Error estimation, convergence of finite element approximations

Questions:

- $\|u-u_h\| \to 0$ when $h \to 0$?
  where "h" is the mesh size
- At what rate?
- What is the error?

Plan: • Make use of "best approximation" property of Finite element solutions:

$$\|u-u_h\|_E \leq \inf_{v_h \in V_h} \|u-v_h\|_E$$

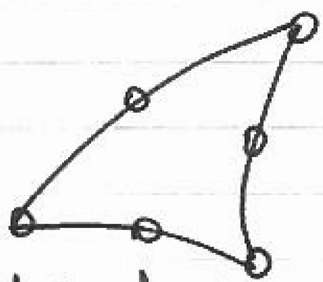
- Devise error estimates for $u_I \in V_h$, where u_I is the unique element in V_h that interpolates the exact solution u.

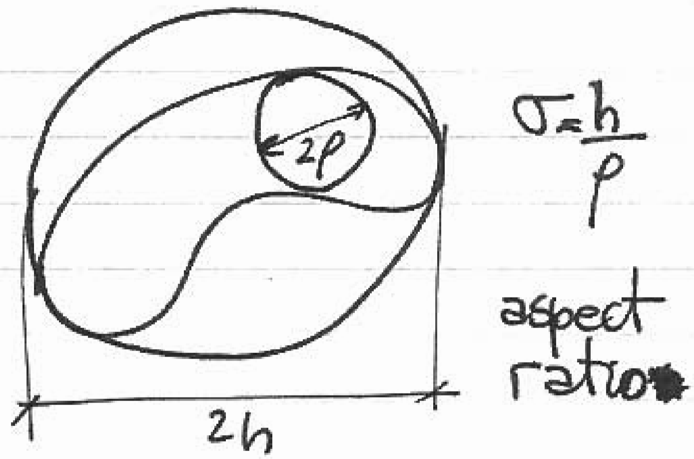
- $\|u-u_h\|_E \leq \|u-u_I\|_E \leq C\|u-u_I\|_1$

$\Longrightarrow$ error estimation problem is delegated to interpolation theory.

Error estimates from interpolation theory

$B = \bigcup_{e=1}^{E} \Omega_h^e$, Ω_h^e d-simplex of order "k"





Let $u \in H^m(B)$, $u^e \in H^{k+1}(\Omega_h^e)$ $e = 1, \dots, E$

Let u_I^e be the polynomial of degree $\leq k$,

i.e., $u_I^e \in P_k(\Omega_h^e)$, which interpolates u^e.

Extend $u_I^e(x) = 0$ over $B - \Omega_h^e$

Let $u_I = \sum_{e=1}^{E} u_I^e \equiv$ global interpolant ($\neq u_h$)

Assume $u_I \in H^m(B)$ (V_h satisfies C^{m-1}-conformity)

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$$\|u - u_I\|_m \leqslant \sum_{e=1}^{E} C \left(\sigma_h^e\right)^m \left(h^e\right)^{k+1-m} |u^e|_{k+1}$$