16.346 Astrodynamics
Fall 2008

Terminology

Pages 419–423

Sphere of Influence Radius r: $r = \text{planet mean distance} \times \left(\dfrac{\text{planet mass}}{\text{solar mass}}\right)^{\frac{2}{5}}$

Inbound relative velocity vector: $\mathbf{v}_{\infty i}$

Outbound relative velocity vector: $\mathbf{v}_{\infty o}$ $\qquad v_{\infty i} = v_{\infty o} = \sqrt{\dfrac{\mu}{-a}}$

Turn angle 2ν: $\csc\nu = e \qquad v_\infty^2 \sin 2\nu = |\mathbf{v}_{\infty o} \times \mathbf{v}_{\infty i}|$

Minimum passing distance: $r_m = q = a(1-e) = \dfrac{\mu}{v_\infty^2}(e-1) = \dfrac{\mu}{v_\infty^2}(\csc\nu - 1)$

Circular speed v_{om} at radius r_m: $v_{om}^2 = \dfrac{\mu}{r_m}$

Point of aim r_a: $r_a = -ae\cos\nu = -a\cot\nu = r_m\sqrt{1 + 2\dfrac{v_{om}^2}{v_\infty^2}}$

$$\sin\nu = \frac{1}{1 + \dfrac{v_\infty^2}{v_{om}^2}} \qquad \tan\nu = \frac{\mu}{r_a v_\infty^2} \qquad r_a = r_m\sqrt{1 + 2\frac{v_{om}^2}{v_\infty^2}}$$

Close Pass of a Target Planet

The inbound plane of relative motion is the plane containing the orbital velocity vectors of the spacecraft and the planet. The unit normal to that plane is

$$\mathbf{i}_n = \text{Unit}(\mathbf{v}_{SC} \times \mathbf{v}_P)$$

so that the vector point of aim is

$$\mathbf{r}_a = r_a \mathbf{i}_n \times \mathbf{i}_{\infty i}$$

The vector $\mathbf{r}_a$ can be rotated about the direction of $\mathbf{i}_{\infty i}$ so that the orientation of the passing plane, relative to the planet, can be controlled at will. The method for this is discussed in the following page.

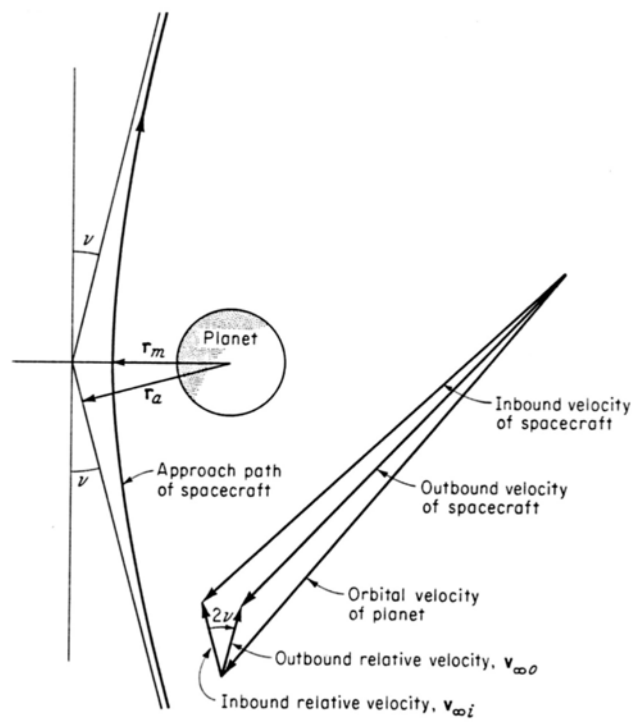


Fig. 9.1 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Let O be a fixed point on an axis whose direction is $\mathbf{i}_\omega = l\,\mathbf{i}_x + m\,\mathbf{i}_y + n\,\mathbf{i}_z$ and let $\mathbf{r}$ be the vector from O to a point P. When the vector $\mathbf{r}$ rotates about this axis, the point P moves in a circle of radius $r\sin\alpha$ where α is the angle between $\mathbf{r}$ and $\mathbf{i}_\omega$.

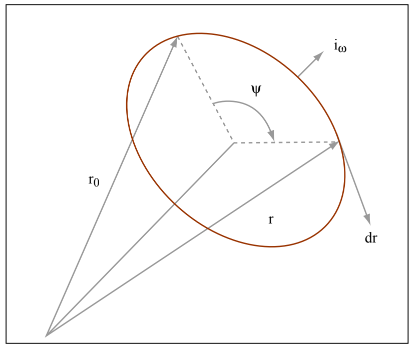


Figure by MIT OpenCourseWare.

A small rotation through an angle $\delta\psi$ produces a change $\delta\mathbf{r}$ given by

$$\delta\mathbf{r} = \delta\psi\,\mathbf{i}_\omega \times \mathbf{r} = \delta\psi\,\mathbf{S}\mathbf{r}$$

where the constant matrix $\mathbf{S}$ is skew-symmetric, i.e., $\mathbf{S} = -\mathbf{S}^{\mathrm{T}}$. Hence

$$\frac{d\mathbf{r}}{d\psi} = \mathbf{S}\mathbf{r}$$

is a linear vector differential equation for $\mathbf{r}$ with constant coefficients whose solution is

$$\mathbf{r} = e^{\psi\mathbf{S}}\mathbf{r}_0 \equiv \left(\mathbf{I} + \psi\mathbf{S} + \frac{1}{2!}\psi^2\mathbf{S}^2 + \frac{1}{3!}\psi^3\mathbf{S}^3 + \cdots\right)\mathbf{r}_0$$

Also, since

$$\mathbf{S} = \begin{bmatrix} 0 & -n & m \\ n & 0 & -l \\ -m & l & 0 \end{bmatrix} \quad \mathbf{S}^2 = \begin{bmatrix} -m^2-n^2 & lm & ln \\ lm & -l^2-n^2 & mn \\ ln & mn & -l^2-m^2 \end{bmatrix} \quad \mathbf{S}^3 = \begin{bmatrix} 0 & n & -m \\ -n & 0 & l \\ m & -l & 0 \end{bmatrix}$$

we conclude that

$$\mathbf{S}^3 = -\mathbf{S}$$

Thus, all powers of the matrix $\mathbf{S}$ are either $\pm\mathbf{S}$ or $\pm\mathbf{S}^2$ and we have

$$\mathbf{R} = \mathbf{I} + \left(\psi - \frac{1}{3!}\psi^3 + \cdots\right)\mathbf{S} + \left(\frac{1}{2!}\psi^2 - \frac{1}{4!}\psi^4 + \cdots\right)\mathbf{S}^2$$

or

$$\mathbf{R} = \mathbf{I} + \sin\psi\,\mathbf{S} + (1-\cos\psi)\mathbf{S}^2$$

Hence $\mathbf{r} = \mathbf{R}\mathbf{r}_0$ is equivalent to

$$\boxed{\mathbf{r} = \mathbf{r}_0 + \sin\psi\,(\mathbf{i}_\omega \times \mathbf{r}_0) + (1-\cos\psi)\,\mathbf{i}_\omega \times (\mathbf{i}_\omega \times \mathbf{r}_0)}$$

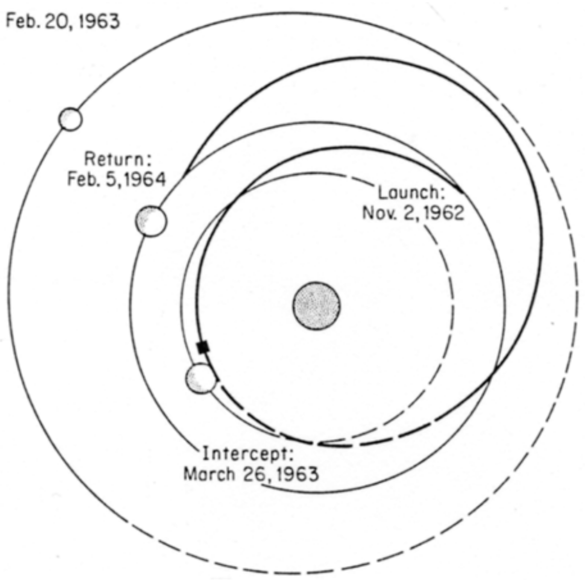
Try this with $\mathbf{r}_0 = \mathbf{i}_x$ and $\mathbf{i}_\omega = \mathbf{i}_z$

Application to Rotating the Point of Aim vector $\mathbf{r}_a$ about the $\mathbf{i}_{\infty i}$ direction

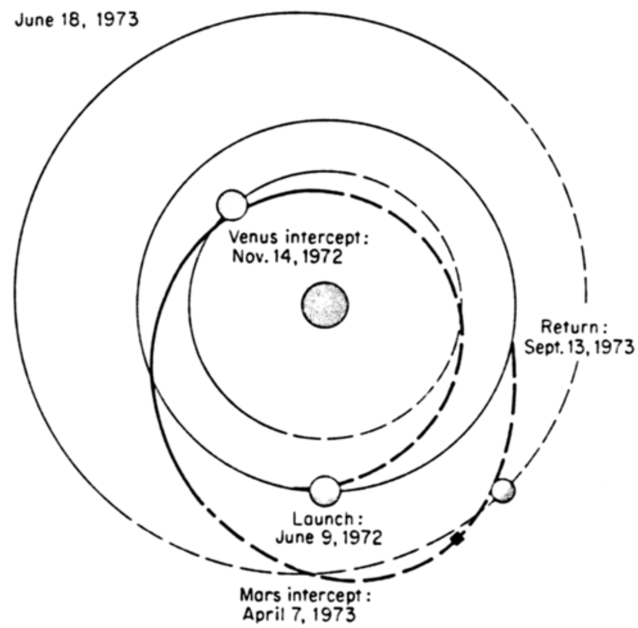
$$\mathbf{r}_a = r_a\,\mathbf{i}_n \times \mathbf{i}_{\infty i} \quad \text{or} \quad \mathbf{r}_a = \frac{\mu}{2v_\infty^2\sin^2\nu}\,\mathbf{i}_{\infty i} \times (\mathbf{i}_{\infty i} \times \mathbf{i}_{\infty o})$$

See Problem 9–4

$$\text{New } \mathbf{r}_a = \mathbf{r}_a + \sin\psi\,(\mathbf{i}_{\infty i} \times \mathbf{r}_a) + (1-\cos\psi)\,\mathbf{i}_{\infty i} \times (\mathbf{i}_{\infty i} \times \mathbf{r}_a)$$

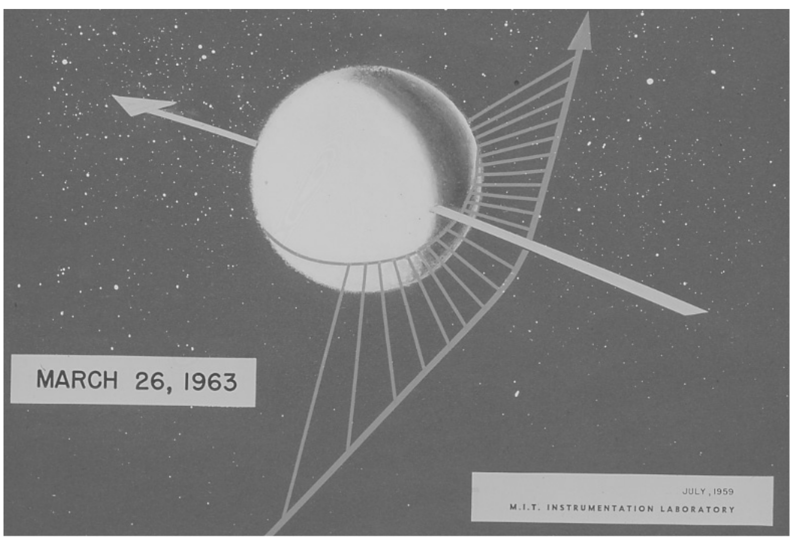


Round-Trip Earth to Venus to Earth
Published August 1959



Round-Trip Earth to Venus to Mars to Earth
Discovered January 1961

Fig. 9.5 (above left); Introduction Fig. 6 (above right); and Fig. 9.6 (below) from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.



Planetary FlyBys are definitely three-dimensional

The normal to the plane in which the spacecraft moves in its hyperbolic swingby of the planet is determined by the cross-product of the inbound and outbound relative velocity vectors.

$$\boxed{\mathbf{v}_{\infty o} \times \mathbf{v}_{\infty i}}$$

The velocity components, which are part of this data, have never been published before. We give them below in units of feet per second:

Launch date: June 9, 1972

Outbound Earth velocity	[−6326 − 12628 − 5051]	Magnitude = 15000 Ft/sec
Inbound Venus velocity	[28253 2698 335]	Magnitude = 28384 Ft/sec
Outbound Venus velocity	[23564 15771 − 1274]	Magnitude = 28383 Ft/sec
Inbound Mars velocity	[19624 − 17633 − 1790]	Magnitude = 26443 Ft/sec
Outbound Mars velocity	[−20418 − 16713 1736]	Magnitude = 26443 Ft/sec
Inbound Earth velocity	[−40370 − 7303 1836]	Magnitude = 41066 Ft/sec

Return date: September 13, 1973

Launch date: February 6, 1966

Outbound Earth velocity	[−7907 − 13922 3985]	Magnitude = 16499 Ft/sec
Inbound Venus velocity	[29837 1164 − 6313]	Magnitude = 30520 Ft/sec
Outbound Venus velocity	[23529 18335 − 6453]	Magnitude = 30519 Ft/sec
Inbound Mars velocity	[6061 − 14285 2957]	Magnitude = 15797 Ft/sec
Outbound Mars velocity	[8659 − 13164 1115]	Magnitude = 15796 Ft/sec
Inbound Earth velocity	[39120 512 2839]	Magnitude = 39226 Ft/sec

Return date: December 17, 1967

Error correction in red was discovered by Steve Block (a student in 16.346) in the fall term of 2003 — over 42 years after the original posting!

The original velocity vector was incorrectly stated as [16713 − 20418 1736].