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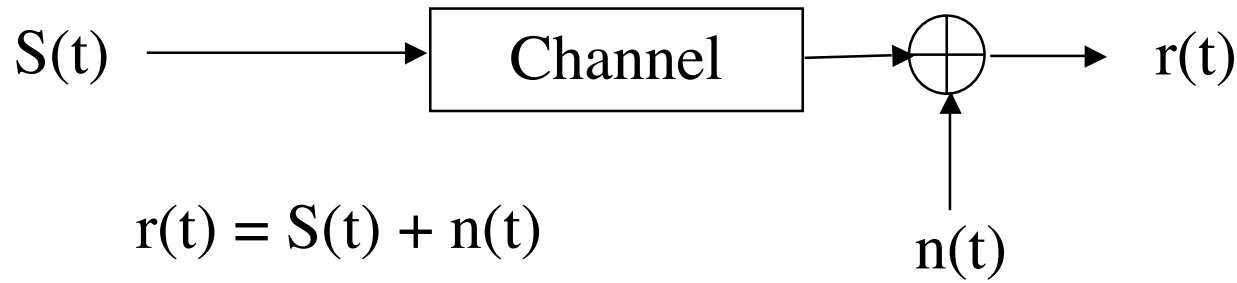
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# **Lectures 9 - 10: Signal Detection in Noise and the Matched Filter**

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# Noise in communication systems

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- **Noise is additional “unwanted” signal that interferes with the transmitted signal**
  - Generated by electronic devices
- **The noise is a random process**
  - Each “sample” of  $n(t)$  is a random variable
- **Typically, the noise process is modeled as “Additive White Gaussian Noise” (AWGN)**
  - **White:** Flat frequency spectrum
  - **Gaussian:** noise distribution

# Random Processes

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- The auto-correlation of a random process  $x(t)$  is defined as
  - $R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$
- A random process is Wide-sense-stationary (WSS) if its mean and auto-correlation are not a function of time. That is
  - $m_x(t) = E[x(t)] = m$
  - $R_{xx}(t_1, t_2) = R_x(\tau)$ , where  $\tau = t_1 - t_2$
- If  $x(t)$  is WSS then:
  - $R_x(\tau) = R_x(-\tau)$
  - $|R_x(\tau)| \leq |R_x(0)|$  (max is achieved at  $\tau = 0$ )
- The power content of a WSS process is:

$$P_x = E\left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt\right] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} R_x(0) dt = R_x(0)$$

# Power Spectrum of a random process

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- If  $x(t)$  is WSS then the power spectral density function is given by:

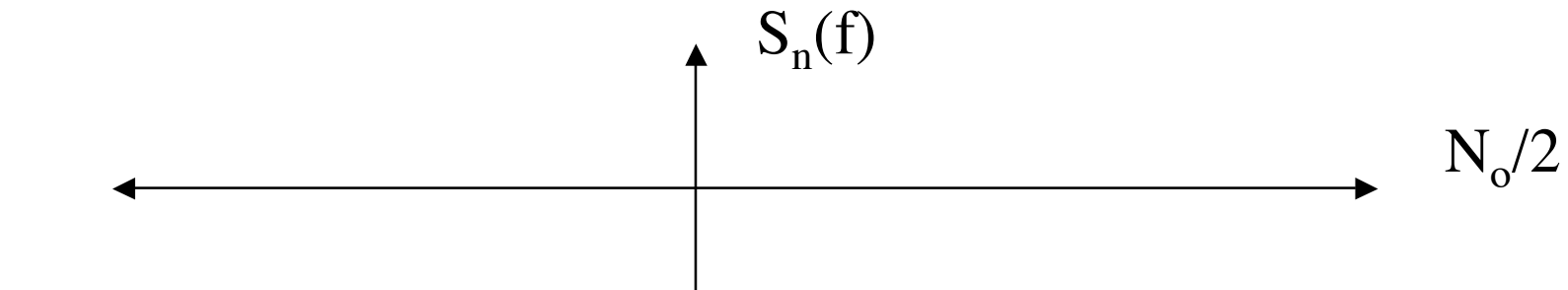
$$\mathbf{S_x(f) = F[R_x(\tau)]}$$

- The total power in the process is also given by:

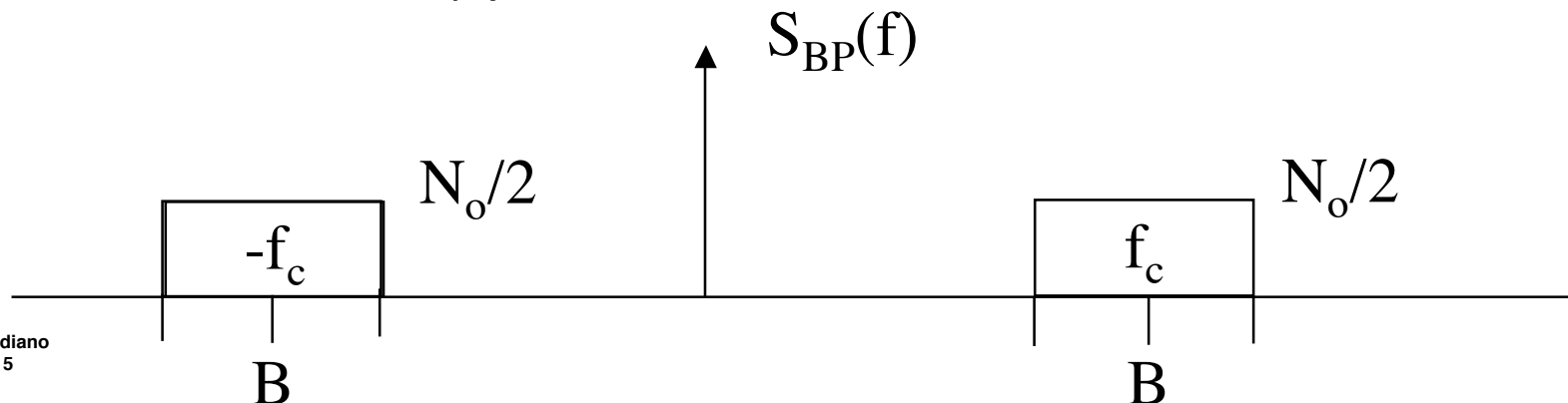
$$\begin{aligned} P_x &= \int_{-\infty}^{\infty} S_x(f) df = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} dt \right] df \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} df \right] dt \\ &= \int_{-\infty}^{\infty} R_x(t) \left[ \int_{-\infty}^{\infty} e^{-j2\pi ft} df \right] dt = \int_{-\infty}^{\infty} R_x(t) \delta(t) dt = R_x(0) \end{aligned}$$

# White noise

- The noise spectrum is flat over all relevant frequencies
  - White light contains all frequencies



- Notice that the total power over the entire frequency range is infinite
  - But in practice we only care about the noise content within the signal bandwidth, as the rest can be filtered out
- After filtering the only remaining noise power is that contained within the filter bandwidth ( $B$ )



# AWGN

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- **The effective noise content of bandpass noise is  $BN_0$** 
  - Experimental measurements show that the pdf of the noise samples can be modeled as zero mean gaussian random variable

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

- AKA Normal r.v.,  $N(0, \sigma^2)$
  - $\sigma^2 = P_x = BN_0$
- **The CDF of a Gaussian R.V.,**

$$F_x(\alpha) = P[X \leq \alpha] = \int_{-\infty}^{\alpha} f_x(x) dx = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$$

- **This integral requires numerical evaluation**
  - Available in tables

# AWGN, continued

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- $X(t) \sim N(0, \sigma^2)$
- $X(t_1), X(t_2)$  are independent unless  $t_1 = t_2$

- $$R_x(\tau) = E[X(t + \tau)X(t)] = \begin{cases} E[X(t + \tau)]E[X(t)] & \tau \neq 0 \\ E[X^2(t)] & \tau = 0 \end{cases}$$
$$= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 & \tau = 0 \end{cases}$$

- $R_x(0) = \sigma^2 = P_x = BN_o$



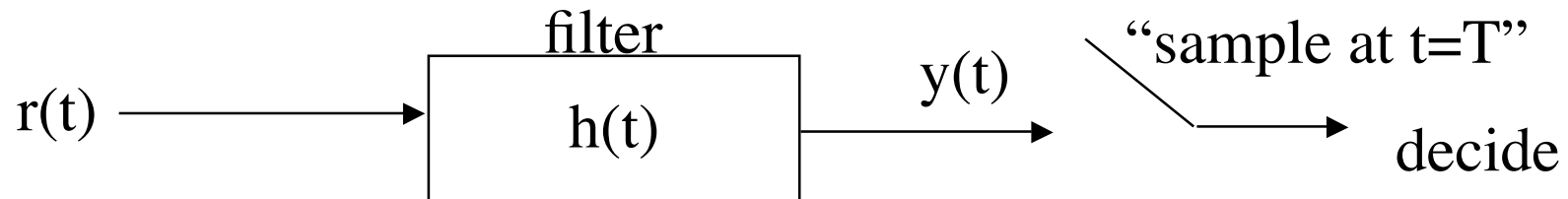
# Detection of signals in AWGN

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Observe:  $r(t) = S(t) + n(t)$ ,  $t \in [0, T]$

Decide which of  $S_1, \dots, S_m$  was sent

- **Receiver filter**
  - Designed to maximize signal-to-noise power ratio (SNR)



- **Goal: find  $h(t)$  that maximized SNR**

# Receiver filter

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$$y(t) = r(t) * h(t) = \int_0^t r(\tau) h(t - \tau) d\tau$$

$$\text{Sampling at } t = T \Rightarrow y(T) = \int_0^T r(\tau) h(T - \tau) d\tau$$

$$r(\tau) = s(\tau) + n(\tau) \Rightarrow$$

$$y(T) = \int_0^T s(\tau) h(T - \tau) d\tau + \int_0^T n(\tau) h(T - \tau) d\tau = Y_s(T) + Y_n(T)$$

$$SNR = \frac{Y_s^2(T)}{E[Y_n^2(T)]} = \frac{\left[ \int_0^T s(\tau) h(T - \tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t) dt} = \frac{\left[ \int_0^T h(\tau) s(T - \tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t) dt}$$

## Matched filter: maximizes SNR

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Cauchy - Schwartz Inequality :

$$\left[ \int_{-\infty}^{\infty} g_1(t) g_2(t) dt \right]^2 \leq \int_{-\infty}^{\infty} (g_1(t))^2 dt \int_{-\infty}^{\infty} (g_2(t))^2 dt$$

Above holds with equality iff:  $g_1(t) = c g_2(t)$  for arbitrary constant  $c$

$$SNR = \frac{\left[ \int_0^T s(\tau) h(T-\tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T-t) dt} \leq \frac{\int_0^T (s(\tau))^2 d\tau \int_0^T h^2(T-\tau) d\tau}{\frac{N_0}{2} \int_0^T h^2(T-t) dt} = \frac{2}{N_0} \int_0^T (s(\tau))^2 d\tau = \frac{2E_s}{N_0}$$

Above maximum is obtained iff:  $h(T-\tau) = cS(\tau)$

$$\Rightarrow h(t) = cS(T-t) = S(T-t)$$

$h(t)$  is said to be “matched” to the signal  $S(t)$

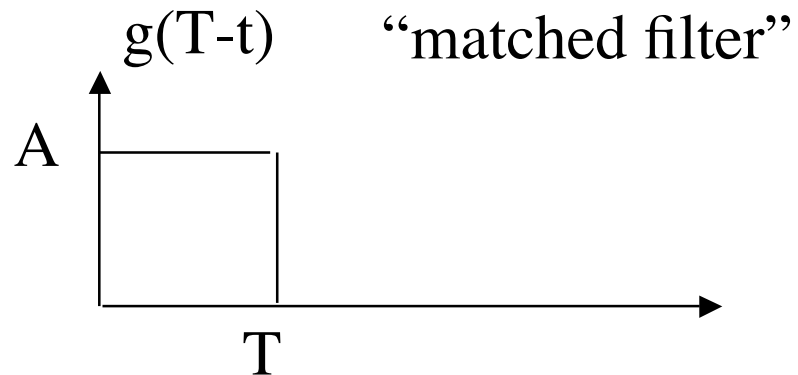
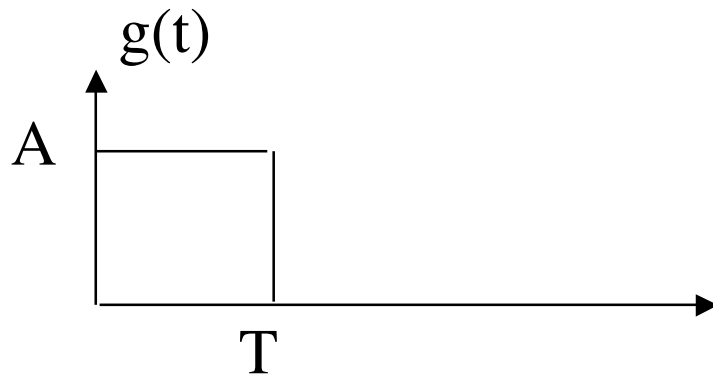
# Example: PAM

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$$S_m(t) = A_m g(t), \quad t \in [0, T]$$

$A_m$  is a constant: Binary PAM  $A_m \in \{0, 1\}$

Matched filter is matched to  $g(t)$



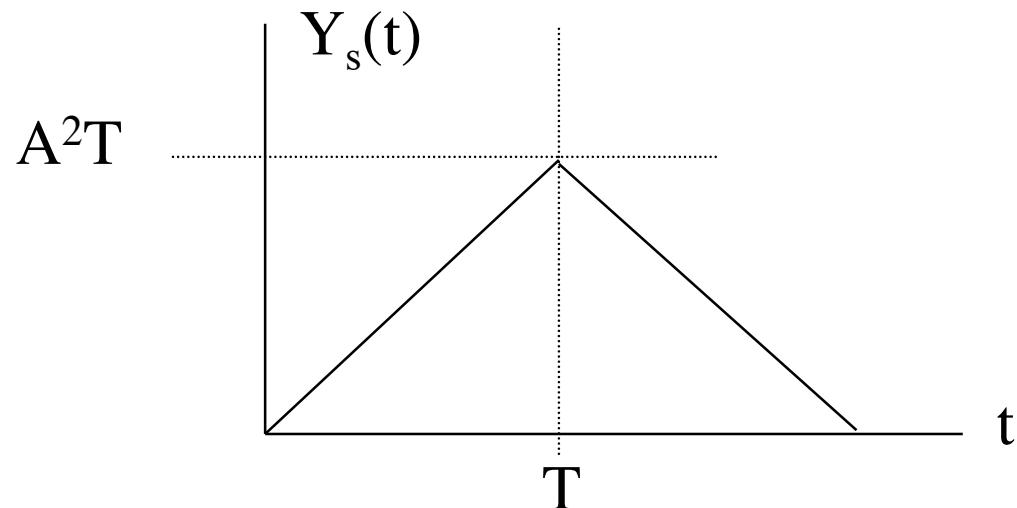
## Example, continued

$$Y_s(t) = \int_0^t S(\tau)h(t-\tau)d\tau, \quad h(t) = g(T-t) \Rightarrow h(t-\tau) = g(T+\tau-t)$$

$$Y_s(t) = \int_0^t g(\tau)g(T+\tau-t)d\tau = \int_0^t g(\tau)g(T-t+\tau)d\tau$$

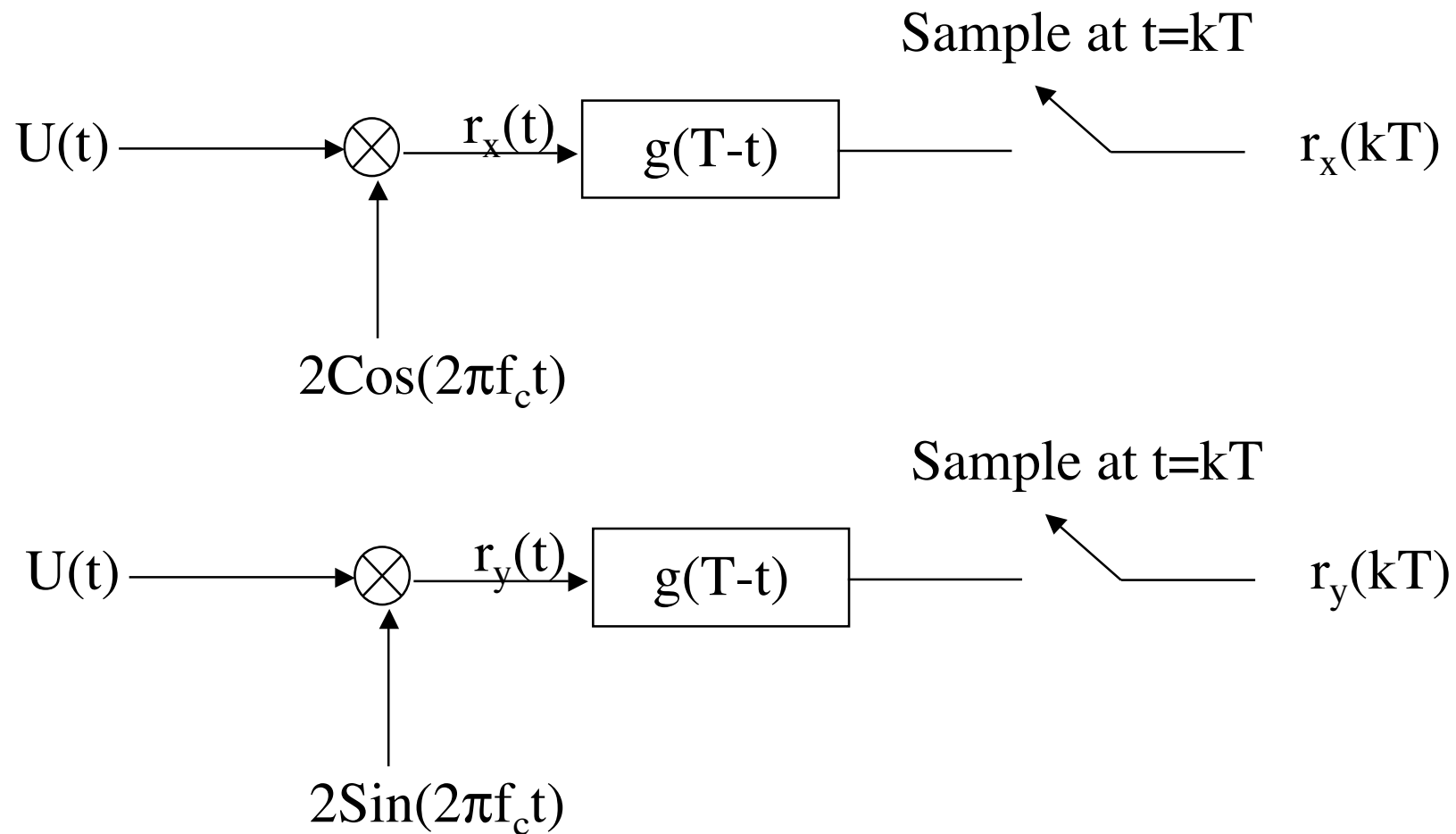
$$Y_s(T) = \int_0^T g^2(\tau)d\tau$$

- **Sample at  $t=T$  to obtain maximum value**



# Matched filter receiver

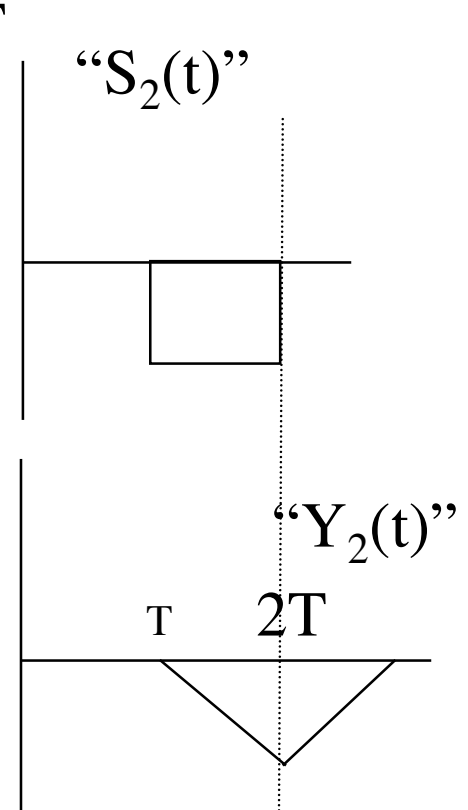
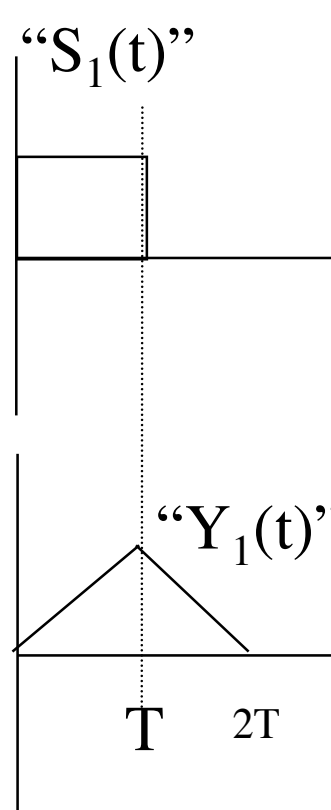
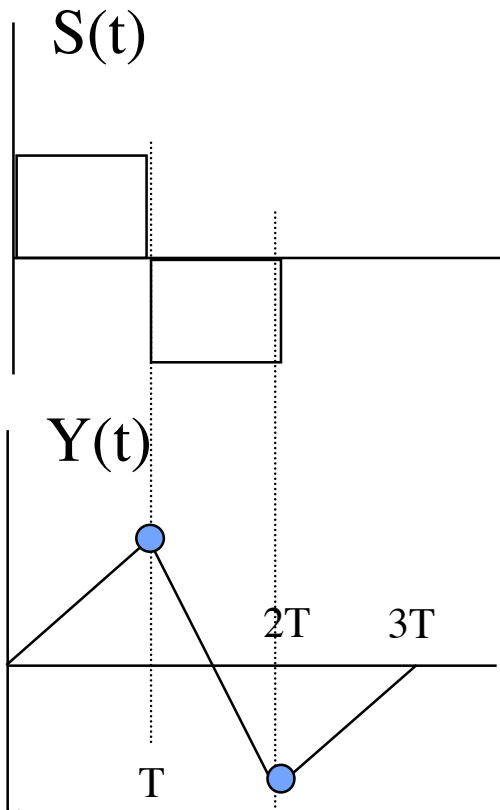
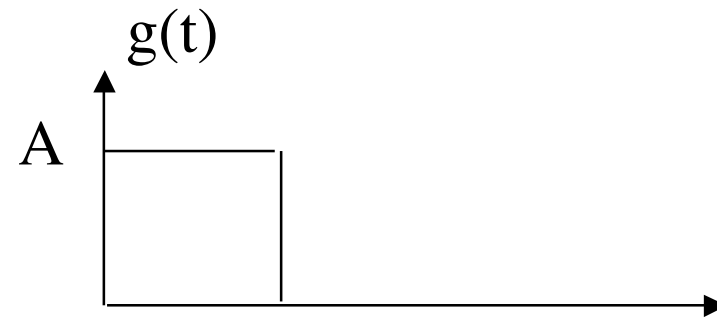
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# Binary PAM example, continued

$$0 \Rightarrow S_1 = g(t)$$

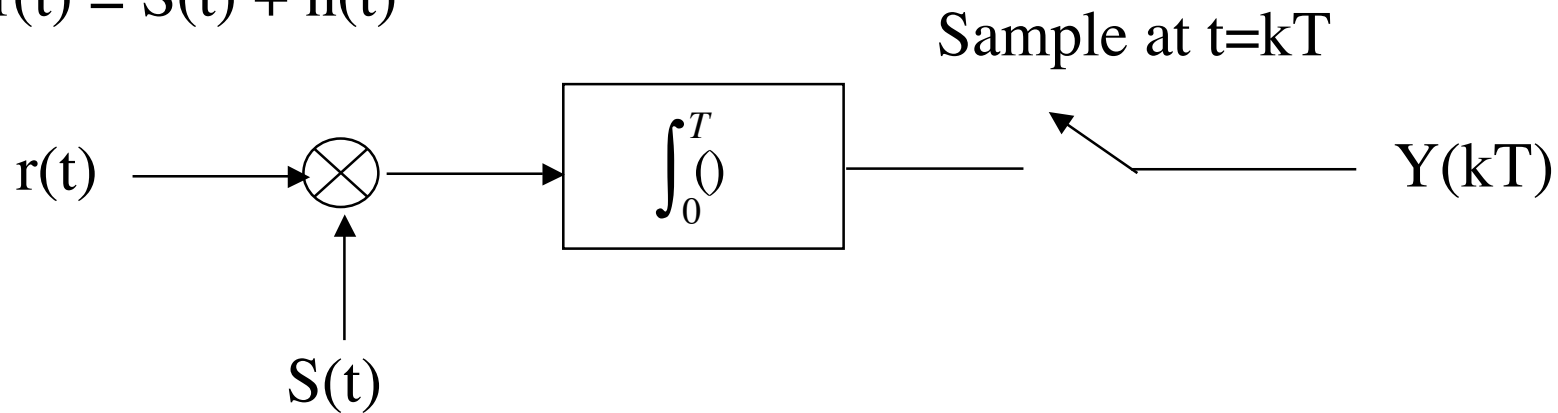
$$1 \Rightarrow S_2 = -g(t)$$



# Alternative implementation: correlator receiver

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$$r(t) = S(t) + n(t)$$



$$Y(T) = \int_0^T r(t)S(t) = \int_0^T S^2(t) + \int_0^T n(t)S(t) = Y_s(T) + Y_n(T)$$

Notice resemblance to matched filter