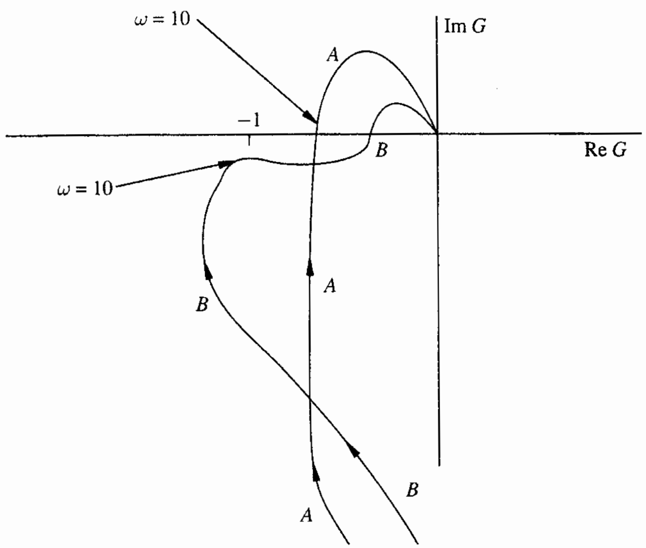


**FIGURE 6.5-14**
Exercise 6.5-3.

## 6.6 ROBUST STABILITY

In the previous section one notion of relative stability was discussed. In the view of that section, one system is considered less stable than another if its time response is more oscillatory and less highly damped. The relative stability decreases as the Nyquist plot of the loop gain transfer function approaches the $-1$ point. Besides an oscillatory time response, a system whose Nyquist plot passes close to the $-1$ point suffers from another problem. If such a system is even slightly mismodeled, the Nyquist plot can be easily perturbed in such a way that the number of encirclements of the $-1$ point changes without changing the number of right half-plane poles in the loop gain transfer function. The Nyquist theorem indicates that the number of right half-plane poles in the perturbed closed-loop system changes, so that, if the nominal closed-loop system is stable, the perturbed closed-loop system is unstable.

It is important that a control system that is designed to be stable and perform well when used in conjunction with a nominal plant model still works well when used in conjunction with an actual physical plant. The output of the physical plant can be expected to behave in a manner similar to but not exactly the same as the nominal model. A controller design that works well with a large set of plant models is said to be *robust*. It is apparent from the preceding discussion that a design with a loop gain Nyquist plot that passes close to the $-1$ point is not robust. In this section, this notion of robustness is formalized using the models of plant uncertainty developed in Sec. 5.7. First, conditions assuring that a design remains *stable* in the face of plant perturbations is developed. Clearly, maintaining stability for expected modeling errors is an absolute requirement. In addition, it is desirable to maintain adequate



performance in the face of modeling errors. After the question of stability robustness is resolved in this section the question of robust performance is addressed in Sec. 6.5.

In Sec. 5.7, we developed techniques for describing a set of possible plant models using a nominal plant model and a perturbation transfer function with a known magnitude bound. Consider first the description using an additive perturbation as described in Sec. 5.7

$$\tilde{G}_p(s) = G_p(s) + L_a(s) \tag{6.6-1a}$$

where $L_a(s)$ is itself a stable transfer function containing no right half-plane poles. A bound $l_a(j\omega)$ on the magnitude of $L_a(j\omega)$ is known, i.e.,

$$|L_a(j\omega)| < l_a(j\omega) \tag{6.6-1b}$$

but otherwise $L_a(s)$ is a completely unknown transfer function. We are interested in the stability of the $G$ configuration system of Fig. 6.6-1.

Let $G(s) = G_C(s)G_p(s)$ be the nominal loop gain, and $\tilde{G}(s) = G_C(s)\tilde{G}_p(s)$ be the perturbed loop gain.

Assume that the nominal design is stable, that is, that the closed-loop system is stable when $L_a(s) = 0$. The robust stability question is formulated as follows: What conditions must be placed on $G(s)$ so that the configuration of Fig. 6.6-1 remains stable for all $\tilde{G}_p(s)$ satisfying Eq. (6.6-1)?

The robust stability question can be answered using the same tool that was used to answer the nominal stability and relative stability questions—the Nyquist diagram. Consider Fig. 6.6-2, which contains a typical Nyquist plot of $G(s)$ and $\tilde{G}(s)$. From the definitions above we can see that

$$\tilde{G}(s) = G_C(s)\left(G_p(s) + L_a(s)\right) = G(s) + G_C(s)L_a(s) \tag{6.6-2}$$

The plot of $\tilde{G}(j\omega)$ can be obtained from the plot of $G(j\omega)$ at each value of $\omega$ by adding a vector corresponding to $G_C(j\omega)L_a(j\omega)$ to $G(j\omega)$. Indeed, if each possible $L_a(j\omega)$ satisfying Eq. (6.6-1$b$) is used in turn, the set of all possible $\tilde{G}(j\omega_o)$ at the frequency $\omega_o$ is given by the interior of a circle centered at $G(j\omega_o)$ with radius $G_C(j\omega_o)l_a(j\omega_o)$. This circle is shown on Fig. 6.6-2 assuming that the $L_a(j\omega)$ chosen for display has the maximum magnitude. The key observation for the desired result is that each $L_a(s)$ is assumed to be stable itself, so that the number of right half-plane poles of the loop gain $\tilde{G}(s)$ is unchanged from the number of right half-plane poles of the nominal loop gain $G(s)$. Therefore, the closed-loop stability assumed for the configuration in Fig. 6.6-1 with $L_a(s) = 0$ is maintained for nonzero $L_a(s)$ if and only if the number of encirclements of the $-1$ point is unchanged in going from $G(j\omega)$ to

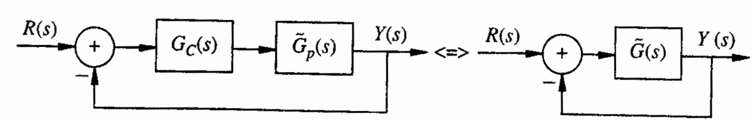


**FIGURE 6.6-1**
The perturbed $G$ configuration.

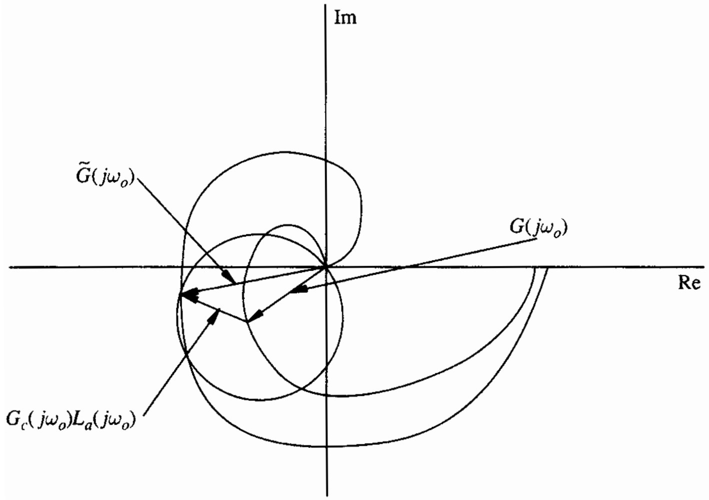


**FIGURE 6.6-2**
Nyquist plots of $\tilde{G}(s)$ and $G(s)$.

$\tilde{G}(j\omega)$. One way to assure that the number of encirclements remains unchanged is to assure that, at every single frequency, $l_a(j\omega)$ is small enough so that $\tilde{G}(j\omega)$ cannot reach the $-1$ point.

At each $\omega$, the distance between $G(j\omega)$ and the $-1$ point is given by

$$|G(j\omega) - (-1)| = |1 + G(j\omega)|$$

Therefore, if at every $\omega$ we have the condition

$$|G_C(j\omega)|\, l_a(j\omega) < |1 + G(j\omega)|$$

or

$$l_a(j\omega) < \frac{|1 + G(j\omega)|}{|G_C(j\omega)|} = \left|G_C^{-1}(j\omega) + G_p(j\omega)\right| \qquad (6.6\text{-}3)$$

then the perturbed system $\tilde{G}(j\omega)$ is stable since the perturbation in the Nyquist plot cannot change the number of encirclements.

From Eq. (6.6-3) we see that, as expected, a measure of the robustness of a control system is given by the distance the Nyquist plot of the loop gain transfer function maintains from the $-1$ point. This distance can be expressed as the magnitude of the return difference transfer function, a quantity that we already have seen should be kept large for such performance requirements as reference input tracking, low parameter sensitivity, and disturbance rejection.

Equation (6.6-3) provides the condition to guarantee maintenance of stability in the face of stable additive perturbations satisfying Eq. (6.6-1). It is interesting to see what can be said if Eq. (6.6-3) is violated. If Eq. (6.6-3) is violated the closed-loop system remains stable for many of the possible $L_a(s)$ satisfying Eq. (6.6-1); however,



there is no longer any guarantee that stability is maintained for every perturbation satisfying Eq. (6.6-1). Indeed, if $l_a(j\omega)$ is continuous in $\omega$ and if, for some $\omega_o$ and $\epsilon > 0$,

$$l_a(j\omega_o) \geq \frac{|1 + G(j\omega_o)|}{|G_C(j\omega_o)|} + \epsilon \qquad (6.6\text{-}4)$$

then there exists a perturbation $L_a(j\omega)$ satisfying Eq. (6.6-1) that causes instability. The destabilizing perturbation is constructed by choosing $L_a(s)$ stable so that

$$|L_a(j\omega_o)| = \frac{|1 + G(j\omega_o)|}{|G_C(j\omega_o)|} + \epsilon/2$$

and

$$\arg(L_a(j\omega_o)) = \arg(1 + G(j\omega_o)) - \arg G_C(j\omega_o) + 180^\circ.$$

The ensuing Nyquist plot of $\tilde{G}(j\omega)$ has a different number of encirclements of the $-1$ point than the Nyquist plot of $G(j\omega)$ and the closed-loop stability is lost.

As discussed in Sec. 5.7 it is often easier to express a class of possible plant models using a multiplicative perturbation rather than an additive perturbation. This is particularly true for stability robustness conditions since the condition of Eq. (6.6-3) has the compensator appearing separately from the loop gain. The robust stability condition for a multiplicative perturbation is a function of only the loop gain. Assume a set of possible plant models is

$$\tilde{G}_p(s) = G_p(s)\,(1 + L_m(s)) \qquad (6.6\text{-}5)$$

where $L_m(s)$ is itself a stable transfer function containing no right half-plane poles. A bound $l_m(j\omega)$ on the magnitude of $L_m(j\omega)$ is known, i.e.,

$$|L_m(j\omega)| < l_m(j\omega) \qquad (6.6\text{-}6)$$

Then Eqs. (6.6-5) and (6.6-6) are equivalent to Eqs. (6.6-1$a$) and (6.6-1$b$) with the identification

$$L_a(s) = G_p(s)L_m(s) \qquad (6.6\text{-}7)$$

However, the equivalent expression for the loop gain is

$$\tilde{G}(s) = G_C(s)G_p(S)\,(1 + L_m(s)) = G(s)\,(1 + L_m(s)) = G(s) + G(s)L_m(s) \qquad (6.6\text{-}8)$$

The Nyquist plot is perturbed from $G(s)$ to $\tilde{G}(s)$ by the addition of $G(s)L_m(s)$. The condition for robustness in the multiplicative perturbation setting is obtained by observing again that if, at each frequency, the distance that the Nyquist plot of $G(j\omega)$ can be perturbed is less than the distance to the $-1$ point, a nominally stable closed-loop system is guaranteed to remain stable. Thus, if the closed loop system of Fig. 6.6-1 is stable for $L_m(s) = 0$, if each $L_m(s)$ is stable, and if for all $\omega$

$$|l_m(j\omega)G(j\omega)| < |1 + G(j\omega)| \qquad (6.6\text{-}9)$$

then the closed-loop system of Fig. 6.6-1 remains stable. Equation (6.6-9) can be manipulated into the equivalent forms

$$l_m(j\omega) < \frac{|1+G(j\omega)|}{|G(j\omega)|} \tag{6.6-10}$$

$$\left|\frac{G(j\omega)}{1+G(j\omega)}\right| < \frac{1}{l_m(j\omega)} \tag{6.6-11}$$

$$l_m(j\omega)\left|\frac{G(j\omega)}{1+G(j\omega)}\right| < 1 \tag{6.6-12}$$

As with the robustness condition for the additive perturbation formulation, something can be said if Eq. (6.6-11) is violated. If Eq. (6.6-11) is violated, then there is a multiplicative perturbation satisfying Eq. (6.6-6) that causes the perturbed closed-loop system of Fig. 6.6-1 to be unstable.

The robustness condition of Eq. (6.6-11) usually poses a constraint on the allowable bandwidth of a control system as can be seen by the following argument. Recall from Sec. 5.7 that usually a multiplicative perturbation is small at low frequencies and grows to be larger than unity at higher frequencies. Let $\omega_1$ be the frequency where $l_m(j\omega_1) = 2$. Assume that for $\omega > \omega_1$, $l_m^{-1}(j\omega) < \frac{1}{2}$. Now notice that the left hand side of Eq. (6.6-11) is simply the nominal closed-loop transfer function. It is usually desirable to keep the closed-loop transfer function close to unity for as large a range of frequencies as possible. However, the constraint of Eq. (6.6-11) dictates that the magnitude of the nominal closed-loop transfer function be less than $\frac{1}{2}$ and the nominal loop gain be less than 1 and for all $\omega > \omega_1$. Thus the maximum bandwidth of the system is limited to less than $\omega_1$ if the controller is to result in a stable closed-loop system for all possible models as given by Eqs. (6.6-5) and (6.6-6).

**Example 6.6-1.** Let a collection of possible plant models be given by Eqs. (6.6-5) and (6.6-6) with

$$G_p(s) = \frac{1}{s} \tag{6.6-13}$$

$$l_m(s) = \frac{1}{10}|(s+1)| \tag{6.6-14}$$

A series compensator given by

$$G_C(s) = a,\ a \text{ constant} \tag{6.6-15}$$

makes

$$G(s) = \frac{a}{s} \tag{6.6-16}$$

By straight calculation, the closed-loop transfer function is

$$\frac{G(s)}{1+G(s)} = \frac{a}{s+a} \tag{6.6-17}$$



which includes a single closed-loop pole at $s = -a$. The closed-loop bandwidth covers $\omega = 0$ to $\omega = a$.

Figure 6.6-3 plots are derived from Eq. (6.6-14) and the closed-loop frequency response for various values of $a$. From the figure and a few calculations we can see that the robustness condition of Eq. (6.6-11) is satisfied for all $a < 10$. Thus, the maximum bandwidth is roughly equal to the frequency where $l_m^{-1}(j\omega) = 1$.

Now, let's take $a = 11$ so that the robustness condition is violated. We show that there is a multiplicative perturbation satisfying the magnitude bound given by Eq. (6.6-14) which produces an unstable closed-loop system.

To construct a destabilizing perturbation first choose a frequency $\omega_o$ where the robustness constraint of Eq. (6.6-11) is violated. In this example, we can choose $\omega_o = 100$ since

$$\left|\frac{G(j100)}{1+G(j100)}\right| = \left|\frac{11}{j100+11}\right| = \frac{11}{\sqrt{10121}} \geq \frac{1}{l_m(j100)} = \frac{10}{\sqrt{10001}}$$

The concept is to select $l_m(j\omega_o)$ so that the equivalent additive perturbation given by $l_m(j\omega_o)G(j\omega_o)$ moves the Nyquist plot of $G(j\omega)$ to and past the $-1$ point at the frequency $\omega_o$. Thus we select the magnitude of $L_m(j\omega_o)$ to be slightly larger than $|1+G(j100)|/|G(j100)|$ while remaining smaller than $l_m(j\omega_o)$ and we select the phase of the $L_m(j\omega_o)$ to point the vector $L_m(j\omega_o)G_m(j\omega_o)$ in the Nyquist plane from $G(j\omega_o)$ towards $-1$. This can be achieved using the relationship

$$\arg(L_m(j\omega_o)) = \arg(1+G(j\omega_o)) - 180^\circ - \arg(G(j\omega_o))$$

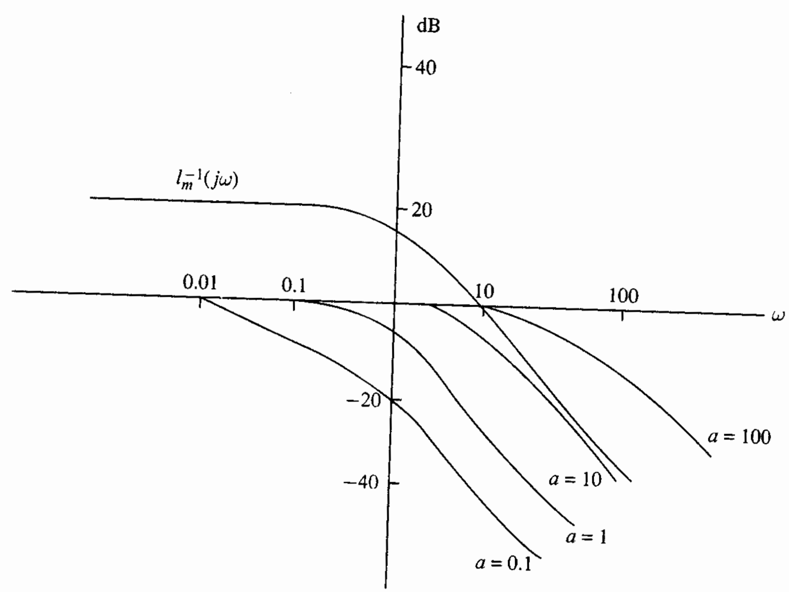


**FIGURE 6.6-3**
Bode magnitude plots of Eq. (6.6-14) and Eq. (6.6-17) for various $a$.

Performing these calculations the phase of the destabilizing $L_m(j\omega_o)$ for this example is $-96°$ while 9.5 is a destabilizing magnitude. Once we know what $L_m(j\omega_o)$ should equal we only need to fit a transfer function that also satisfies the upper bound of Eq. (6.6-6) to that point. For example, one destabilizing $L_m(s)$ is given by

$$L_m(s) = \frac{0.95(s+1)}{(0.001s+10)} \frac{(-0.0103s+1)^2}{(0.0103s+1)^2}$$

First note that $L_m(s)$ is stable and that

$$|L_m(s)| < \frac{0.95|j\omega + 1|}{10} < l_m(j\omega)$$

Then we can draw the Nyquist plot for the loop gain $G(s)(1 + L_m(s))$ as in Fig. 6.6-4 and see that the resulting closed-loop system is unstable.

We have seen in this section that the stability of the nominal system can be guaranteed for a system with a stable multiplicative perturbation if Eq. (6.6-12) is satisfied. Such a system is said to possess robust stability with respect to the perturbation in question. Multiplicative perturbations tend to get large at high frequencies. Equation (6.6-12) requires that the nominal loop gain is made small at high frequencies.

## Exercises 6.6

**6.6-1.** A plant is modeled by

$$G_p(s) = \frac{3}{s+1}$$

but in reality the plant contains unmodeled dynamics. Its perturbed transfer function is

$$\tilde{G}_p(s) = \frac{15(s+5)}{(s+1)\left(s^2+3s+25\right)}$$

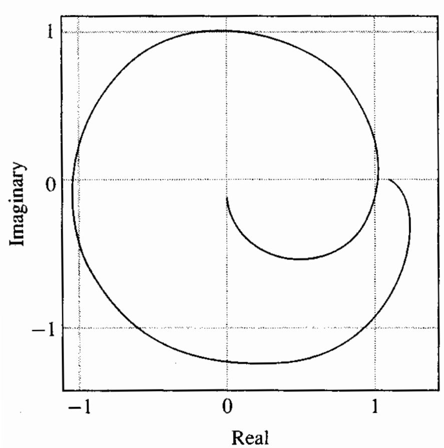


**FIGURE 6.6-4**
Perturbed loop gain for Example 6.6-1.



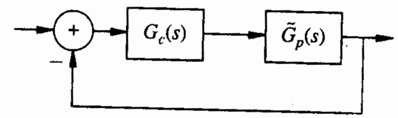


**FIGURE 6.6-5**
Exercise 6.6-1.

(*a*) Find the additive perturbation $L_a(s)$ and the multiplicative perturbation $L_m(s)$ for this case. Sketch the Bode magnitude plots of $L_a(j\omega)$ and $L_m(j\omega)$. Consider the closed-loop system of Figure 6.6-5. Let $G_C(s) = K$, a constant.

(*b*) For $K = 1$, check to see if the design that is clearly stable for the nominal model is stable for the actual plant by sketching the Bode magnitude plot of

$$\frac{|1 + G_C(j\omega)G_p(j\omega)|}{|G_C(j\omega)|}$$

on the same plot as $L_a(j\omega)$. Also, sketch the Bode magnitude plot of

$$\frac{G_C(j\omega)G_p(j\omega)}{1 + G_C(j\omega)G_p(j\omega)}$$

on the same plot as $|L_m^{-1}(j\omega)|$.

(*c*) Repeat (*b*) for $K = 8$.

*Answers:*

(*a*) $L_a(s) = \dfrac{\frac{6}{25}s(1-0.5s)}{(s+1)\left(1 + \frac{0.6}{5}s + \left(\frac{s}{5}\right)^2\right)}$ $\qquad L_m(s) = \dfrac{2}{25}\dfrac{s(1-0.5s)}{\left(\left(\frac{s}{5}\right)^2 + \frac{0.6}{5}s + 1\right)}$

(*b*) Robustly stable

(*c*) Not robustly stable

Remember, the magnitude tests in (*b*) and (*c*) are sufficient for stability, but not necessary. If a system passes the magnitude test, then the perturbed system is stable. However, if the system fails the magnitude test the perturbed system may be stable or unstable. (Option: To see this analyze (*b*) and (*c*) with $K = 2$. Check the robust stability test and check stability directly.)

**6.6-2.** The magnitude of the possible multiplicative perturbations that are likely to occur between a plant $\tilde{G}_p(s)$ and a model $G_p(s)$ can be bounded by $|L_m(s)| < l_m(s)$, where $l_m(s)$ is shown in Fig. 6.6-6.

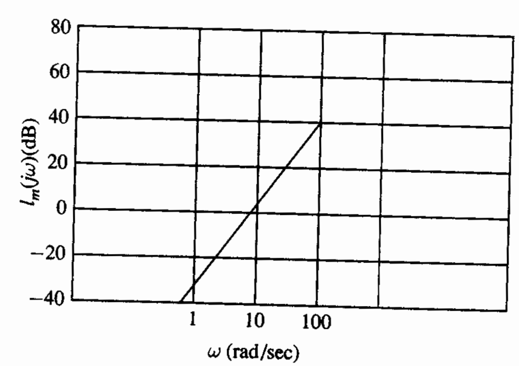


**FIGURE 6.6-6**
Exercise 6.6-2.

The model is given by $G_P(s) = \frac{1}{s+1}$. If a proportional controller $G_c(s) = K$ is used the nominal closed-loop system is stable for all $K > -1$. For what values of $K > 0$ can stability be guaranteed in the presence of all multiplicative perturbations obeying the above bound? Explain your answer briefly.

*Answer:* $K < 1$

## 6.7 PERFORMANCE AND ROBUSTNESS

In Sec. 6.6, we developed the constraint of Eq. (6.6-11), which characterizes a collection of loop gain transfer functions for which a perturbed closed-loop system remains stable. If such a constraint is satisfied and the actual plant's behavior is adequately described by one of the models in the collection of models given by Eqs. (6.6-5) and (6.6-6), the implementation of the control system produces a stable closed-loop response. However, even though that response is stable it may produce unacceptable oscillatory responses. We need some way to assure that the performance aspects required for the system and met by the nominal design are also met when the controller is implemented on the actual plant. We cannot assure 100 percent performance on the actual plant since even the collection of plants described by Eq. (6.6-5) and (6.6-6) cannot perfectly model the actual plant. We can arrive at a constraint that will assure that some performance measure is achieved for any plant that is a member of the collection of plants described by Eqs. (6.6-5) and (6.6-6).

We must first decide on a performance measure that suitably describes such diverse control system performance requirements as possessing desirable transient responses to command inputs, quickly and completely rejecting certain disturbances, and providing a small sensitivity in response to small parameter changes. One way to abstract these goals into a single performance specification is to recall that, in Sec. 3.2, we came to the realization that all these performance objectives can be met if we can make the return difference transfer function large enough. While the return difference cannot capture every aspect of performance that might be specified, a controller with a large return difference over a broad frequency band can be expected to perform well in most common tests of control system performance. With that in mind we can write down an interesting general performance requirement for a control system.

We can say that a control system in the $G$ configuration performs adequately if

$$|1 + \tilde{G}(j\omega)| > p(j\omega) \tag{6.7-1}$$

where $p(j\omega)$ is some specified function of frequency. We now show an example which demonstrates how $p(j\omega)$ may be derived from typical performance requirements.

**Example 6.7-1.** One performance aspect of interest in a control system is how well disturbances are rejected. A typical specification would be that a control system must reject all constant output disturbances completely as time goes to infinity, and, in addition, it must attenuate the effect of all output disturbances of frequency less than 1 rad/sec so that the output is disturbed by less than 1 percent of the magnitude of the disturbance. The effect of an output disturbance on the plant output is taken from Table 3.2-2.

$$Y(s) = \frac{1}{1 + \tilde{G}(s)} D(s) \tag{6.7-2}$$



where the class of perturbed plants $\tilde{G}(s)$ is used to indicate the desire that the system reject disturbances when used with any of the possible plant models. The disturbance rejection requirement above can be translated to the form of Eq. (6.7-1) by requiring

$$p(j\omega) > 100 \text{ for } \omega < 1 \tag{6.7-3}$$

and

$$\lim_{\omega \to 0} p(j\omega) = \infty \tag{6.7-4}$$

If Eqs. (6.7-1) and (6.7-3) are satisfied, then

$$|Y| \le \frac{1}{|p(j\omega)|}|D| \le 0.01|D| \qquad \text{for } \omega < 1$$

A second common specification involves the closed-loop transfer function's sensitivity to changes in the plant transfer function. A specification may read that for all input frequencies less than 10 rad/sec the closed loop transfer function's magnitude response should not differ from a nominal design by more than 1 percent of the change in the plant model. Again, from Table 3.2-2 it is seen that

$$S_{\tilde{G}_p}^{M} = \frac{1}{1 + \tilde{G}(s)} \tag{6.7-5}$$

The sensitivity requirement is met if

$$p(j\omega) > 100 \text{ for } \omega < 10 \tag{6.7-6}$$

Specifications on the transient response of the system translate into specifications on $p(j\omega)$ less directly than do specifications on disturbance rejection and sensitivity reduction. To translate typical step response information into a specification on $p(j\omega)$, two intermediate steps are used. First, the transient response specifications are translated into desired dominant closed-loop pole positions. Then, the desired closed-loop pole positions are translated into a desired closed-loop frequency response. Finally, the closed-loop frequency response is translated into a specification on the return difference function or, equivalently, $p(j\omega)$.

Suppose that there is a requirement to produce a closed-loop step response which has less than 20 percent overshoot and a 5 percent settling time of less than 5.5 sec. From Eqs. (4.4-13) and (4.4-14), it is seen that these specifications are satisfied by a closed-loop transfer function whose dominant behavior is characterized by a single pole pair with damping ratio $\zeta = 0.47$ and natural frequency $\omega_n = 1.2$. From Sec. 5.3 it is known that the closed-loop frequency response for such a transfer function has magnitude very close to unity for all frequencies less than 1.2 rad/sec and rolls off at higher frequencies. From Eq. (5.3-9) the peak of the magnitude plot is computed to be less than 1.2.

In the $G$ configuration the magnitude of the closed-loop frequency response $|\tilde{M}_c(j\omega)|$ is related to the magnitude of the loop gain $|\tilde{G}(j\omega)|$ and the magnitude of the return difference $|1 + \tilde{G}(j\omega)|$ by the expression

$$\left|\tilde{M}_c(j\omega)\right| = \frac{\left|\tilde{G}(j\omega)\right|}{\left|1 + \tilde{G}(j\omega)\right|} \tag{6.7-7}$$

If $|1 + \tilde{G}(j\omega)|$ is much greater than one, then $|\tilde{G}(j\omega)|$ is much greater than one and $|\tilde{M}_c(j\omega)|$ is very close to one. This logic can be quantified using the inequalities

$$\left|1 + \tilde{G}(j\omega)\right| - 1 < \left|\tilde{G}(j\omega)\right| < \left|1 + \tilde{G}(j\omega)\right| + 1 \tag{6.7-8}$$

Dividing Inequalities (6.7-8) by $|1 + \tilde{G}(j\omega)|$ and using Eq. (6.7-7) we can arrive at an expression relating the magnitude of the closed-loop frequency response to the magnitude of the return difference.

$$1 - \frac{1}{\left|1 + \tilde{G}(j\omega)\right|} < \left|\tilde{M}_c(j\omega)\right| < 1 + \frac{1}{\left|1 + \tilde{G}(j\omega)\right|} \tag{6.7-9}$$

If the magnitude of the return difference is greater than 5 for $\omega < 1.2$, the magnitude of the closed-loop frequency response remains between 0.8 and 1.2 for those frequencies. The inequalities given by (6.7-9) provide only a rough guideline to what is needed to achieve a certain closed-loop response. To meet the step response specifications, not only must the magnitude of the closed-loop response be kept near one for the appropriate frequencies ($\omega < 1.2$ in this case) but also peaks in the magnitude of the closed-loop frequency response at higher frequencies must be avoided. This requires that the return difference be kept from being too small at any frequency.

The guidelines needed to produce an adequate closed-loop frequency response become clearer by looking at Fig. 6.7-1. This figure shows a polar frequency plot complete with $M$-circles for a typical loop gain transfer function. Recall from Section 6.5 that, from the $M$-circles, the magnitude of the closed-loop frequency response can be read off at any frequency. The plot of Fig. 6.7-1 would be an acceptable loop gain for this

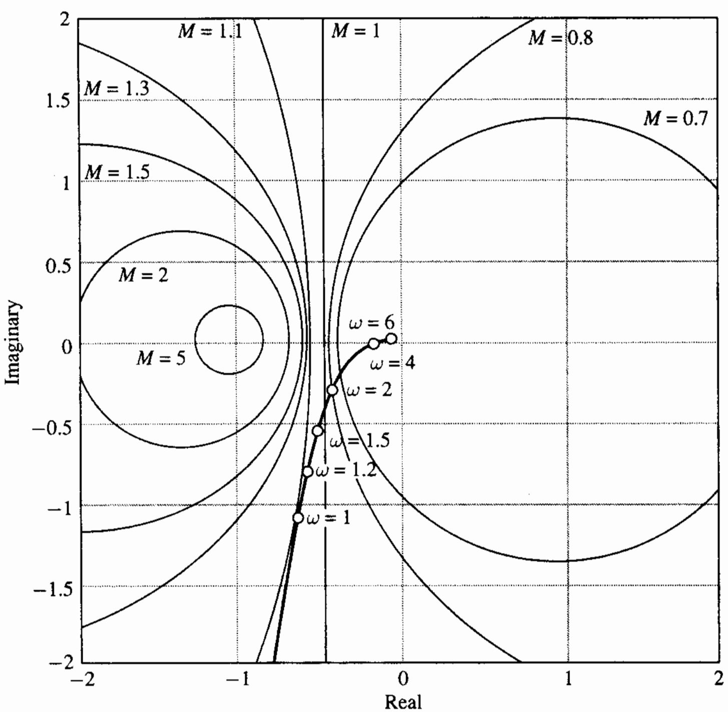


**FIGURE 6.7-1**
Performance specifications for transient response illustrated with $M$-circles.



example since the peak $M$-circle reached is $M = 1.1$ near $\omega = 1.2$ and the plot moves through $M$-circles of smaller values for higher frequencies.

In this typical example, the guidelines of Eq. (6.7-9) are conservative in that it is not required to have the return difference greater than 5 for all $\omega \le 1.2$. The guidelines are somewhat lacking as mentioned above in that they don't guarantee that a peak in the closed-loop frequency won't occur at a higher frequency, causing oscillations.

In spite of the shortcomings in precise guarantees, it is still useful to provide a guideline for adequate transient response of the closed-loop system by bounding the return difference from below. For many control systems, the loop gain transfer function decreases monotonically producing a smooth polar frequency plot similar to Fig. 6.7-1. For these systems the range of frequencies where the magnitude of the return difference stays greater than one provides a good estimate of the bandwidth of the controller. The greater the frequency range over which the magnitude of the return difference stays larger than one, the wider is the bandwidth of the system and the faster can the system respond to step inputs. In addition, the magnitude of the return difference should be kept as large as possible over all frequencies to guard against high frequency oscillations.

In the example, we have shown that the concept of providing a performance measure for a control system by bounding the magnitude of the return difference as in Eq. (6.7-1) works well for specifications involving disturbance rejection and sensitivity reduction. The bound of Eq. (6.7-1) also provides a guideline for transient response specifications. This lack of precision for transient response specification is acceptable because the transient response of a robustly stable control loop that responds almost fast enough can usually be modified to precisely meet specifications using a prefilter on the command input.

Equation (6.7-1) can be rewritten as

$$\left|\frac{1}{1 + \tilde{G}(j\omega)}\right| < \frac{1}{p(j\omega)} \tag{6.7-10}$$

or

$$p(j\omega)\left|\frac{1}{1 + \tilde{G}(j\omega)}\right| < 1 \tag{6.7-11}$$

The function $\tilde{S}(j\omega) = (1 + \tilde{G}(j\omega))^{-1}$ is called the *sensitivity function* of the perturbed control system since the response of the control system is relatively insensitive to parameter changes and disturbances if this function is small.

Notice that the performance requirement of Eq. (6.7-1) is written as a function of $\tilde{G}(j\omega)$. It is desirable to design a controller based upon the nominal model of the plant $G(j\omega)$ that meets the performance requirement of Eq. (6.7-1) for any plant $\tilde{G}(j\omega)$ in the collection given by the multiplicative perturbation model of Eqs. (6.6-5) and (6.6-6).

To assure that the sensitivity function of the perturbed control system $\tilde{S}(j\omega)$ is small for any allowable perturbation $L_m(j\omega)$ it is useful to find how large $\tilde{S}(j\omega)$ can become when facing the most damaging perturbation allowable. The goal is to find the maximum value of $\tilde{S}(j\omega)$ as $L_m(j\omega)$ is allowed to vary over all allowable perturbations.

$$\max_{|L_m(j\omega)|<l_m(j\omega)} \left|\tilde{S}(j\omega)\right|$$

$$= \max_{|L_m(j\omega)|<l_m(j\omega)} \frac{1}{\left|1+\tilde{G}(j\omega)\right|}$$

$$= \max_{|L_m(j\omega)|<l_m(j\omega)} \frac{1}{|1+G(j\omega)+G(j\omega)L_m(j\omega)|}$$

$$= \frac{1}{\min\limits_{|L_m(j\omega)|<l_m(j\omega)} |1+G(j\omega)+G(j\omega)L_m(j\omega)|} \tag{6.7-12}$$

Clearly, minimizing the denominator maximizes the expression of Eq. (6.7-12) where various definitions have been substituted. The minimization is accomplished by realizing that the worst case $L_m(j\omega)$ has maximal magnitude and has phase aligned to subtract this maximal magnitude away from the first two terms.

$$\min_{|L_m(j\omega)|<l_m(j\omega)} |1+G(j\omega)+G(j\omega)L_m(j\omega)|$$

$$= |1+G(j\omega)| - |G(j\omega)l_m(j\omega)| \tag{6.7-13}$$

So

$$\max_{|L_m(j\omega)|<l_m(j\omega)} \left|\tilde{S}(j\omega)\right| = \frac{1}{|1+G(j\omega)| - |G(j\omega)|l_m(j\omega)}$$

$$= \frac{1}{|1+G(j\omega)|}\left(\frac{1}{1-\left|\dfrac{G(j\omega)}{1+G(j\omega)}\right| l_m(j\omega)}\right) \tag{6.7-14}$$

The nominal *complementary sensitivity function*, $T(j\omega)$ is defined as

$$T(j\omega) = \frac{G(j\omega)}{1+G(j\omega)} = 1 - \frac{1}{1+G(j\omega)} = 1 - S(j\omega) \tag{6.7-15}$$

The complementary sensitivity function equals one minus the sensitivity function. (The complementary sensitivity function is also equal to the closed-loop response function for the $G$ configuration.) A final expression for the sensitivity under the worst case perturbation in terms of the nominal sensitivity and complementary sensitivity functions can be written as

$$\max_{|L_m(j\omega)|<l_m(j\omega)} \left|\tilde{S}(j\omega)\right| = |S(j\omega)|\left(\frac{1}{1-|T(j\omega)|l_m(j\omega)}\right) \tag{6.7-16}$$



If the worst case perturbed sensitivity function remains less than $p^{-1}(j\omega)$ then the sensitivity functions for all allowable perturbations are less than $p^{-1}(j\omega)$. The robust performance condition of Eq. (6.7-10) is satisfied if

$$|S(j\omega)|\left(\frac{1}{1-|T(j\omega)|l_m(j\omega)}\right) < \frac{1}{p(j\omega)} \tag{6.7-17}$$

Equation (6.7-17) can be rewritten as

$$|S(j\omega)|p(j\omega) + |T(j\omega)|l_m(j\omega) < 1 \tag{6.7-18a}$$

or, more explicitly,

$$\left|\frac{1}{1+G(j\omega)}\right| p(j\omega) + \left|\frac{G(j\omega)}{1+G(j\omega)}\right| l_m(j\omega) < 1 \tag{6.7-18b}$$

Equation (6.7-18) provides some interesting information. First note that keeping the second term less than one matches Eq. (6.6-12) and guarantees stability robustness. Keeping the first term less than one matches Eq. (6.7-10) with $G(j\omega)$ replacing $\tilde{G}(j\omega)$ and this provides for acceptable nominal performance, i.e., acceptable performance would result if the plant actually responded like $G_p(j\omega)$. When there is mismodeling as represented by $l_m(j\omega)$ the nominal design must exceed the performance specification by enough of a margin to account for modeling error. Alternatively, the nominal design must not only allow enough stability margin to ensure stability but must allow a greater margin to maintain performance.

It is perhaps even more interesting to view Eq. (6.7-18) as a weighted tradeoff between two terms. The first term contains the magnitude of the nominal sensitivity function, weighted by the performance requirement, which is large at frequencies where good performance is required. The second term contains the magnitude of the complementary sensitivity function weighted by the bound on the modeling error, which is large at frequencies where the plant is not well modeled.

Since by Eq. (6.7-15) the sensitivity function and the complementary sensitivity function sum to one they cannot both be small at the same frequency. Thus, by using Eq. (6.7-18) it can be seen that good control system performance can be maintained only at frequencies where the plant is well modeled. The modeling error quantified by $l_m(j\omega)$ is usually large at high frequencies. The complementary sensitivity function is then required to be small at high frequencies. A small complementary sensitivity function means a sensitivity function very near one, dictating that the achievable performance function $p(j\omega)$ be somewhat less than one at high frequencies. The resulting implication that the magnitude of the sensitivity function cannot be kept smaller than one for all frequencies means poor performance at some frequencies in the areas of disturbance rejection, sensitivity reduction and reference input tracking. Luckily, in most situations, a large performance bound is required only for low frequency reference inputs and disturbances. Similar logic then dictates that $l_m(j\omega)$, the modeling error, be small at low frequencies. If the control designers are asked to produce strong performance results at frequencies where the modeling error is large, they must reply that they can not. Either the performance requirements must be relaxed at those