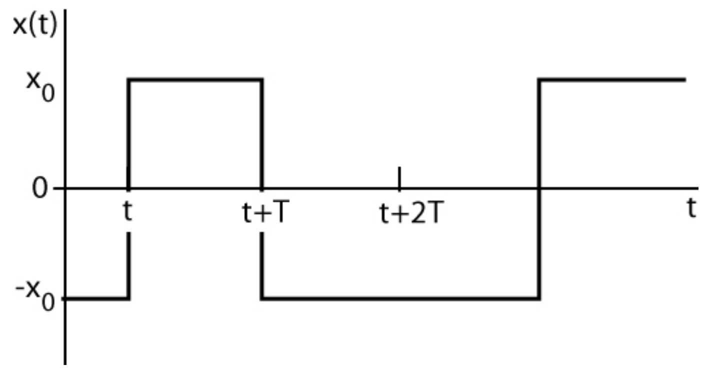
Lecture 11

Last time: Ergodic processes

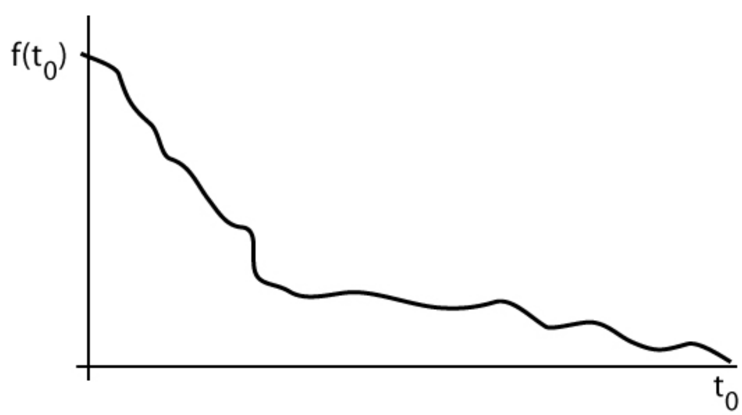
An ergodic process is necessarily stationary.

Example: Binary process

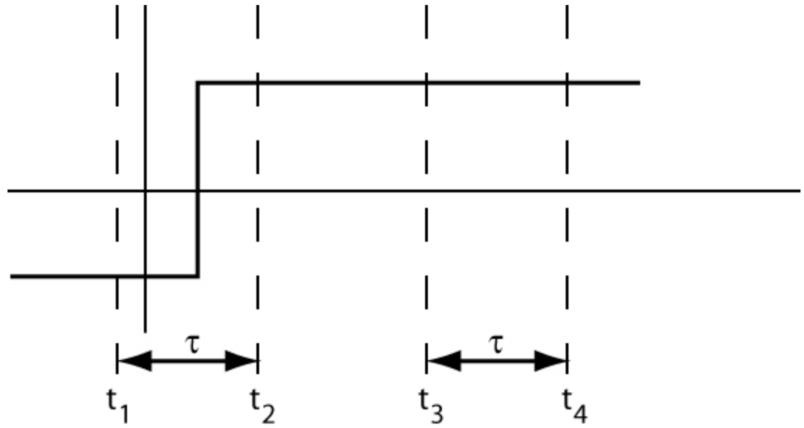


At each time step the signal may switch polarity or stay the same. Both $+x_0$ and $-x_0$ are equally likely.

Is it stationary and is it ergodic?



For this distribution, we expect most of the members of the ensemble to have a change point near t=0.



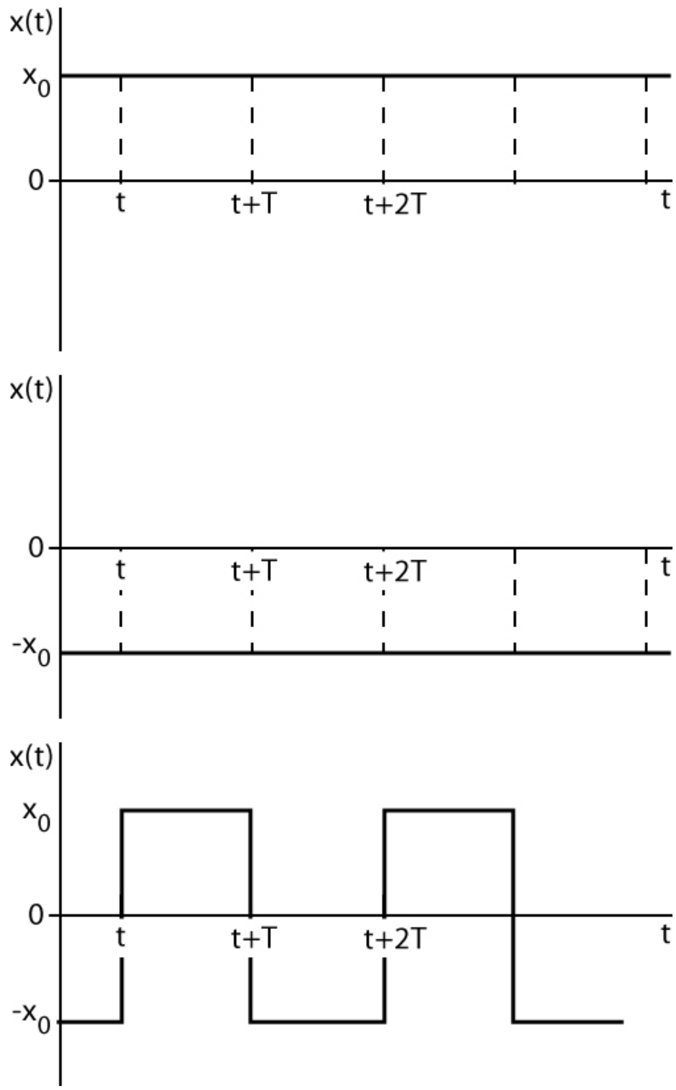
$$
\begin{aligned}
R_{xx}(t_1,t_2) &= E\left[x(t_1)x(t_2)\right] \\
&\approx \overline{x(t_1)}\,\overline{x(t_2)} \\
&= 0
\end{aligned}
$$

$$R_{xx}(t_3,t_4) \approx \overline{x(t_3)^2}$$

$$t_4 - t_3 = t_2 - t_1 = x_0^2$$

Chance of spanning a change point is the same over each regular interval, so the process is stationary. Is it ergodic?

Some ensemble members possess properties which are not representative of the ensemble as a whole. As an infinite set, the probability that any such member of the ensemble occurs is zero.



All of these exceptional points are associated with rational points. They are a countable set of infinity which constitute a set of zero measure. The complementary set of processes are an uncountable infinity associated with irrational numbers which constitute a set of measure one.

For ergodic processes:

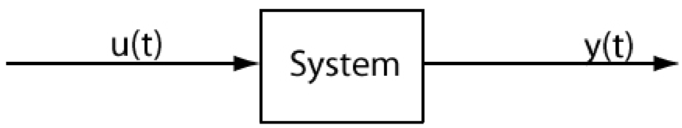
$$\overline{x} = E\left[x(t)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$\overline{x^2} = E[x(t)^2] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)^2 dt$$

$$R_{xx}(\tau) = E\left[x(t)x(t+\tau)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$R_{xy}(\tau) = E\left[x(t)y(t+\tau)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt$$

A time invariant system may be defined as one such that any translation in time of the input affects the output only by an equal translation in time.



This system will be considered time invariant if for every τ, the input $u(t+\tau)$ causes the output $y(t+\tau)$. Note that the system may be either linear or non-linear.

It is proved directly that if $u(t)$ is a stationary random process having the ergodic property and the system is time invariant, then $y(t)$ is a stationary random process having the ergodic property, in the steady state. This requires the system to be stable, so a defined steady state exists, and to have been operating in the presence of the input for all past time.

Example: Calculation of an autocorrelation function

Ensemble: $x(t) = A\sin(\omega t + \theta)$

θ, A are independent random variables
θ is uniformly distributed over $0, 2\pi$

This process is stationary (the uniform distribution of θ hints at this) but not ergodic. Unless we are certain of stationarity, we should calculate:

$$R_{xx}(t_1,t_2) = \overline{x(t_1)x(t_2)}$$

$$= \int_0^\infty dA \int_0^{2\pi} d\theta f(a) \frac{1}{2\pi} A^2 \sin\underbrace{\left(\omega t_1 + \theta\right)}_{"B"} \sin\underbrace{\left(\omega t_2 + \theta\right)}_{"A"}$$

$$\sin A \sin B = \frac{1}{2}\left[\cos\left(A-B\right) - \cos\left(A+B\right)\right]$$

$$R_{xx}(t_1,t_2) = \frac{1}{2}\int_0^{2\pi} \overline{A^2} \frac{1}{2\pi}\left\{\cos\left[\omega\left(t_2-t_1\right)\right] - \cos\left[\omega\left(t_1+t_2\right)+2\theta\right]\right\} d\theta$$

$$= \frac{1}{2}\overline{A^2}\cos\omega\tau, \quad \tau = t_2 - t_1$$

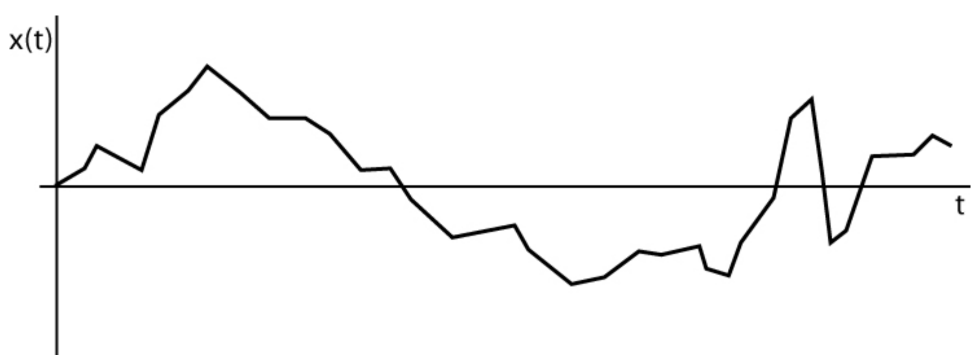
So the autocorrelation function is sinusoidal with the same frequency.

This periodic property is true in general. If all members of a stationary ensemble are periodic, $x(t+nT) = x(t)$

$$R_{xx}(\tau + nT) = \overline{x(t)x(t+\tau+nT)}$$
$$= \overline{x(t)x(t+\tau)}$$
$$= R_{xx}(\tau)$$

Identification of a periodic signal in noise

We have recorded a signal from an experimental device which looks like just hash.

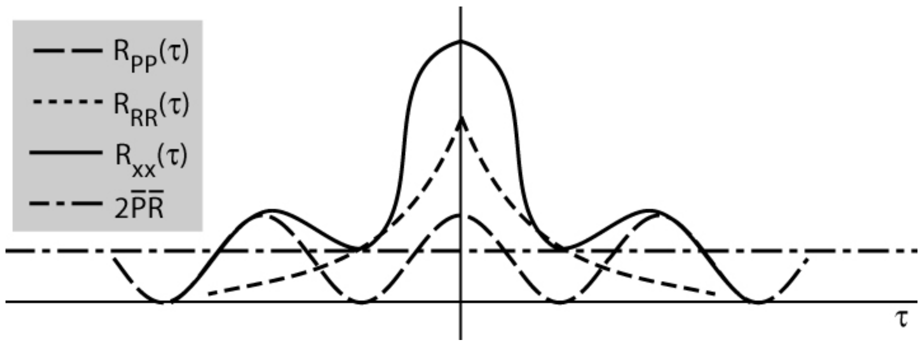


It is of interest to know if there are periodic components contained in it.

Consider:
$x(t) = R(t) + P(t)$
where $P(t)$ is any periodic function of period T and $R(t)$ is a random process independent of P.

If R is a stationary random process, use $\tau = t_2 - t_1$

$$R_{xx}(\tau) = R_{RR}(\tau) + R_{PP}(\tau) + R_{RP}(\tau) + R_{PR}(\tau)$$
$$= R_{RR}(\tau) + R_{PP}(\tau) + 2\overline{R}\overline{P}$$

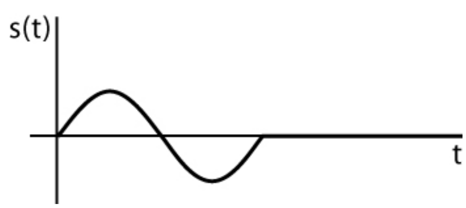


This usually makes the periodic component obvious. If P contains <u>more than one frequency component</u>, $R_{PP}(\tau)$ will contain the same components.
Note that this depends on $P(t)$ being truly periodic, not just oscillatory.
Cohesion time (over which phase is maintained) must exceed correlation time of signal.

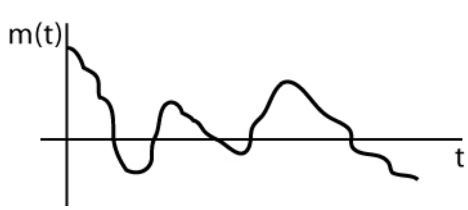
<u>Detection of a known signal in noise</u>

Communication systems depend on this technique for the detection of very weak signals of known form in strong noise. This is how the Lincoln Laboratory radar engineers "touched" Venus by radar, and the RLE last people touched the moon by laser.

A signal of known form is transmitted, $s(t)$. Upon receipt it is badly corrupted with noise so that no recognizable waveform appears.



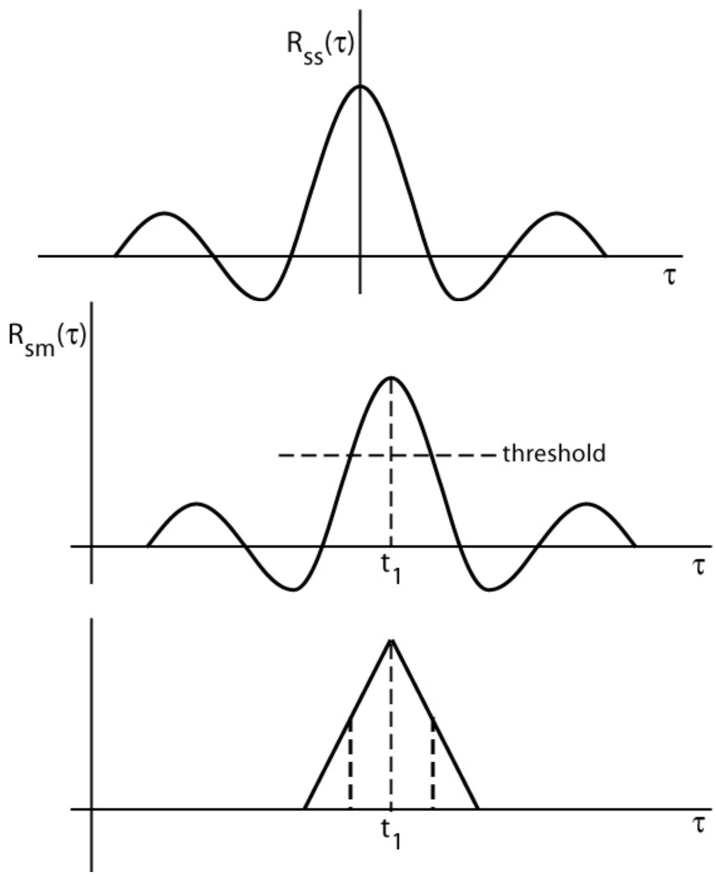
Received message $m(t) = ks(t - t_1) + n(t)$



The received message is cross-correlated with a signal of the known waveform. If the time of arrival is not known, the cross-correlation is carried out for various values of τ.

$$\begin{aligned}
R_{sm}(\tau) &= \overline{s(t)m(t+\tau)} \\
&= k\overline{s(t)s(t+\tau-t_1)} + \overline{s(t)n(t+\tau)} \\
&= k\overline{s(t)s(t+\tau-t_1)} + \overline{s(t)}\,\overline{n(t+\tau)} \\
&= kR_{ss}(\tau - t_1)
\end{aligned}$$

The signal is designed with zero mean to eliminate the second term.



The use of these correlation functions imply signals which continue for all time. Actually it is finite data in these cases and functions similar to the R functions are used which involve integration without averaging. However, the notions are analogous to these. This is the basis for <u>correlation detection</u>. The same result is obtained, starting from a different point of view, with the <u>matched filter</u>.

The also provides and estimate of k and of t_1. GPS uses the t_1 estimate.

Two-sided Fourier transform is used as it defines the behavior of the signal for negative time. This is important so that this can be set to zero for causal systems.

You know that use of transforms is convenient in the analysis of time-invariant linear systems. The same is true of the study of stationary processes in invariant linear systems.

Using the two-sided Fourier transform, the transform of the autocorrelation function is

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

This is called the power spectral density function.
We reject the real part of the argument of the exponential function, as this will diverge for negative time.

Properties of $S_{xx}(\omega)$

$$\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} R_{xx}(\tau)\left[\cos\omega\tau - j\sin\omega\tau\right] d\tau \\ &= \int_{-\infty}^{\infty} R_{xx}(\tau)\cos\omega\tau d\tau \\ &= 2\int_{0}^{\infty} R_{xx}(\tau)\cos\omega\tau d\tau \end{aligned}$$

We see that a PSD is a real function, even in ω, and non-negative.

$$S_{xx}(\omega) \in \mathrm{Re}\,\forall\omega$$

$$S_{xx}(\omega) \geq 0 \forall\omega$$

$$S_{xx}(-\omega) = S_{xx}(\omega)$$

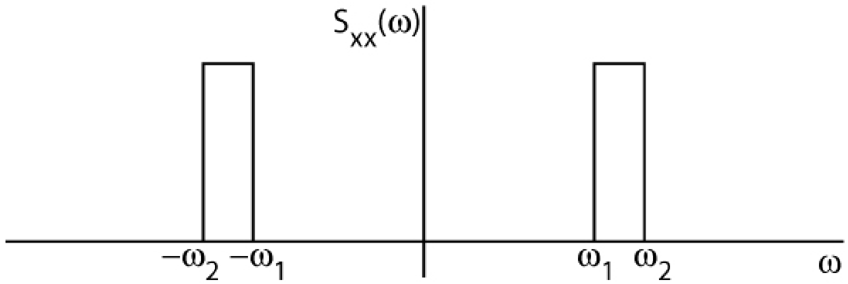
The inverse relation between $s_{xx}(\omega)$ and $R_{xx}(\tau)$ is

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega$$

Note that

$$\begin{aligned} E\left[x(t)^2\right] &= R_{xx}(0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega \end{aligned}$$

Power means mean squared value. The PSD gives the spectral distribution of power density.



$\overline{y(t)^2} = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} S_{xx}(\omega)d\omega$ = mean squared value of the frequency components in $x(t)$ in the range ω_1 to ω_2.

If $x(t)$ has a non-zero mean,

$$x(t) = \overline{x} + r(t), \qquad \overline{r(t)} = 0$$

$$\begin{aligned} R_{xx}(\tau) &= \overline{x(t)x(t+\tau)} \\ &= \overline{\left[\overline{x} + r(t)\right]\left[\overline{x} + r(t+\tau)\right]} \\ &= \overline{x}^2 + R_{rr}(\tau) \end{aligned}$$

Corresponding to $\overline{x}^2$ term, the PSD for $x(t)$ will have an additive term $\int_{-\infty}^{\infty} \overline{x}^2 e^{-j\omega\tau} d\tau = 2\pi\overline{x}^2\delta(\omega)$.