LECTURE # 13

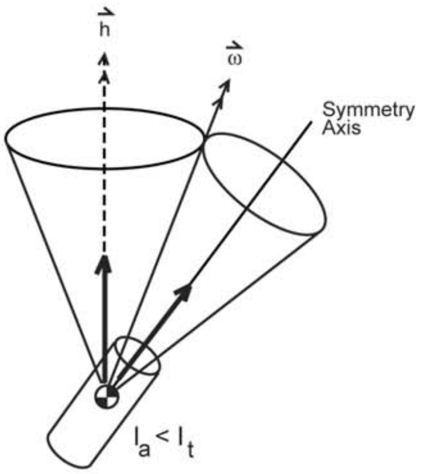
- AXISYMMETRIC ROTATIONS
  - BODY CONES
  - SPACE CONES
  - PRECESSION , NUTATION
- GEOMETRIC INTERPRETATIONS

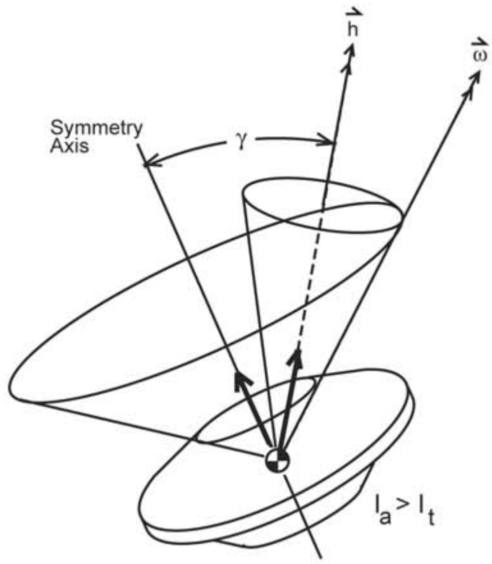
ATTITUDE MOTION - TORQUE FREE

- HAVE DISCUSSED THE ROTATIONAL MOTION FROM THE PERSPECTIVE OF THE "BODY FRAME"
  - NEED TO FIND A WAY TO CONNECT THE MOTION TO THE INERTIAL FRAME SO WE CAN DESCRIBE THE ACTUAL MOTION.

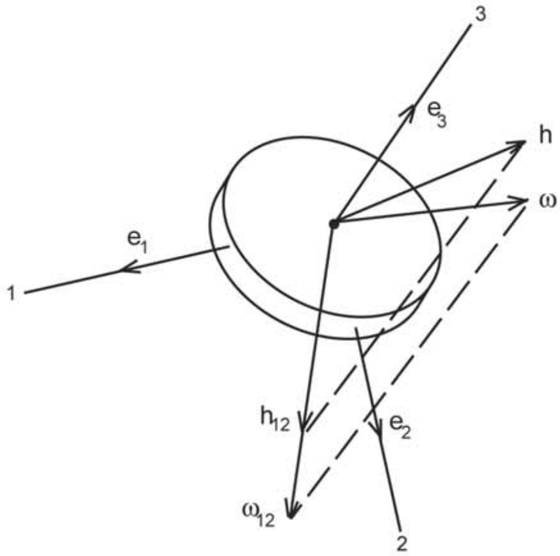
- TYPICALLY DONE BY DESCRIBING MOTION OF VEHICLE ABOUT THE $\vec{H}$ SINCE THIS IS FIXED IN THE INERTIAL FRAME ($\vec{M} \equiv 0$)
  - CONSIDER AXISYMMETRIC BODIES
    - "PLATES"
    - "TUBES"

- CAN DEVELOP SIMPLE, FAIRLY INTUITIVE GEOMETRIC INTERPRETATIONS FOR THE RESULTING MOTION.
  - CLASSIC PROBLEM IN CLASSICAL MECHANICS





- AXISYMMETRIC WITH PRIMARY SPIN ABOUT THE e_3 - AXIS $\Rightarrow$ $I_1 \equiv I_2$



- EULERS E.O.M REDUCE TO:

$$I_1 \dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3 = 0$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0$$

$$I_3 \dot{\omega}_3 = 0$$

$\Rightarrow$ ω_3 = CONSTANT = ν

- REWRITE:

$$\dot{\omega}_1 + \lambda \omega_2 = 0$$
$$\dot{\omega}_2 - \lambda \omega_1 = 0$$

$$\lambda = \left(\frac{I_1 - I_3}{I_1}\right)\nu$$

"RELATIVE SPIN RATE"

$$\Rightarrow \quad \ddot{\omega}_1 + \lambda^2 \omega_1 = 0$$

- SOLUTION OF THE FORM

$$\left.\begin{aligned} \omega_1(t) &= \omega_{10} \cos \lambda t + \omega_{20} \sin \lambda t \\ \omega_2(t) &= \omega_{20} \cos \lambda t - \omega_{10} \sin \lambda t \end{aligned}\right\}$$

EASY TO SHOW

$$\omega_{12}^2 \equiv \omega_1^2 + \omega_2^2 = \omega_{10}^2 + \omega_{20}^2 = \text{CONSTANT.}$$

- SO, CONSTANTS IN THIS PROBLEM ARE i) ν
  ii) ω_{12}

AND TIME t_0 AT WHICH $\omega_1 = 0$, $\omega_2 = \omega_{12}$

$$\Rightarrow \quad t_0 = \frac{1}{\lambda} \tan^{-1}\left(-\frac{\omega_{10}}{\omega_{20}}\right)$$

JUST DEFINES THE "START" TIME

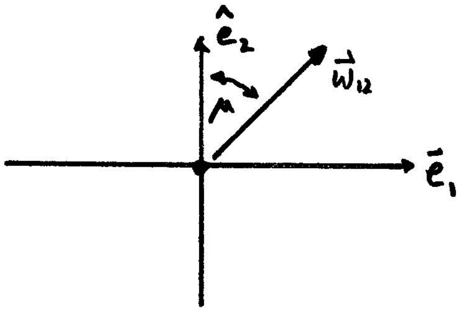
- So ω_{12} corresponds to the projection of the $\vec{\omega}$ instantaneously into the body frame.
  - Body frame is rotating in 3-D
  - The $\vec{\omega}$ is also moving in 3-D

$\Rightarrow$ Only thing that is fixed is the $\vec{H}$

- What can we say about the relative motions of the body ($\vec{e}_3$) and the $\vec{\omega}$?

  $\Rightarrow$ Can answer this by studying the motion of $\vec{\omega}$ projected onto the $\vec{e}_1, \vec{e}_2$ plane.

$$\Rightarrow \quad \vec{\omega}_{12} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2$$



Body principal axes $\vec{e}_1, \vec{e}_2$

- Recall: $|\vec{\omega}_{12}| =$ constant
- Direction that $\vec{\omega}_{12}$ points (size of ω_1, ω_2 components) will change as a function of time.
- Define $\mu = \lambda (t - t_0)$

$$\Rightarrow \quad \omega_1 = \omega_{12} \sin \mu$$
$$\omega_2 = \omega_{12} \cos \mu$$

Note: λ can be either +ve or -ve

$\Rightarrow$ gives relative spin rate.

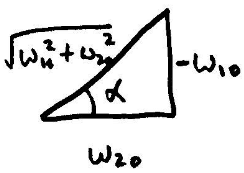
$$\omega_1 = \omega_{10} \cos \lambda t + \omega_{20} \sin \lambda t \qquad \lambda t_0 = \tan^{-1}\left(-\frac{\omega_{10}}{\omega_{20}}\right)$$

$$\omega_1 = \omega_{12} \sin \mu = \omega_{12} \sin \lambda (t - t_0)$$

$$= \omega_{12} \sin (\lambda t - \lambda t_0)$$

$$= \omega_{12}\left(\sin \lambda t \cos \lambda t_0 - \sin \lambda t_0 \cos \lambda t\right)$$

BUT



$$\alpha = \tan^{-1}\left(\frac{-\omega_{10}}{\omega_{20}}\right) \equiv \lambda t_0$$

$$\therefore \cos \lambda t_0 = \frac{\omega_{20}}{\sqrt{\omega_{10}^2 + \omega_{20}^2}} = \frac{\omega_{20}}{\sqrt{\omega_{12}^2}} = \frac{\omega_{20}}{\omega_{12}}$$

$$\sin \lambda t_0 = \frac{-\omega_{10}}{\omega_{12}}$$

$$\therefore \omega_1 = \sin \lambda t \left(\omega_{12} \cos \lambda t_0\right) - \cos \lambda t \left(\omega_{12} \sin \lambda t_0\right)$$

$$= \omega_{20} \sin \lambda t + \omega_{10} \cos \lambda t$$

- SUMMARY

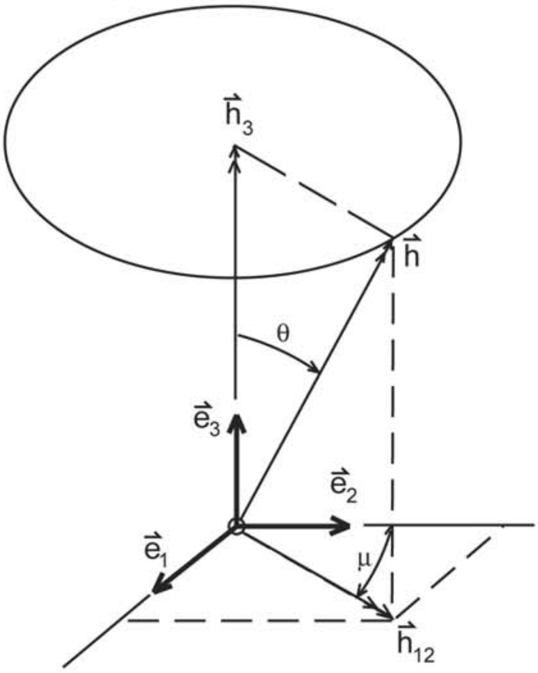
$$\left.\begin{aligned} \omega_1 &= \omega_{12} \sin\mu \\ \omega_2 &= \omega_{12} \cos\mu \\ \omega_3 &= \nu \end{aligned}\right\} \quad \begin{aligned} \omega^2 &= \omega_1^2 + \omega_2^2 + \omega_3^2 \\ &= \nu^2 + \omega_{12}^2 = \text{CONSTANT.} \end{aligned}$$

- NOW CONSIDER ANGULAR MOMENTUM.
  - $\vec{H}$ FIXED, BUT
  - AS BODY ROTATES, EXPECT THAT THE PROJECTION OF $\vec{H}$ INTO THE BODY FRAME $\vec{e}_1, \vec{e}_2, \vec{e}_3$ WILL CHANGE WITH TIME.

- DETAILS:

$$H_1 = I_1 \omega_1 = I_1 \omega_{12} \sin\mu$$
$$H_2 = I_2 \omega_2 = I_2 \omega_{12} \cos\mu = I_1 \omega_{12} \cos\mu$$
$$H_3 = I_3 \omega_3 = I_3 \nu$$

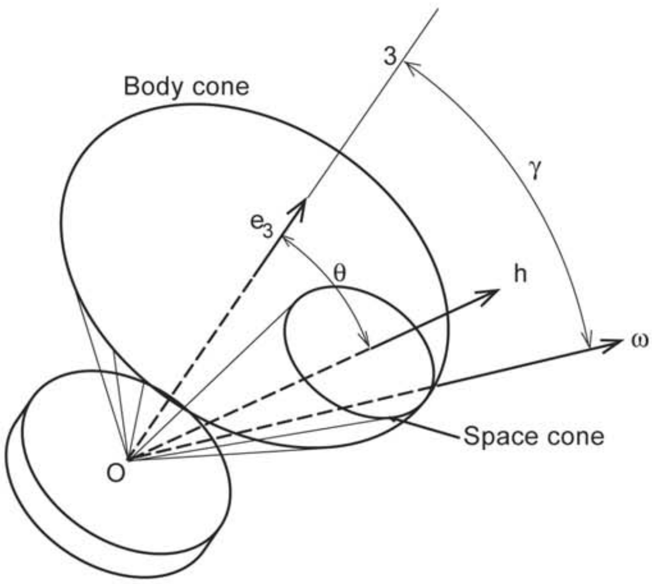
LET $H_{12} = I_1 \omega_{12}$ $\Rightarrow$ $\left\{\begin{aligned} H_1 &= H_{12} \sin\mu \\ H_2 &= H_{12} \cos\mu \\ H_3 &= I_3 \nu \end{aligned}\right.$



NOTE: μ STILL DEFINES ANGLE FROM $\vec{e}_2$ TO $\vec{H}_{12} = H_1\vec{e}_1 + H_2\vec{e}_2$

- MORE ON θ LATER

- FOR THE GEOMETRY, LET:
  - γ BE THE ANGLE BETWEEN THE $\vec{\omega}$ AND THE 3-AXIS OF THE BODY FRAME $(\vec{e}_3)$
  - θ BE THE ANGLE BETWEEN THE $\vec{H}$ AND THE 3-AXIS OF THE BODY FRAME. $(\vec{e}_3)$



- THEN WE HAVE:

$$\tan\theta = \frac{H_{12}}{H_3} = \frac{I_1 \omega_{12}}{I_3 \nu}$$

$$\tan\gamma = \frac{\omega_{12}}{\omega_3} = \frac{\omega_{12}}{\nu}$$

KEY EQUATION.

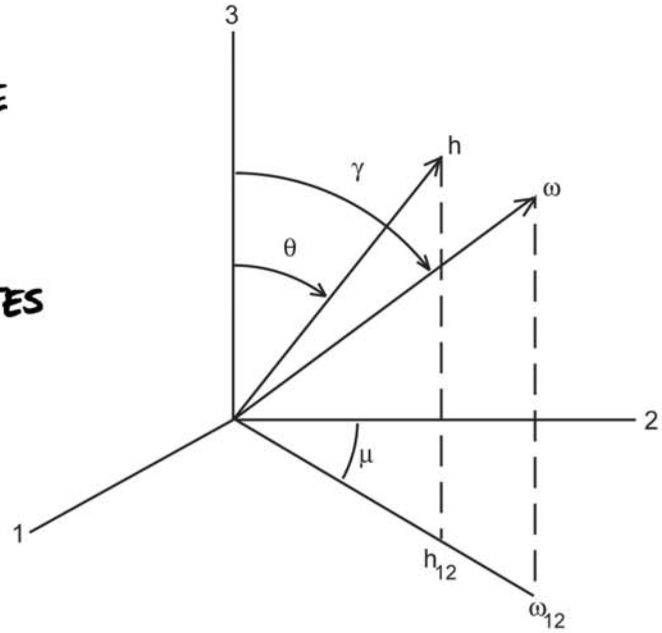
$$\therefore \quad \tan\theta = \left(\frac{I_1}{I_3}\right)\tan\gamma$$

- IF $I_1 > I_3$ (ROD) THEN $\theta > \gamma$

  $I_1 < I_3$ (DISC) THEN $\theta < \gamma$

- <u>NOTE:</u> θ GIVES BODY AXIS ORIENTATION WRT INERTIAL DIRECTION, AND IS OFTEN CALLED THE <u>NUTATION ANGLE</u>.

- NOTE - FAIRLY EASY TO SHOW THAT $\vec{\omega}, \vec{H}, \vec{e}_3$ ALL LIE IN ONE PLANE.



① SINCE $\vec{H}$ FIXED, THIS PLANE ROTATES ABOUT $\vec{H}$.

- PATH OF $\vec{\omega}$ IN 3-D CREATES A BODY CONE AND A SPACE CONE

- BODY CONE: - ATTACHED TO $\vec{e}_3$ OF BODY + ALIGNED WITH SYMMETRY AXIS
  - AT AN ANGLE γ FROM $\vec{e}_3$ TO $\vec{\omega}$

- SPACE CONE: - ATTACHED TO $\vec{H}$, SO FIXED IN INERTIAL SPACE.
  - AT AN ANGLE $|\gamma - \theta|$ FROM $\vec{H}$ TO $\vec{\omega}$

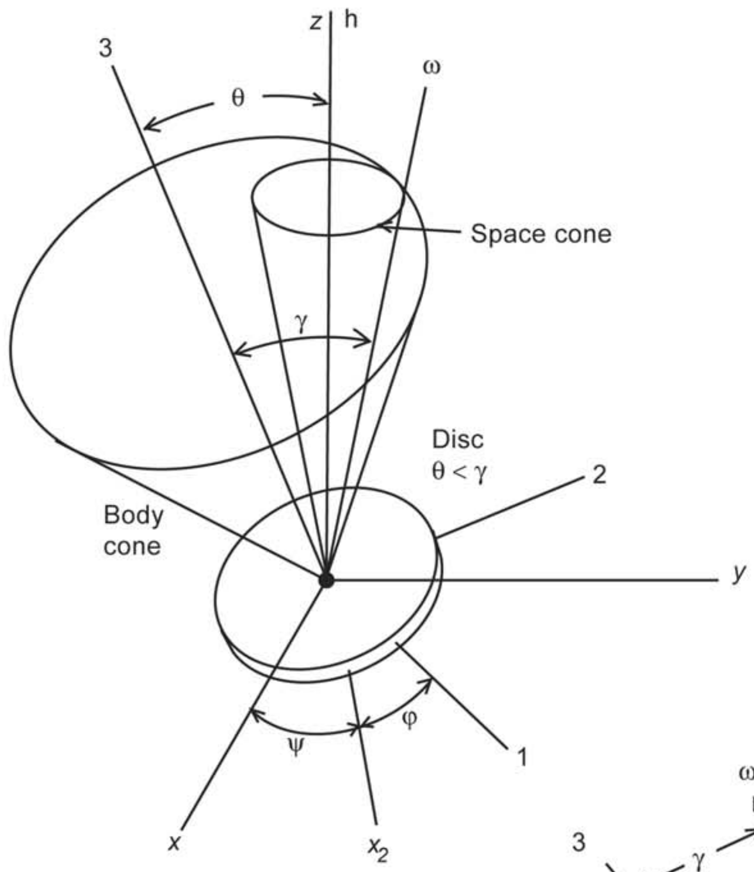
- $\vec{\omega}$ IS AT THE LINE OF TANGENCY OF THE TWO CONES

  $\Rightarrow$ BODY ATTITUDE MOTION CAN BE VISUALIZED BY ROLLING ONE CONE (BODY) ON THE OTHER.

- RECALL FROM BEFORE

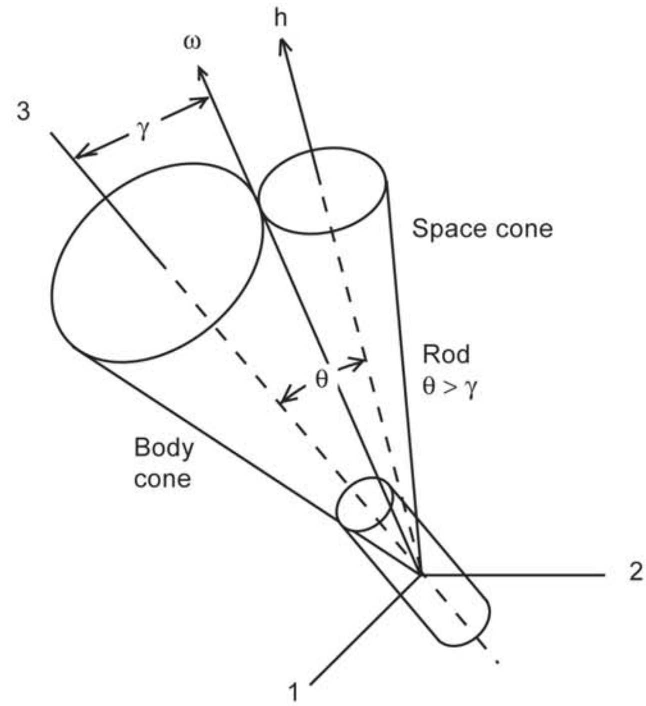
$I_1 > I_3 \Rightarrow \theta > \gamma$ $\quad e_3 \rightarrow \omega \rightarrow H$ (ROD)

$I_3 > I_1 \Rightarrow \theta < \gamma$ $\quad e_3 \rightarrow H \rightarrow \omega$ (DISC)



BODY CONE ROLLS ON FIXED SPACE CONE

$\vec{\omega}$ ALWAYS AT LINE OF TANGENCY OF THE 2 CONES.



- THE ROTATION OF $\vec{e}_3$ AND $\vec{\omega}$ ABOUT $\vec{H}$ IS CALLED <u>PRECESSION</u>
  - BUT WE HAVE TWO DIFFERENT TYPES OF PRECESSION HERE
  - DIFFERENTIATE BETWEEN THEM BY HOW $\vec{e}_3$ AND $\vec{\omega}$ ARE MOVING WRT TO EACH OTHER. $\Rightarrow$ DETERMINED BY $\lambda \leftrightarrow \mu$

- SINCE $\lambda = \left(\frac{I_1 - I_3}{I_1}\right)\nu$, THEN IF

$I_3 > I_1$ (DISC) $\lambda < 0$

$I_3 < I_1$ (ROD) $\lambda > 0$

- WHEN $\lambda < 0$ CALLED <u>RETROGRADE PRECESSION</u>

  " $\lambda > 0$ " <u>DIRECT PRECESSION</u>

- THIS DIFFERENCE IS NOT SOMETHING THAT CAN NORMALLY BE SEEN.

- FINAL STEP IS TO CONNECT THE BODY TO THE INERTIAL FRAME MORE CONCRETELY USING <u>EULER ANGLES.</u>

  - ROTATE BY ψ ABOUT $\vec{H}$ $\quad x_1, y_1, z_1 \to x_2, y_2, z_2$
  - ROTATE BY θ ABOUT x_2 $\quad \to x_3, y_3, z_3$
  - ROTATE BY ϕ ABOUT $z_3 \equiv \vec{e}_3$

NOTE: θ CONSTANT.

$\dot{\phi} \sim$ "BODY SPIN RATE"

- CAN RELATE $\vec{\omega} = \dot{\psi}\,\vec{z}_1 + \dot{\phi}\,\vec{e}_3$

PROJECT INTO BODY FRAME COMPONENTS:

$$\left.\begin{aligned} \omega_1 &= \dot{\psi}\sin\theta\sin\phi \\ \omega_2 &= \dot{\psi}\sin\theta\cos\phi \\ \omega_3 &= \dot{\phi} + \dot{\psi}\cos\theta \end{aligned}\right\}$$

CAN SHOW $\dot{\psi}$ = CONSTANT.

$\dot{\psi} \sim$ PRECESSION SPEED - RATE OF ROTATION OF x_2 IN INERTIAL SPACE

$$\Rightarrow \dot{\omega}_1 = \dot{\psi}\dot{\phi}\sin\theta\cos\phi$$

$$\therefore I_1\dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3 = I_1(\dot{\psi}\dot{\phi}\sin\theta\cos\phi) + (I_3 - I_1)(\dot{\psi}\sin\theta\cos\phi)(\dot{\phi} + \dot{\psi}\cos\theta) = 0$$

$$\Rightarrow I_1\dot{\phi} + (I_3 - I_1)(\dot{\phi} + \dot{\psi}\cos\theta) = 0$$

$$\dot{\psi} = \frac{I_3}{(I_1 - I_3)\cos\theta}\dot{\phi}$$

$I_3 > I_1$, $\dot{\psi}, \dot{\phi}$ HAVE OPPOSITE SIGNS.

SUMMARY

- SPACE AND BODY CONES GIVE A LOT OF INSIGHT INTO THE MOTION OF THE BODY - NO DIRECT INTEGRATION
  ⇒ COMPLEX BEC $\vec{\omega}, \vec{H}$ NOT ALIGNED.
- "CONING" MOTION OF BODY AROUND THE $\vec{H}$ IS VERY COMMON
  - POORLY THROWN SPIRAL ON A FOOTBALL
- OFTEN HEAR ABOUT "SPIN STABILIZATION"
  - REFERS TO GIVING A BODY A LARGE SPIN RATE → LARGE $\vec{H}$
    ⇒ MAKES IT RELATIVELY IMMUNE TO THE INFLUENCE OF SMALL EXTERNAL TORQUES.
  - USED EXTENSIVELY IN EARLY SPACECRAFT. LESS SO NOW.