Topic #4

16.30/31 Feedback Control Systems

Control Design using Bode Plots

- Performance Issues
- Synthesis
- Lead/Lag examples

Bode's Gain Phase Relationship

- Control synthesis by classical means would be very hard if we had to consider both the magnitude and phase plots of the loop, but that is not the case.

- **Theorem**: For any stable, minimum phase system with transfer function $G(s)$, $\measuredangle G(\mathbf{j}\omega)$ is **uniquely related** to the slope of $|G(\mathbf{j}\omega)|$.
  - Relationship is that, on a log-log plot, if slope of the magnitude plot is constant over a decade in frequency, with slope n, then

$$\measuredangle G(\mathbf{j}\omega) \approx 90^\circ n$$

- So in the crossover region, where $L(\mathbf{j}\omega) \approx 1$ if the magnitude plot is (locally):

s^0	slope of 0, so no crossover possible
s^{-1}	slope of -1, so about 90° PM
s^{-2}	slope of -2, so PM very small

- **Basic rule** of classical control design:

  Select $G_c(s)$ so that **LTF crosses over with a slope of -1.**

Performance Issues

- Step error response

$$e_{ss} = \frac{1}{1 + G_c(0)G_p(0)}$$

and we can determine $G_c(0)G_p(0)$ from the low frequency Bode plot for a type 0 system.

- For a type 1 system, the DC gain is infinite, but define

$$K_v = \lim_{s \to 0} sG_c(s)G_p(s) \;\Rightarrow\; e_{ss} = 1/K_v$$

  - So can easily determine this from the low frequency slope of the Bode plot.

Performance Issues II

- Classic question: how much phase margin do we need? Time response of a second order system gives:
  1. Closed-loop pole damping ratio $\zeta \approx PM/100$, $PM < 70^\circ$
  2. Closed-loop resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \approx \frac{1}{2\sin(PM/2)}$, near
  $$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$
  3. Closed-loop bandwidth
  $$\omega_{BW} = \omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}}$$
  $$\text{and} \quad \omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}$$

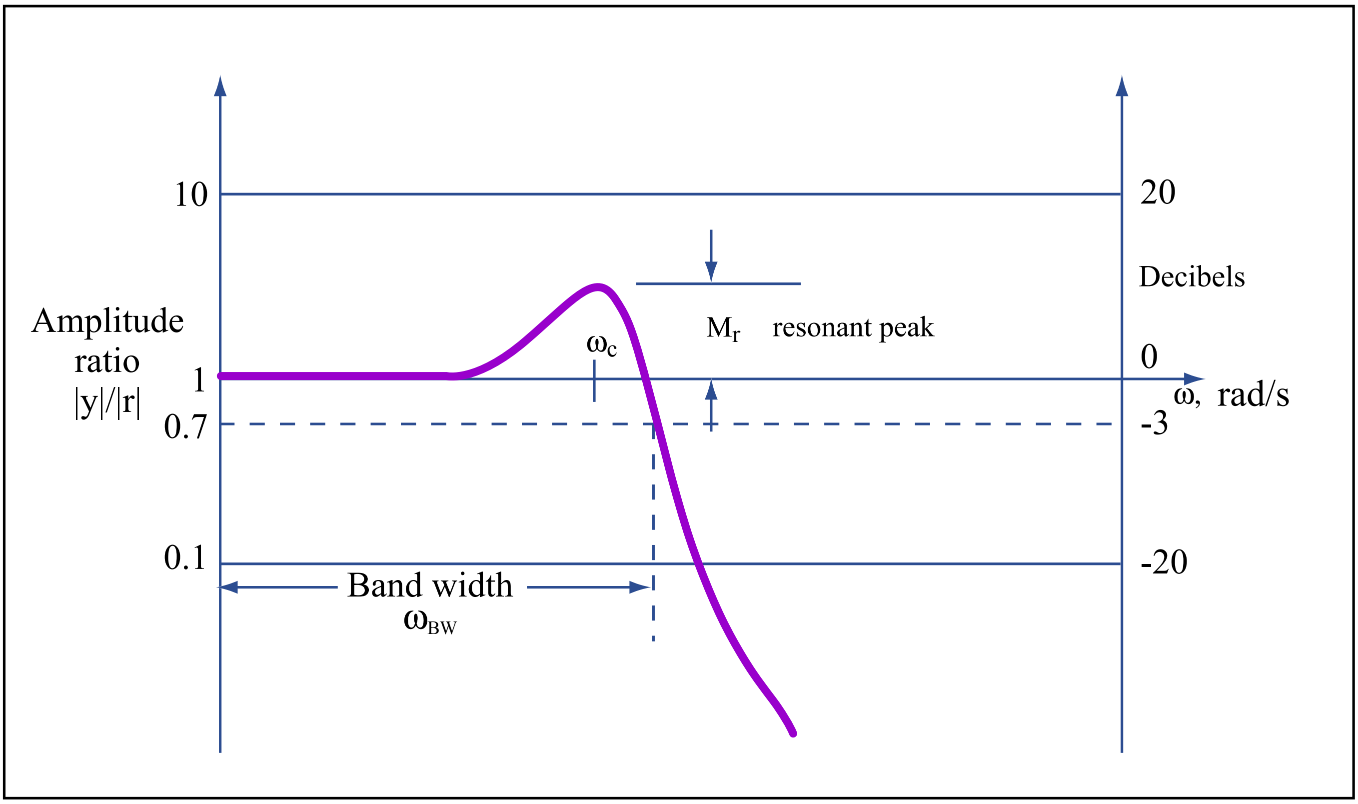

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Fig. 1: Frequency domain performance specifications.

- So typically specify ω_c, PM, and error constant as design goals

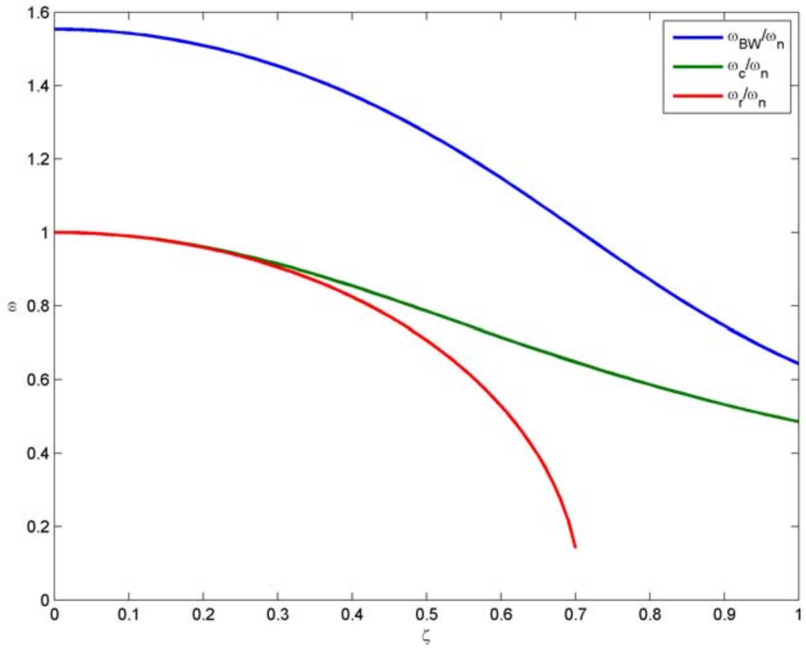

Fig. 2: Crossover frequency for second order system

```
figure(1)%
z=[0:.01:1]'; zs=[0:.01:1/sqrt(2)-eps]';
wbw=sqrt(1-2*z.^2+sqrt(2-4*z.^2+4*z.^4));%
wc=sqrt(sqrt(1+4*z.^4)-2*z.^2);%
wr=sqrt(1-2*zs.^2);%
set(gcf,'DefaultLineLineWidth',2) %
plot(z,wbw,z,wc,zs,wr)%
legend('\omega_{BW}/\omega_n','\omega_c/\omega_n','\omega_r/\omega_n') %
xlabel('\zeta');ylabel('\omega')%
print -dpng -r300 fresp.png
```

- Other rules of thumb come from approximating the system as having a 2nd order dominant response:

10-90% rise time	$t_r = \dfrac{1 + 1.1\zeta + 1.4\zeta^2}{\omega_n}$
Settling time (5%)	$t_s = \dfrac{3}{\zeta\omega_n}$
Time to peak amplitude	$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Peak overshoot	$M_p = e^{-\zeta\omega_n t_p}$

Frequency Domain Design

- Looked at the building block

$$G_c(s) = K_c \frac{s+z}{s+p}$$

- Question: how choose $G_c(s)$ to modify the LTF $L(s) = G_c(s)G_p(s)$ to get the desired bandwidth, phase margin, and error constants?

- Lead Controller ($|z| < |p|$)
  - Zero at a lower frequency than the pole
  - Gain increases with frequency (slope +1)
  - Phase positive (i.e. this adds phase lead)

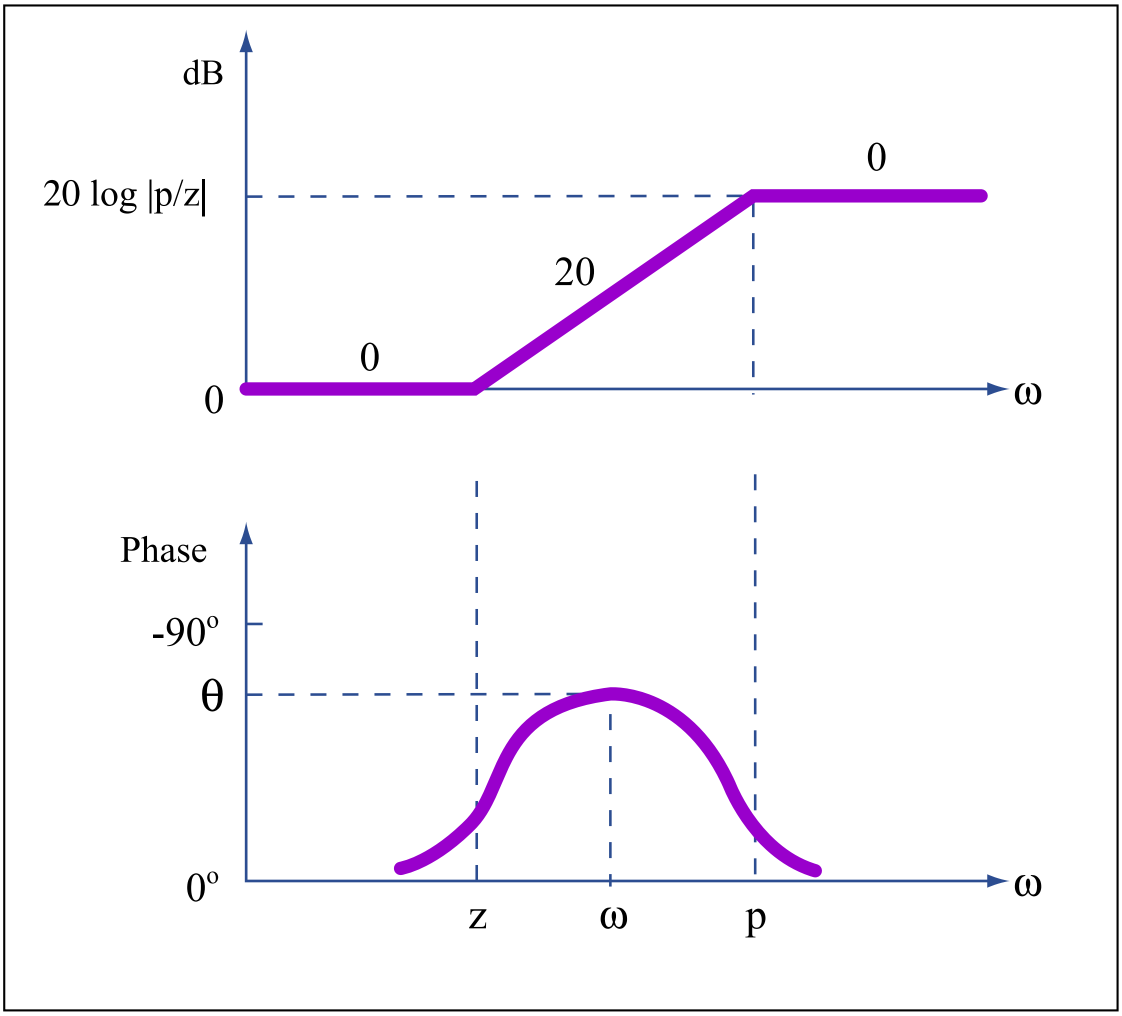

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Fig. 3: Lead: frequency domain.

Lead Mechanics

- Maximum phase added

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}$$

where $\alpha = |z|/|p|$, which implies that

$$\alpha = \frac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}}$$

- Frequency of the maximum phase addition is $\omega_{\max} = \sqrt{|z|\cdot|p|}$
  - Usually try to place this near ω_c
- High frequency gain increase is by $1/\alpha$

- So there is a compromise between wanting to add a large amount of phase (α small) and the tendency to generate large gains at high frequency
  - So try to keep $1/\alpha \leq 10$, which means $|p| \leq 10|z|$ and

$$\phi_{\max} \lesssim 60^\circ$$

- If more phase lead is needed, use multiple lead controllers

$$G_c(s) = k_c \frac{(s+z_1)}{(s+p_1)}\frac{(s+z_2)}{(s+p_2)}$$

- Select of the overall gain is problem specific.
  - One approach is to force the desired crossover frequency so that $|L(\mathbf{j}\omega_c)| = 1$

- Use Lead to add phase $\Rightarrow$ increases PM $\Rightarrow$ improves transient response.

Lead Mechanics II

- Adding a lead to the LTF changes both the magnitude and phase, so it is difficult to predict the new crossover point (which is where we should be adding the extra phase).

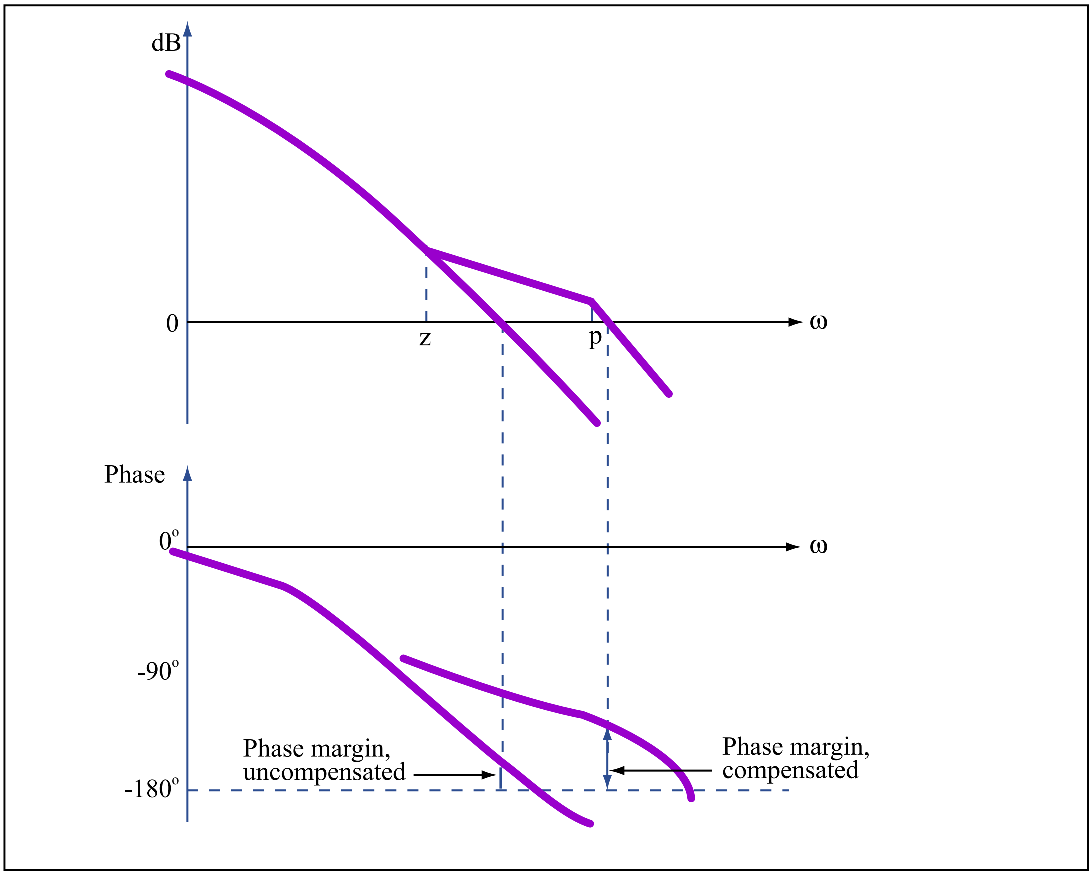

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Fig. 4: Lead example

- The process is slightly simpler if we target the lead compensator design at a particular desired ω_c

1. Find the $\phi_{\max}$ required
   - Note that $\phi_{required} = PM - (180^\circ + \measuredangle G(\mathbf{j}\omega_c))$
2. Put $\phi_{\max}$ at the crossover frequency, so that

$$\omega_c^2 = |p| \cdot |z|$$

3. Select K_c so that crossover is at ω_c

Design Example

- Design a compensator for the system $G(s) = \frac{1}{s(s+1)}$
  - Want $\omega_c = 10$rad/sec and $PM \approx 40^\circ$
- Note that at 10rad/sec, the slope of $|G|$ is -2, corresponding to a plant phase of approximately 180°
- So need to add a lead compensator ⇒ adds a slope of $+1$ (locally), changing the LTF slope to -1, and thus increasing the phase margin

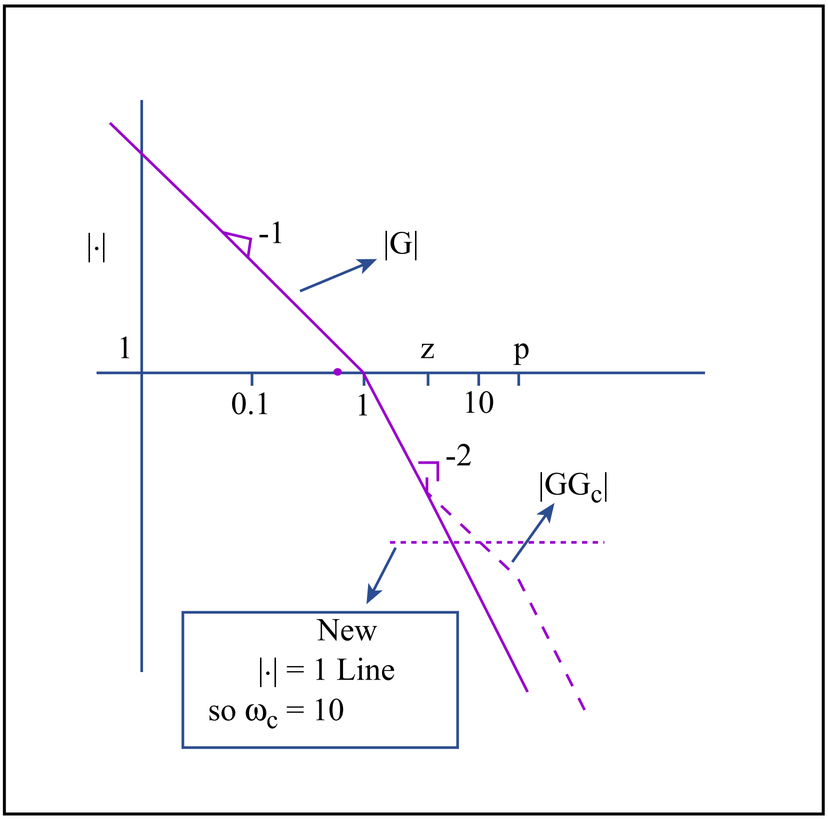

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Fig. 5: Lead example

- Design steps:

① $\left\{\frac{z}{p} = \frac{1-\sin\phi_m}{1+\sin\phi_m}\right\}$, $\left\{\begin{array}{c}\angle G(\mathbf{j}\omega_c) \approx -180^\circ \\ PM = 40^\circ\end{array}\right\} \Rightarrow \phi_m = 40^\circ$

$\Rightarrow \frac{z}{p} = 0.22$

② $\omega_c^2 = z \cdot p = 10^2 \quad \Rightarrow \quad z = 4.7, p = 21.4$

③ Pick k_c so that $|G_cG(\mathbf{j}\omega_c)| = 1$

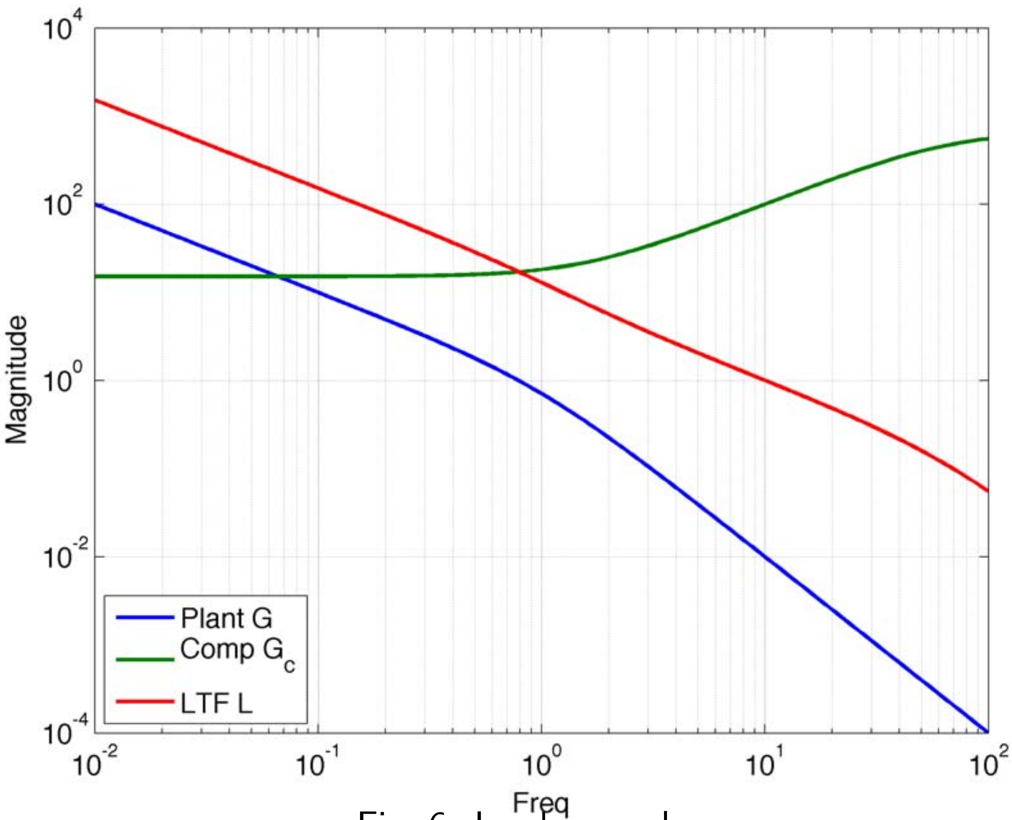


Fig. 6: Lead example

Code: Lead Example

```
%
% Lead Example 2009
%
close all
set(0,'DefaultLineLineWidth',2)
set(0,'DefaultlineMarkerSize',10)
set(0,'DefaultlineMarkerFace','b')
set(0, 'DefaultAxesFontSize', 14, 'DefaultAxesFontWeight','demi')
set(0, 'DefaultTextFontSize', 14, 'DefaultTextFontWeight','demi')
%
% g=1/s/(s+1)
figure(1);clf;
wc=10;
PM=40*pi/180;

G=tf(1,conv([1 0],[1 1]));
phi_G=phase(evalfr(G,j*wc))*180/pi;
%
phi_m=PM-(180+phi_G);
%
zdp=(1-sin(phi_m))/(1+sin(phi_m));
z=sqrt(wc^2*zdp);
p=z/zdp;
%
Gc=tf([1 z],[1 p]);
k_c=1/abs(evalfr(G*Gc,j*wc));
Gc=Gc*k_c;
w=logspace(-2,2,300);
f_G=freqresp(G,w);f_Gc=freqresp(Gc,w);f_L=freqresp(G*Gc,w);
f_G=squeeze(f_G);f_Gc=squeeze(f_Gc);f_L=squeeze(f_L);
loglog(w,abs(f_G),w,abs(f_Gc),w,abs(f_L))
xlabel('Freq');ylabel('Magnitude');grid on
legend('Plant G','Comp G_c','LTF L','Location','SouthWest')
print -dpng -r300 lead_examp2.png
```

Lag Mechanics

- If pole at a lower frequency than zero:
  - Gain decreases at high frequency – typically scale lag up so that HF gain is 1, and thus the low frequency gain is higher.
  - Add negative phase (i.e., adds lag)

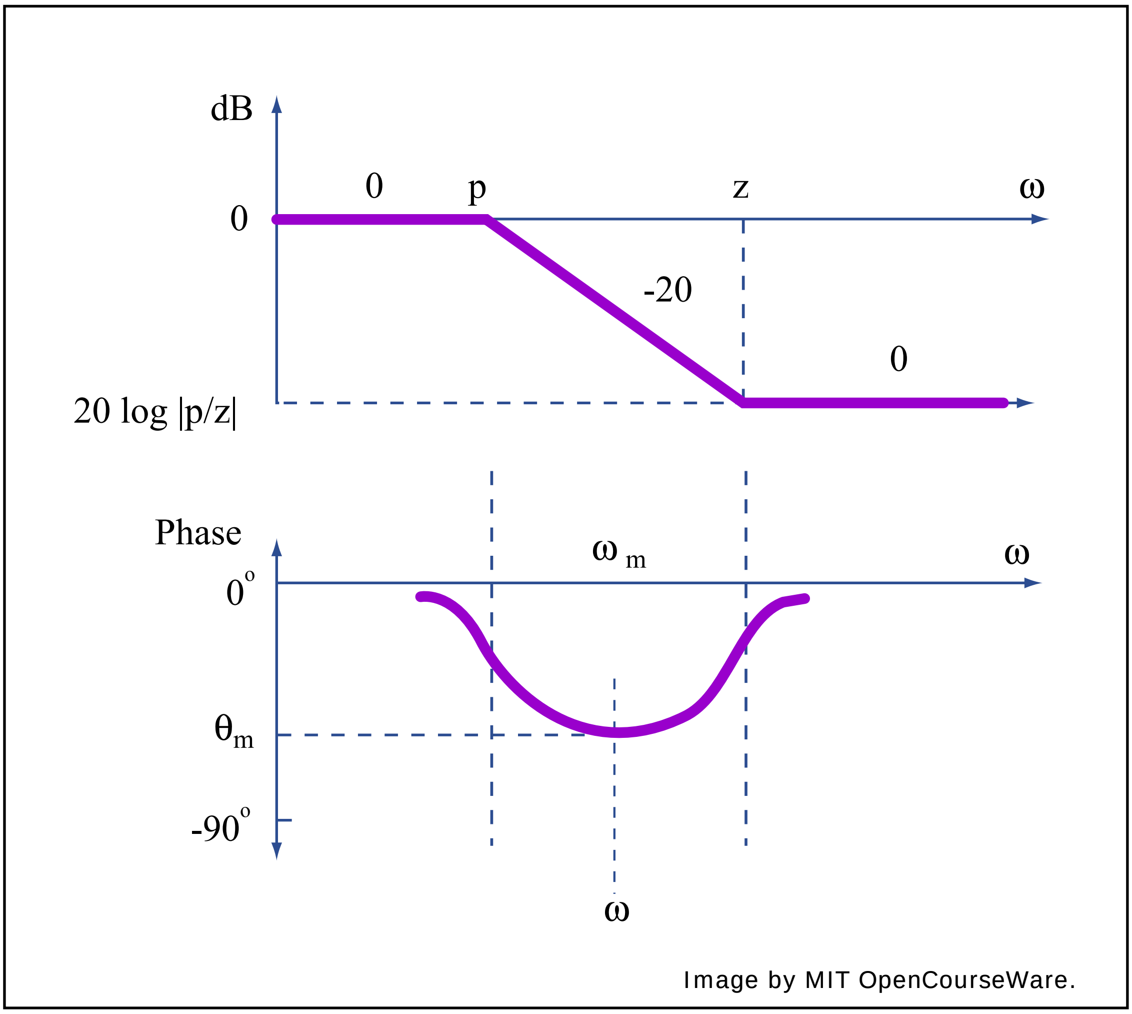


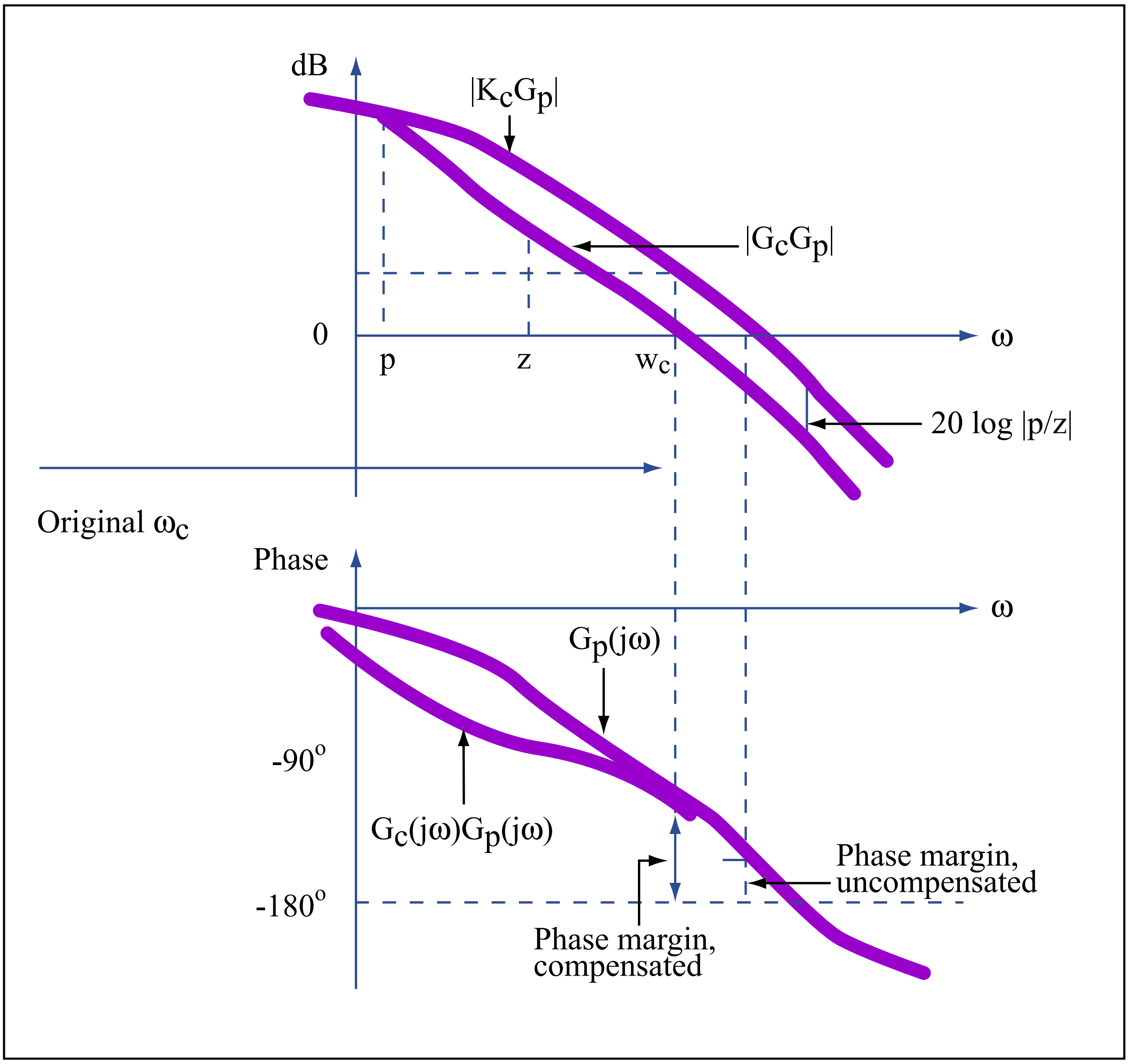
Fig. 7: Lag: frequency domain $G_{lag} = k_c \frac{s/z+1}{s/p+1}$

- Typically use a lag to add $20 \log \alpha$ to the low frequency gain with (hopefully) a small impact to the PM (at crossover)
  - Pick the desired gain reduction at high frequency $20 \log(1/\alpha)$, where $\alpha = |z|/|p|$
  - Pick $|G_{lag}|_{s=0} = k_c$ to give the desired low frequency gain increase for the LTF (shift the magnitude plot up)
  - Heuristic: want to limit frequency of the zero (and pole) so that there is a minimal impact of the phase lag at $\omega_c \Rightarrow z = \omega_c/10$

Lag Compensation

- Assumption is that we need to modify (increase) the DC gain of the LTF to reduce the tracking error.
- Two ways to get desired low frequency gain:

  ① Using just a gain increase, which increases ω_c and decrease PM
  ② Add Lag dynamics that increase increase the gain at low frequency and leave the gain near and above ω_c unchanged



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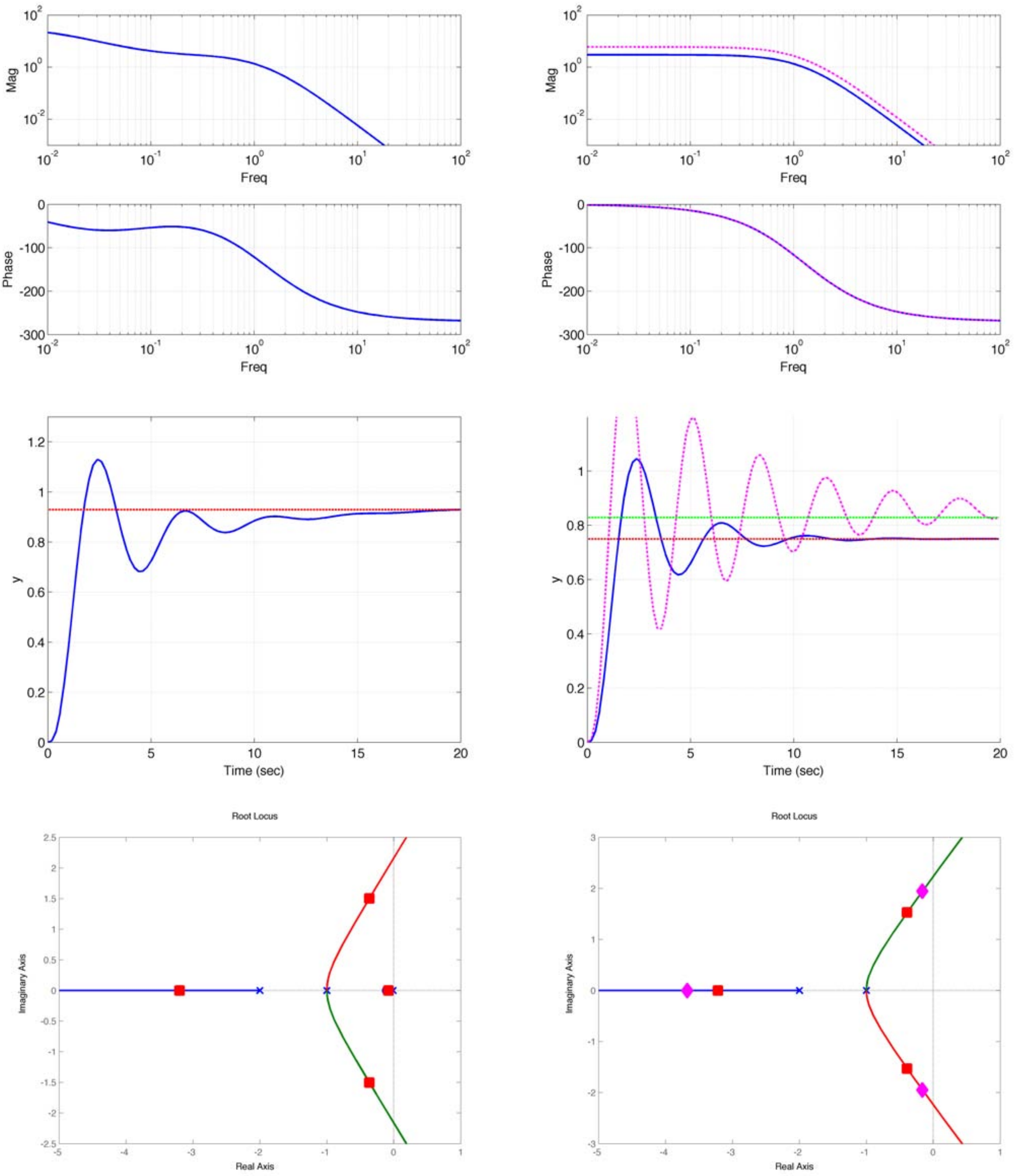
Lag Example

Lag example with $G(s) = 3/((s+1)^2(s/2+1))$ and on right $G_c(s) = 1$, left $G_c(s) = (s+0.1)/(s+0.01)$, which should reduce steady state step error from 0.25 to 0.032



Code: Lag Example

```
%
% Lag Example 2009
%
close all
set(0,'DefaultLineLineWidth',2)
set(0,'DefaultlineMarkerSize',10)
set(0,'DefaultlineMarkerFace','b')
set(0, 'DefaultAxesFontSize', 14, 'DefaultAxesFontWeight','demi')
set(0, 'DefaultTextFontSize', 14, 'DefaultTextFontWeight','demi')

G=tf([3],conv(conv([1/1 1],[1/2 1]),[1 1]));%
wc=1;zl=wc/10; pl=zl/10;
close all;

Gc=zl/pl*tf([1/(zl) 1],[1/(pl) 1]);%

figure(1);clf; %
set(gcf,'DefaultLineLineWidth',2)
L=Gc*G;Gcl=(L)/(1+L);%
[y,t]=step(Gcl,20);plot(t,y);grid on; axis([0 20 0 1.3])%
hold on;plot([min(t) max(t)],[y(end) y(end)],'r--');%
xlabel('Time (sec)');ylabel('y') %
ess=1/(1+evalfr(L,0*j))
print -dpng -r300 lag_examp1-1.png %

w=logspace(-2,2,300)';
figure(2);clf;%
set(gcf,'DefaultLineLineWidth',2)
[mag,ph]=bode(L,w);grid on; %
subplot(211);loglog(w,squeeze(mag));grid on;%
axis([1e-2 1e2 .001 100]) %
xlabel('Freq');ylabel('Mag')%
subplot(212);semilogx(w,squeeze(ph));%
xlabel('Freq');ylabel('Phase');grid on%
print -dpng -r300 lag_examp1-2.png

figure(3);rlocus(L);rr=rlocus(L,1);hold on; %
plot(rr+j*eps,'rs','MarkerSize',12,'MarkerFace','r');hold off %
print -dpng -r300 lag_examp1-3.png

Gc=1;
Gc2=2;

figure(4); set(gcf,'DefaultLineLineWidth',2) %
L=Gc*G;Gcl=(L)/(1+L);L2=Gc2*G;Gcl2=(L2)/(1+L2);%
[y,t]=step(Gcl,20);%
[y2,t]=step(Gcl2,t);%
hold on; %
plot(t,y,'b-',t,y2,'m-.');grid on; axis([0 20 0 1.2])%
plot([min(t) max(t)],[y(end) y(end)],'r--');%
plot([min(t) max(t)],[y2(end) y2(end)],'g--');%
xlabel('Time (sec)');ylabel('y') %
ess=1/(1+evalfr(L,0*j)) %
hold off
print -dpng -r300 lag_examp1-4.png %

w=logspace(-2,2,300)';
figure(5);clf;set(gcf,'DefaultLineLineWidth',2)
[mag,ph]=bode(L,w);
[mag2,ph2]=bode(L2,w);
subplot(211);loglog(w,squeeze(mag),'b-',w,squeeze(mag2),'m-.');grid on;%
axis([1e-2 1e2 .001 100]) %
xlabel('Freq');ylabel('Mag')%
subplot(212);semilogx(w,squeeze(ph),'b-',w,squeeze(ph2),'m-.');%
xlabel('Freq');ylabel('Phase');grid on%
print -dpng -r300 lag_examp1-5.png

figure(6);rlocus(L);rr=rlocus(L,Gc);rr2=rlocus(L,Gc2);hold on; %
plot(rr+j*eps,'rs','MarkerSize',12,'MarkerFace','r');
plot(rr2+j*eps,'md','MarkerSize',12,'MarkerFace','m');hold off %
print -dpng -r300 lag_examp1-6.png
```

Summary

- The design summary online is old, but gives an excellent summary of the steps involved.
- Typically find that this process is highly iterative because the final performance doesn't match the specifications (second order assumptions)
- Practice is the only way to absorb these approaches.