{vol} B^e$$

$$= \left(P^{vol} B^e\right)^T \left(P^{vol} B^e\right) = \left[\left(B^e\right)^{vol}\right]^T \left(B^e\right)^{vol}$$

$$K_h u_h = K_h^{dev} u_h + 3K \sum_{e=1}^{E} \sum_{q=1}^{Q} w_q^e \, (B^e)^{vol^T} \underbrace{(B^e)^{vol} u_h^e}_{\varepsilon_h^{e\,vol} = \frac{1}{3}\theta_h^e I}$$

$$I^T = \{1, 1, 1, 0, 0, 0\}$$

$$\Rightarrow \boxed{K_h u_h = K_h^{dev} u_h + 3K \sum_{e=1}^{E} \sum_{q=1}^{Q} w_q^e \, (B^e)^{vol^T} \frac{1}{3} \theta_h(\xi_q^e) I = f_h}$$

Note: $K \theta_h(\xi_q^e) = p(\xi_q^e)$

Take limit $K \rightarrow \infty$. For $p(\xi_q^e) < \infty \Rightarrow$

$$\underbrace{\theta(\xi_q^e) \rightarrow 0}, \text{ in the limit} \Rightarrow$$

Each quadrature point introduces a volumetric constraint. Too many constraints results in LOCKING