22A

$$\boxed{\text{Constraints ratio} = \frac{\#\,\text{displacements dof}}{\#\,\text{volumetric constraints}} = r}$$

$$\boxed{\text{Locking} \Leftrightarrow r \leqslant 1}$$

r^{opt}:

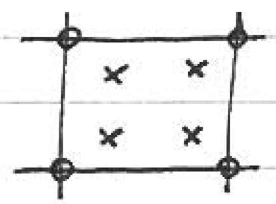
$$\begin{cases} \mu u_{i,jj} + p_{,i} + f_i = 0 \\ u_{i,i} = 0 \end{cases}$$

every point: d - displacement dof
1 - volumetric constraint

$$\Rightarrow \boxed{r^{opt} = d}$$

Examples

1) 4-node quad: $d=2$, $n=4$, $Q=4$ (for no zero energy modes)

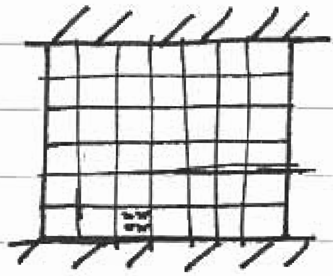


can analyze by elements since regular mesh.
Each node shared by 4 elements $\Rightarrow$

$$\text{nodes/element} = \frac{1}{4} 4 = 1$$

$$\#\text{dof/element} = 2 \qquad \boxed{r = \frac{1}{2}} \quad \text{Locking!}$$
$$\#\text{constraints/element} = 4$$

How does it manifest?

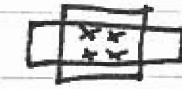


$\downarrow u$

$\frac{K}{\mu} \to 0$

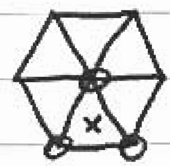
$\frac{\Delta V}{V} = 0$

$\to u_h = 0$



only deformation mode

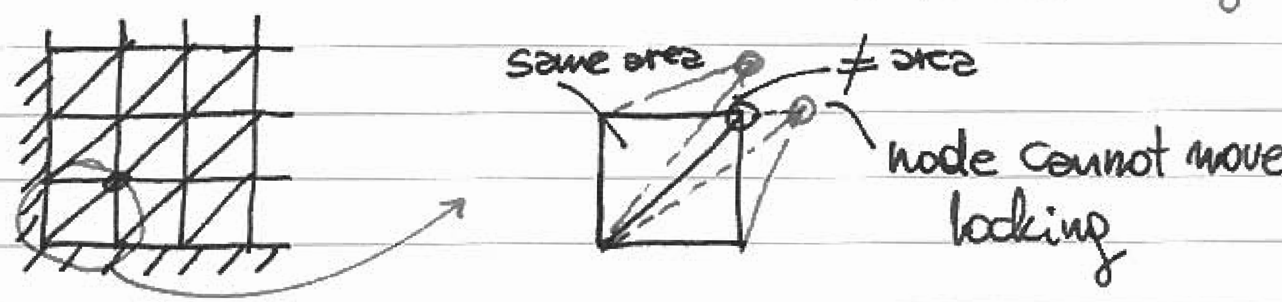
2) 3-node simplex: $n=3$, $d=2$, $Q=1$; hexagonal mesh



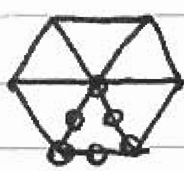
$$\#\text{nodes/element} = \frac{1}{6}3 = \frac{1}{2}$$

$$\#\text{dof/element} = 2\frac{1}{2} = 1$$

$$\#\text{constraints} = 1 \implies \boxed{r=1} \text{ locking!}$$



3) 6-node simplex: $n=6$, $d=2$, $Q=3$, hexagonal mesh



$$\#\text{nodes/element} = \frac{1}{6}3 + \frac{1}{2}3 = 2$$

$$\#\text{dof/element} = 4$$

$$\#\text{constraints/element} = 3 \qquad \boxed{r = \frac{4}{3}} \text{ no locking!}$$

working element, suboptimal.

Constrained problems

Solution approaches:

1) Selective, reduced integration
2) Assumed strain methods
3) Mixed-methods

Variational principles for incompressible elasticity

Recipe: weighted residuals

$$\int_{\Omega} (\mu u_{i,jj} + p_{,i} + f_i)\, \eta_i \, dV = 0 \,, \quad \forall \text{ admissible } \eta_i$$

$u \in V$ (suitable Sobolev Space)

$u = \bar{u}_i$ on S_1

Weak form:

$$\int_{B} (-\mu u_{i,j} \eta_{i,j} - p\, \eta_{i,i} + f_i \eta_i)\, dV +$$

$$+\int_{S_2} (\mu\, u_{i,j} + p\,\delta_{ij})\, n_j\, \eta_i\, dv$$

$$\boxed{\int_B (\mu\, u_{i,j}\, \eta_{i,j} + p\, \eta_{i,i})\, dv - \int_B f_i\, \eta_i\, dv - \int_{S_2} \bar{t}_i\, \eta_i\, ds}$$

$\forall$ admissible η

Look for $u \in V$, $p \in Q \, (\equiv L^2(B))$

the incompressibility condition

$$u_{i,i} = 0 \quad \text{in } B \rightarrow$$

$$\boxed{\int_B u_{i,i}\, q\, dV = 0, \ \forall q \in Q}$$

Variational statement

$$L(u,p) = \underbrace{\int_B \mu\, u_{i,j}\, u_{i,j}\, dV - \int_B f_i\, u_i\, dV - \int_{S_2} \bar{t}_i\, u_i\, dV}_{J(u)} -$$

$$-\int_B p \cdot u_{i,i}\, dV$$

$J(u)$: unconstrained potential

$$\boxed{L(u,p) = J(u) - \int_B p\, u_{i,i}\, dV}$$

First variation of "L":

$$\langle D L(u,p), \eta \rangle = 0$$

$$\langle D L(u,p), q \rangle = 0$$

$(u,p) \in V \times Q$ exact solution $\Rightarrow$ (u,p) NOT an extremum of "L" but a saddle point:

$$\boxed{L(u,q) \leq L(u,p) \leq L(v,p)\, ; \ \forall v, q \in V \times Q}$$

Saddle point problem:

$$\boxed{L(u,p) = \max_{q \in Q} \min_{v \in V} L(v,q) = \min_{v \in V} \max_{q \in Q} L(v,q)}$$

Constrained minimization problem

$$\boxed{J(u) = \min_{v \in V \,/\, \nabla \cdot v = 0} J(v)}$$

space of trial functions is constrained to $\nabla \cdot v = 0$

1) Reduced selective integration

$$L(u,p) = \frac{1}{2}\int_B \underbrace{2\mu\varepsilon_{ij}^{dev}}_{\sigma_{ij}^{dev}} \varepsilon_{ij}^{dev}\,dV + \int_B p\, u_{i,i}\,dV - FT$$

unconstrained potential:

$$J(u) = \frac{1}{2}\int_B 2\mu\varepsilon_{ij}^{dev}\varepsilon_{ij}^{dev}\,dV + \frac{1}{2}\int_B K(\mathrm{tr}\,\varepsilon)^2\,dV - FT$$

Penalty formulation (1-field)

$\alpha = \frac{1}{K}$, incompressible limit $\alpha \Rightarrow 0$

$$J_\alpha(u) = \frac{1}{2}\int_B 2\mu\varepsilon_{ij}^{dev}\varepsilon_{ij}^{dev}\,dV + \frac{1}{2\alpha}\int_B (\varepsilon_{kk})^2\,dV - FT$$

Finite element discretization:

$$K^e = \int_{\Omega^e} B^T C^{dev} B\,dV + \int_{\Omega^e} B^T C^{vol} B\,dV$$

$$= \sum_{q=1}^{Q} W_q \left(B^T D^{dev} B\right)(\xi_q) + \sum_{p=1}^{\bar{Q}} \bar{W}_p \left(B^T C^{vol} B\right)(\bar{\xi}_p)$$

$\bar{\xi}_p, \bar{W}_p, \bar{Q} \equiv$ reduced integration