

Numerical Methods for PDEs

Integral Equation Methods, Lecture 1
Discretization of Boundary Integral Equations

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Outline for this Module

Overview of Integral Equation Methods

Important for many exterior problems
(Fluids, Electromagnetics, Acoustics)

Quadrature and Cubature for computing integrals

One and Two dimensional basics
Dealing with Singularities

1st and 2nd Kind Integral Equations

Collocation, Galerkin and Nystrom theory

Alternative Integral Formulations

Ansatz approach and Green's theorem

Outline for this Module

Fast Solvers

Fast Multipole and FFT-based methods.

Outline

Integral Equation Methods

Exterior versus interior problems

Start with using point sources

Standard Solution Methods

Collocation Method

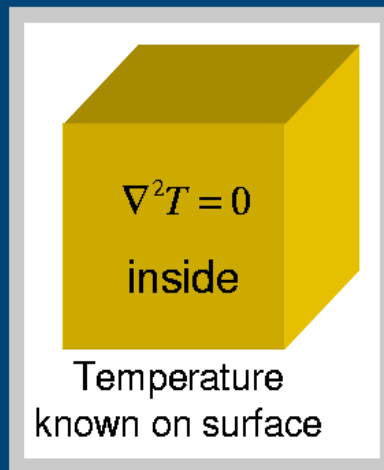
Galerkin Method

Some issues in 3D

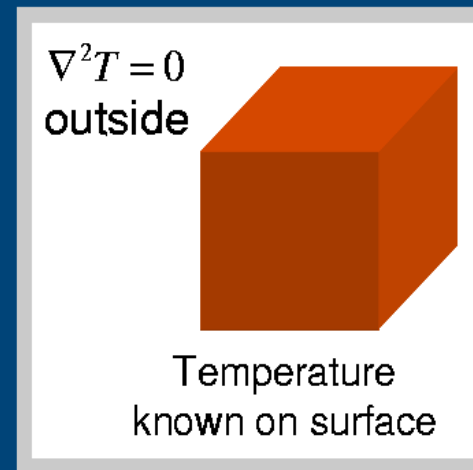
Singular integrals

Interior Vs Exterior Problems

Interior



Exterior

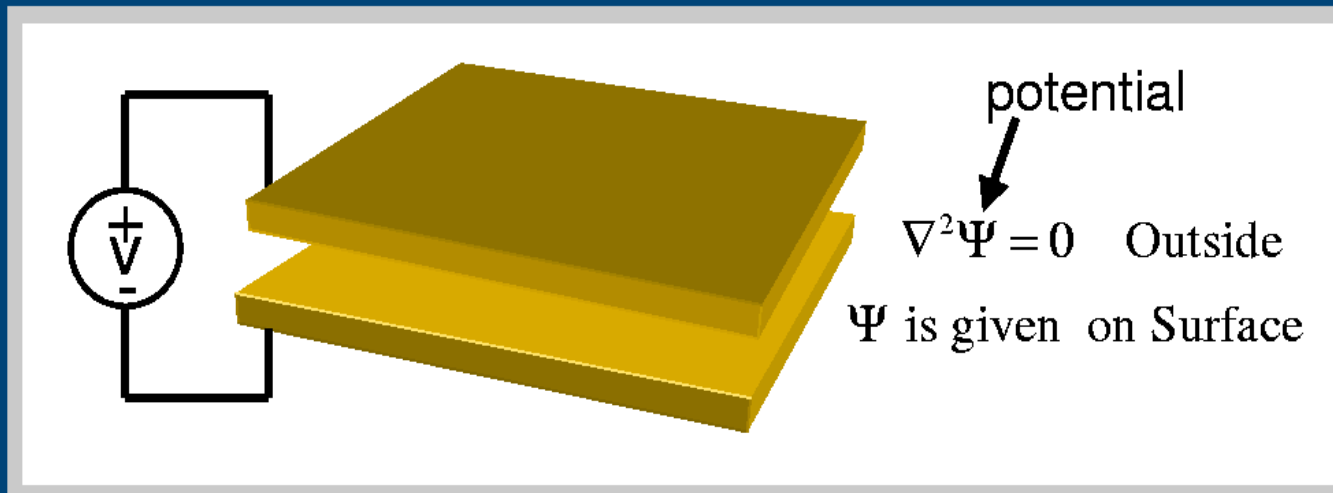


"Temperature in a tank" "Ice cube in a bath"

What is the heat flow?

$$\text{Heat flow} = \text{Thermal conductivity} \int_{\text{surface}} \frac{\partial T}{\partial n}$$

Examples

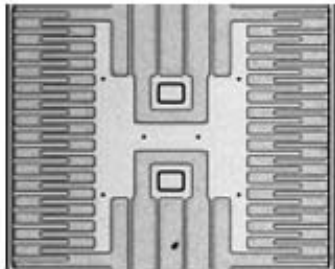


What is the capacitance?

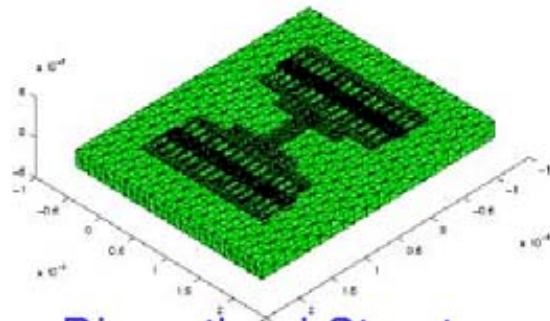
Capacitance = Dielectric Permittivity $\int \frac{\partial \Psi}{\partial n}$

Drag Force in a Microresonator

Examples



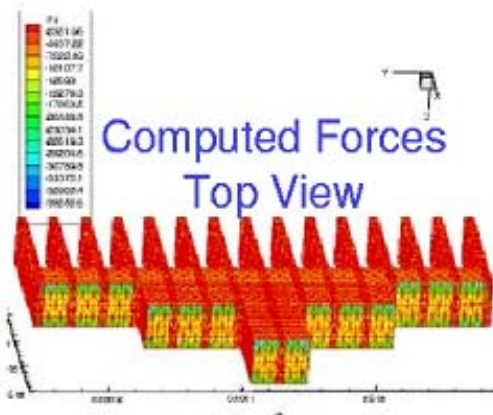
Resonator



Discretized Structure

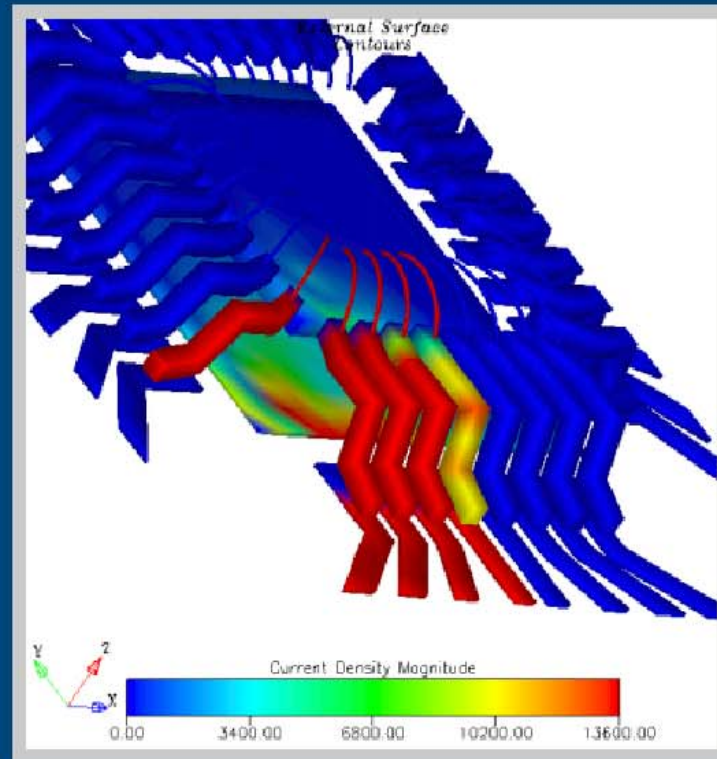


Computed Forces
Bottom View



Computed Forces
Top View

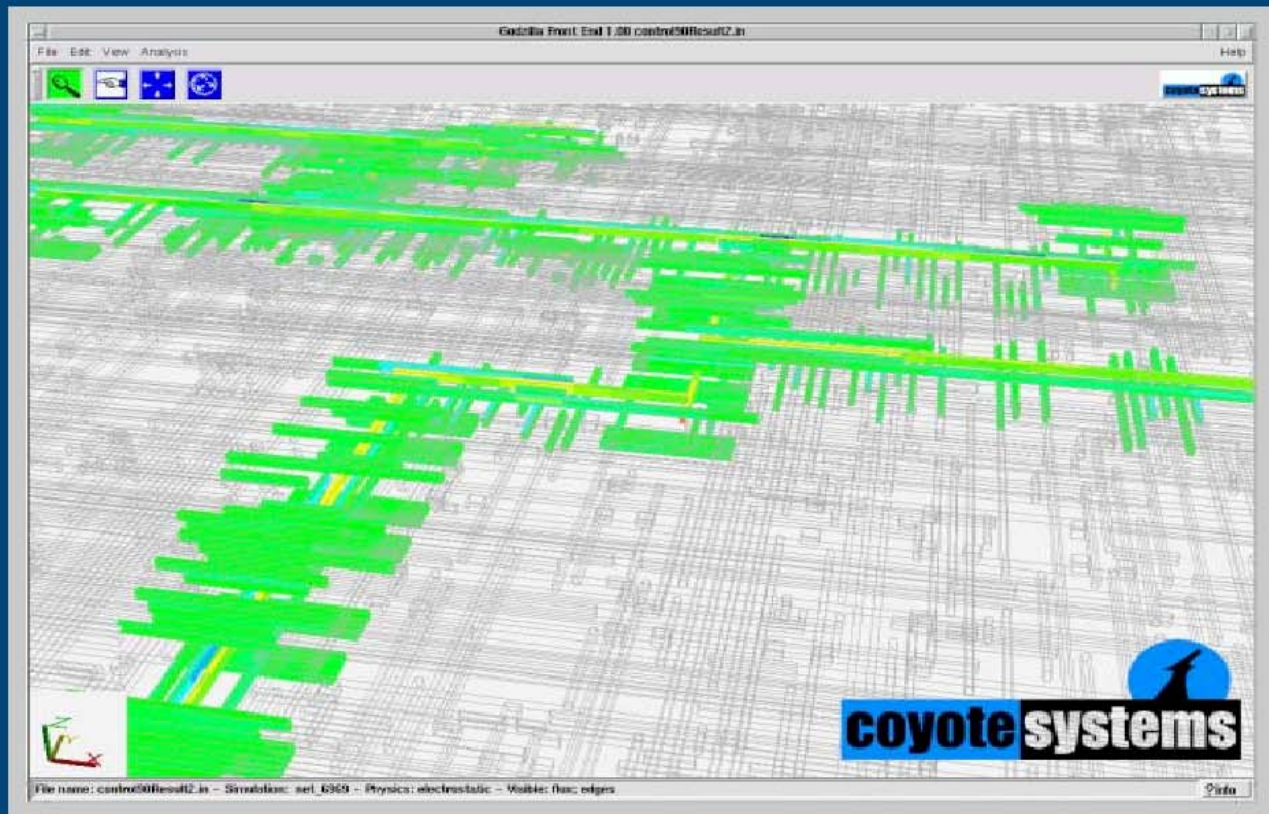
Examples



Picture Thanks to Coventor.

Capacitance of Microprocessor Signal Lines

Examples



What is common about these problems?

Exterior Problems

MEMS device - fluid (air) creates drag

Package - Exterior fields create coupling

Signal Line - Exterior fields.

Quantities of interest are on surface

MEMS device - Just want surface traction force

Package - Just want coupling between conductors

Signal Line - Just want surface charge.

Exterior problem is linear and space-invariant

MEMS device - Exterior Stoke's flow equation (linear)

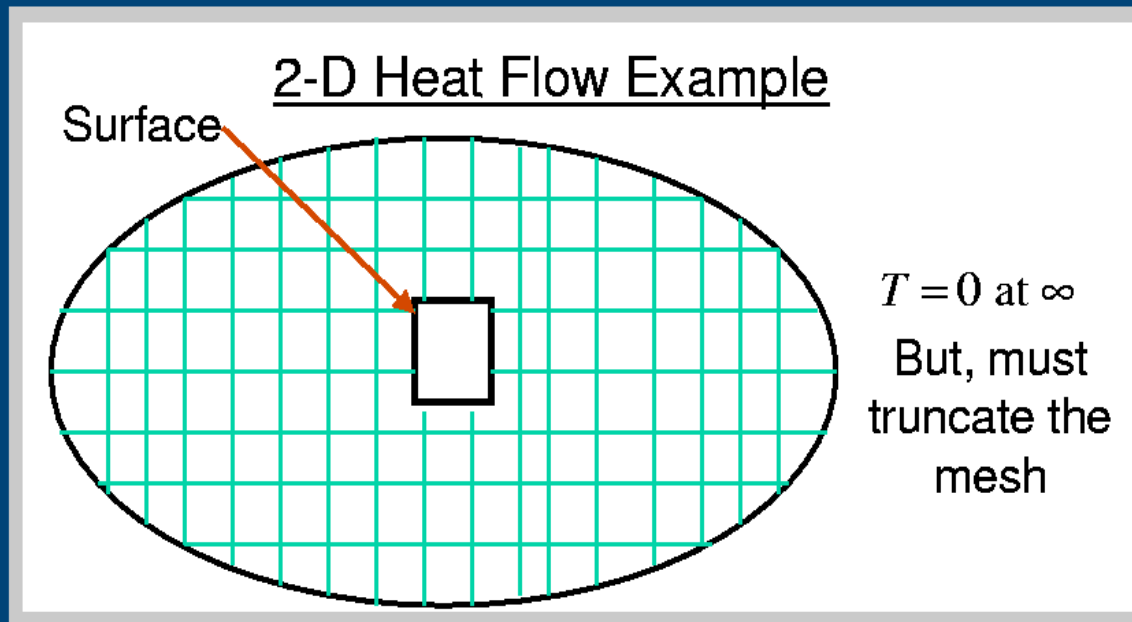
Package - Maxwell's equations in free space (linear)

Signal line - Laplace's equation in free space (linear)

But problems are geometrically very complex

Exterior Problems

Why not use FDM / FEM?



Only need $\frac{\partial T}{\partial n}$ on the surface, but T is computed everywhere.
Must truncate the mesh, $\Rightarrow T(\infty) = 0$ becomes $T(R) = 0$.

Laplace's Equation

Green's Function

In 2D

$$\text{If } u = \log \left(\sqrt{(x - x_0)^2 + (y - y_0)^2} \right)$$
$$\text{then } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \forall \quad (x, y) \neq (x_0, y_0)$$

In 3D

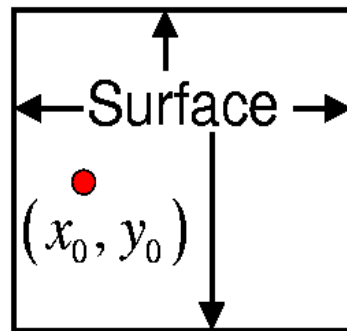
$$\text{If } u = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$
$$\text{then } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \forall \quad (x, y, z) \neq (x_0, y_0, z_0)$$

Proof: Just differentiate and see!

Laplace's Equation in 2D

Simple idea

u is given on surface



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{outside}$$

Let $u = \log \left(\sqrt{(x-x_0)^2 + (y-y_0)^2} \right)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{outside}$$

~~Problem Solved~~

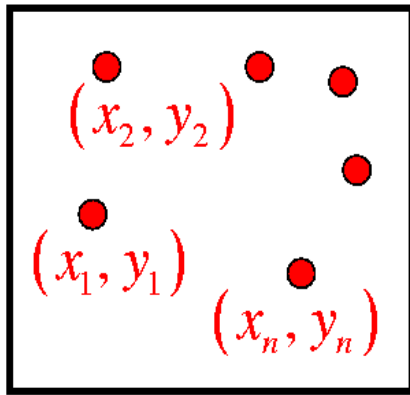
Does not match boundary conditions!

Laplace's Equation in 2D

Simple idea

"More points"

u is given on surface



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{outside}$$

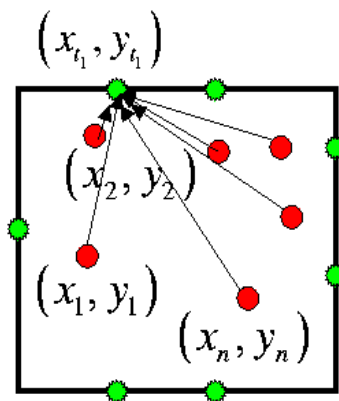
$$\text{Let } u = \sum_{i=1}^n \alpha_i \log \left(\sqrt{(x - x_i)^2 + (y - y_i)^2} \right) = \sum_{i=1}^n \alpha_i G(x - x_i, y - y_i)$$

Pick the α_i 's to match the boundary conditions!

Laplace's Equation in 2D

Simple idea

"More points equations"



Source Strengths selected to give correct potential at **test points.**

$$\begin{bmatrix} G(x_{t_1} - x_1, y_{t_1} - y_1) & \cdots & \cdots & G(x_{t_1} - x_n, y_{t_1} - y_n) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ G(x_{t_n} - x_1, y_{t_n} - y_1) & \cdots & \cdots & G(x_{t_n} - x_n, y_{t_n} - y_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}, y_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_n}, y_{t_n}) \end{bmatrix}$$

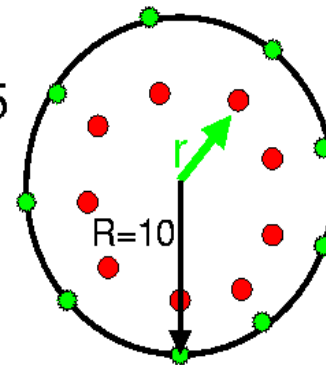
Laplace's Equation in 2D

Simple idea

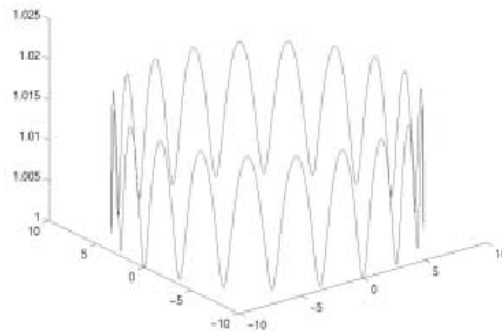
Computational Results

Circle with Charges $r=9.5$

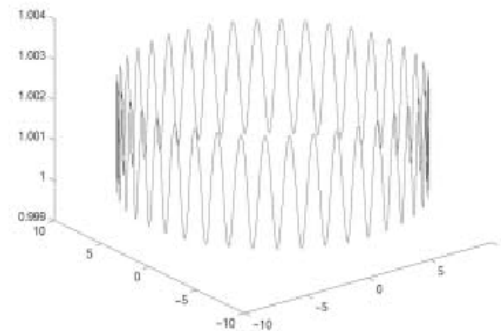
Potentials on the Circle



$n=20$



$n=40$

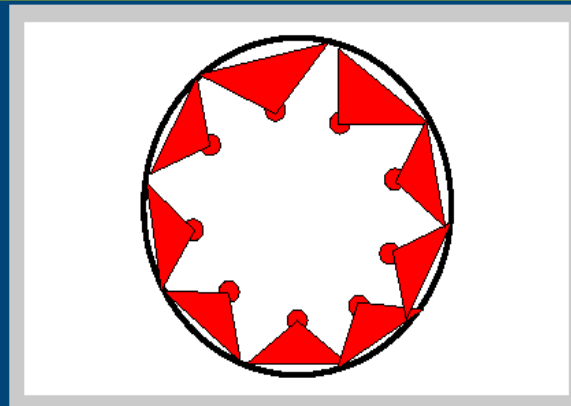


Laplace's Equation in 2D

Integral Formulation

Limiting Argument

Want to smear point charges to the surface



Results in an integral equation

$$\Psi(x) = \int_{\text{surface}} G(x, x') \sigma(x') dS'$$

How do we solve the integral equation?

Laplace's Equation in 2D

Basis Function Approach

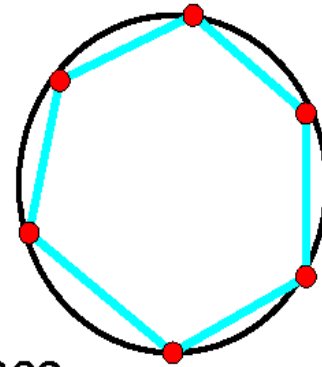
Basic Idea

$$\text{Represent } \sigma(x) = \sum_{i=1}^n \alpha_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$$

Example Basis

Represent circle with straight lines

Assume σ is constant along each line



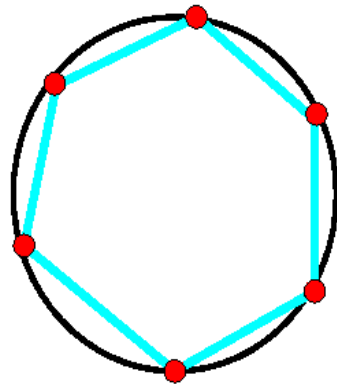
The basis functions are “on” the surface

Can be used to approximate the density
May also approximate the geometry.

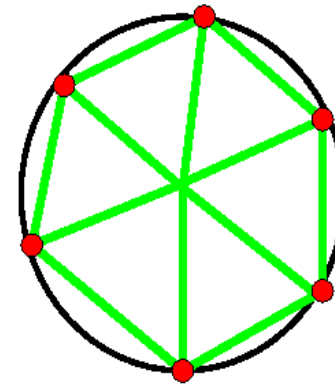
Laplace's Equation in 2D

Basis Function Approach

Geometric Approximation is Not New



Piecewise Straight surface basis
Functions approximate the circle



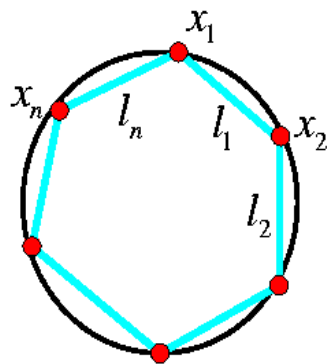
Triangles for 2-D FEM
approximate the circle too!

$$\Psi(x) = \int_{\text{approx surface}} G(x, x') \sum_{i=1}^n \alpha_i \varphi_i(x') dS'$$

Laplace's Equation in 2D

Basis Function Approach

Piecewise Constant Straight Sections Example



- 1) Pick a set of n Points on the surface
- 2) Define a new surface by connecting points with n lines.
- 3) Define $\varphi_i(x) = 1$ if x is on line l_i , otherwise, $\varphi_i(x) = 0$

$$\Psi(x) = \int_{\text{approx surface}} G(x, x') \sum_{i=1}^n \alpha_i \varphi_i(x') dS' = \sum_{i=1}^n \alpha_i \int_{\text{line } l_i} G(x, x') dS'$$

How do we determine the α_i 's?

Laplace's Equation in 2D

Basis Function Approach

Residual Definition and Minimization

$$R(x) = \Psi(x) - \int_{\text{approx surface}} G(x, x') \sum_{i=1}^n \alpha_i \varphi_i(x') dS'$$

We pick the α_i 's to make $R(x)$ small

General Approach: Pick a set of test functions ϕ_1, \dots, ϕ_n and force $R(x)$ to be orthogonal to the set

$$\int \phi_i(x) R(x) dS = 0 \quad \text{for all } i$$

Laplace's Equation in 2D

Basis Function Approach

Residual Minimization Using Test Functions

$$\int \phi_i(x) R(x) dS = 0 \Rightarrow$$

$$\int \phi_i(x) \Psi(x) dS - \int \int_{\text{approx surface}} \phi_i(x) G(x, x') \sum_{j=1}^n \alpha_j \varphi_j(x') dS' dS = 0$$

We will generate different methods by choosing the ϕ_1, \dots, ϕ_n

Collocation : $\phi_i(x) = \delta(x - x_{t_i})$ (point matching)

Galerkin Method : $\phi_i(x) = \varphi_i(x)$ (basis = test)

Weighted Residual Method : $\phi_i(x) = 1$ if $\varphi_i(x) \neq 0$
(averages)

Laplace's Equation in 2D

Basis Function Approach

Collocation

Collocation: $\phi_i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_{t_i})$ (point matching)

$$\int \delta(\mathbf{x} - \mathbf{x}_{t_i}) R(\mathbf{x}) dS = R(\mathbf{x}_{t_i}) = 0 \Rightarrow$$

$$\sum_{j=1}^n \alpha_j \int_{\substack{\text{approx} \\ \text{surface}}} \overbrace{G(\mathbf{x}_{t_i}, \mathbf{x}') \varphi_j(\mathbf{x}') dS'}^{A_{i,j}} = \Psi(\mathbf{x}_{t_i})$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \cdots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(\mathbf{x}_{t_1}) \\ \vdots \\ \vdots \\ \Psi(\mathbf{x}_{t_n}) \end{bmatrix}$$

Laplace's Equation in 2D

Basis Function Approach

Centroid Collocation for Piecewise Constant Bases

$$\Psi(x_i) = \sum_{j=1}^n \alpha_j \int_{\text{approx surface}} G(x_i, x') \phi_j(x') dS'$$

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix}
 \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix}
 =
 \begin{bmatrix} \Psi(x_{t_1}) \\ \vdots \\ \vdots \\ \Psi(x_{t_1}) \end{bmatrix}$$

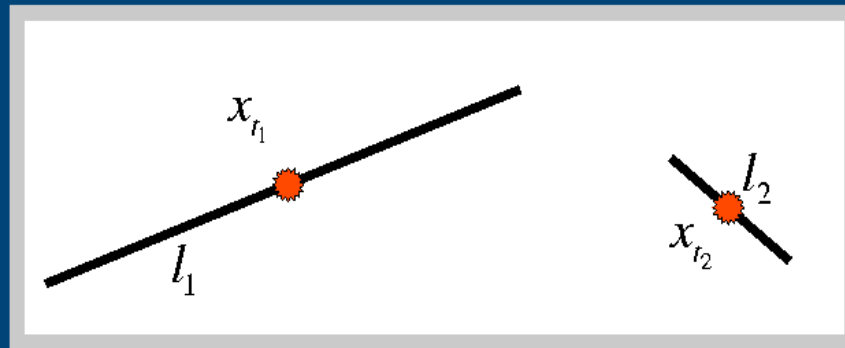
$$\Psi(x_{t_i}) = \sum_{j=1}^n \alpha_j \underbrace{\int_{\text{line } j} G(x_{t_i}, x') dS'}_{A_{i,j}}$$

Laplace's Equation in 2D

Basis Function Approach

Centroid Collocation Generates Nonsymmetric A

$$\Psi(x_{t_i}) = \sum_{j=1}^n \alpha_j \int_{line_j} \overbrace{G(x_{t_i}, x')}^{A_{i,j}} dS'$$



$$A_{1,2} = \int_{line_2} G(x_{t_1}, x') dS' \neq \int_{line_1} G(x_{t_2}, x') dS' = A_{2,1}$$

Laplace's Equation in 2D

Basis Function Approach

Galerkin

Galerkin: $\phi_i(x) = \varphi_i(x)$ (test=basis)

$$\int \varphi_i(x) R(x) dS = \int \varphi_i(x) \Psi(x) dS - \int \int_{\text{approx surface}} \varphi_i(x) G(x, x') \sum_{j=1}^n \alpha_j \varphi_j(x') dS' dS = 0$$

$$\underbrace{\int_{\text{approx surface}} \varphi_i(x) \Psi(x) dS}_{b_i} = \sum_{j=1}^n \alpha_j \underbrace{\int \int_{\text{approx surface}} G(x, x') \varphi_i(x) \varphi_j(x') dS' dS}_{A_{i,j}}$$

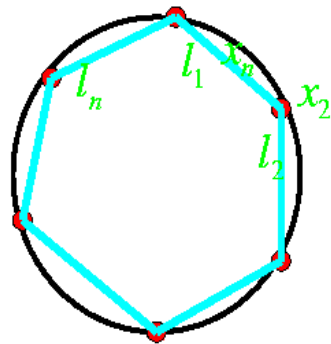
$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

If $G(x, x') = G(x', x)$ then $A_{i,j} = A_{j,i} \Rightarrow \mathbf{A}$ is symmetric

Laplace's Equation in 2D

Basis Function Approach

Galerkin for Piecewise Constant Bases



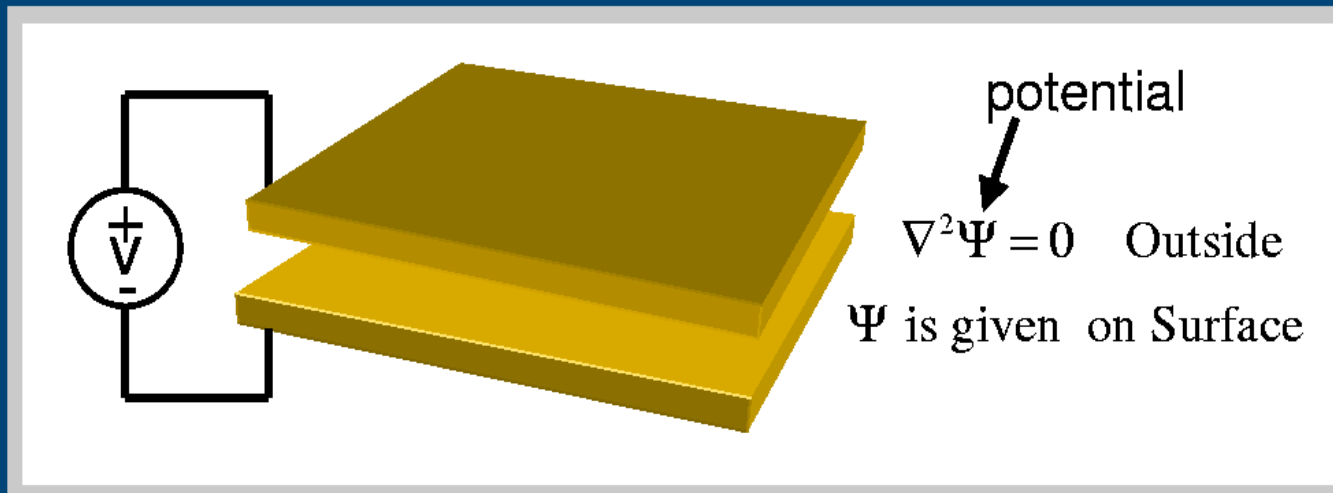
$$\underbrace{\int_{\text{line}_i} \Psi(x) dS}_{b_i} = \sum_{j=1}^n \alpha_j \underbrace{\int_{\text{line}_i} \int_{\text{line}_j} G(x, x') dS' dS}_{A_{i,j}}$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

Laplace's Equation in 3D

Electrostatics Example

Dirichlet Problem



First kind integral equation for charge:

$$\underbrace{\Psi(x)}_{\text{Potential}} = \int_{\text{surface}} \underbrace{\frac{1}{\|x - x'\|}}_{\text{Green's function}} \underbrace{\sigma(x')}_{\text{Charge density}} dS'$$

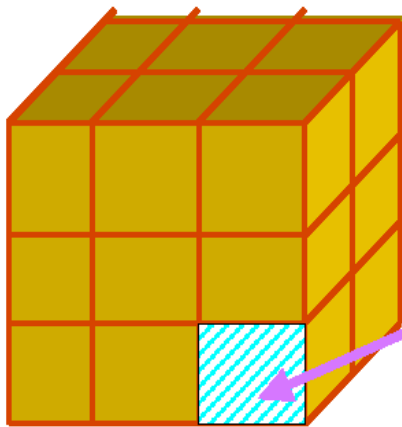
Laplace's Equation in 3D

Basis Function Approach

Piecewise Constant Basis

$$\text{Integral Equation : } \Psi(x) = \int_{\text{surface}} \frac{1}{\|x-x'\|} \sigma(x') dS'$$

Discretize Surface into Panels



Panel j

$$\text{Represent } \sigma(x) \approx \sum_{i=1}^n \alpha_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$$

$$\varphi_j(x) = 1 \quad \text{if } x \text{ is on panel } j$$

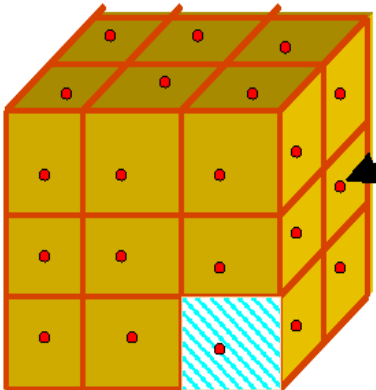
$$\varphi_j(x) = 0 \quad \text{otherwise}$$

Laplace's Equation in 3D

Basis Function Approach

Centroid Collocation

Put collocation points at panel centroids

$$\Psi(x_{c_i}) = \sum_{j=1}^n \alpha_j \underbrace{\int_{\text{panel } j} \frac{1}{\|x_{c_i} - x'\|} dS'}_{A_{i,j}}$$


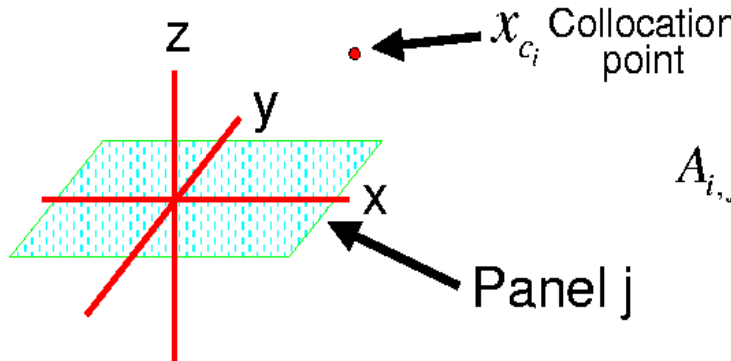
Collocation point x_{c_i}

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{c_1}) \\ \vdots \\ \vdots \\ \Psi(x_{c_n}) \end{bmatrix}$$

Laplace's Equation in 3D

Basis Function Approach

Calculating Matrix Elements



x_{c_i} Collocation point
 Panel j

$$A_{i,j} = \int_{\text{panel } j} \frac{1}{\|x_{c_i} - x'\|} dS'$$

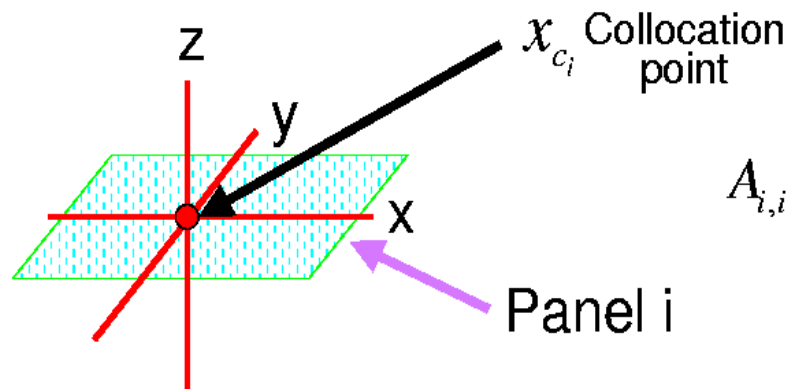
One point quadrature Approximation $A_{i,j} \approx \frac{\text{Panel Area}}{\|x_{c_i} - x_{\text{centroid}_j}\|}$

Four point quadrature Approximation $A_{i,j} \approx \sum_{j=1}^4 \frac{0.25 * \text{Area}}{\|x_{c_i} - x_{\text{point}_j}\|}$

Laplace's Equation in 3D

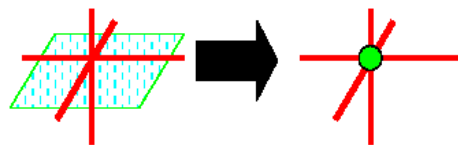
Basis Function Approach

Calculating "Self Term"



$$A_{i,i} = \int_{\text{panel } i} \frac{1}{\|x_{c_i} - x'\|} dS'$$

One point quadrature Approximation



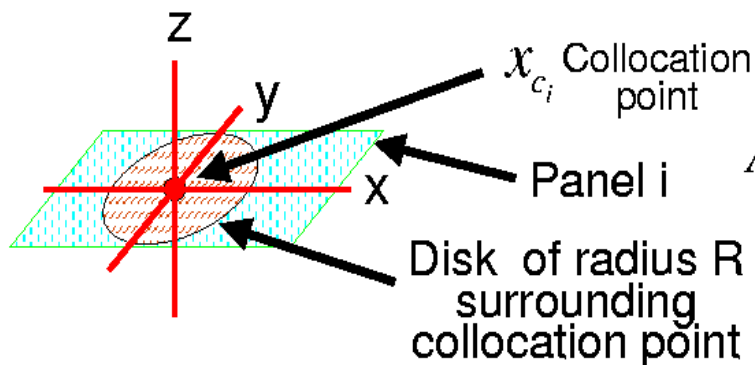
$$A_{i,i} \approx \frac{\text{Panel Area}}{\underbrace{\|x_{c_i} - x_{c_i}\|}_0}$$

$$A_{i,i} = \int_{\text{panel } i} \frac{1}{\|x_{c_i} - x'\|} dS' \text{ is an integrable singularity}$$

Laplace's Equation in 3D

Basis Function Approach

Calculating "Self Term" Tricks of the Trade



$$A_{i,i} = \int_{\text{panel } i} \frac{1}{\|x_{c_i} - x'\|} dS'$$

Integrate in two pieces

$$A_{i,i} = \int_{\text{disk}} \frac{1}{\|x_{c_i} - x'\|} dS' + \int_{\text{rest of panel}} \frac{1}{\|x_{c_i} - x'\|} dS'$$

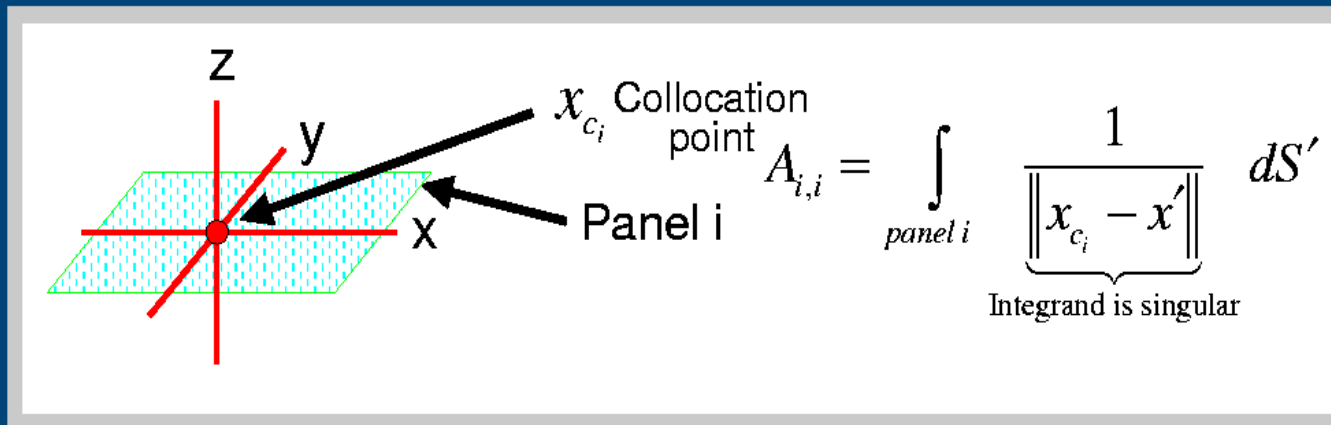
Disk Integral has singularity but has analytic formula

$$\int_{\text{disk}} \frac{1}{\|x_{c_i} - x'\|} dS' = \int_0^R \int_0^{2\pi} \frac{1}{r} r dr d\theta = 2\pi R$$

Laplace's Equation in 3D

Basis Function Approach

Calculating "Self Term" Other Tricks of the Trade



1. If panel is a flat polygon, analytical formulas exist.
2. Curved panels can be handled with projection.

Summary

Integral Equation Methods

- Exterior versus interior problems

- Start with using point sources

Standard Solution Methods

- Collocation Method

- Galerkin Method

Integrals for 3D Problems

- Singular Integrals

We will examine computing integrals next time, and then examine integral equation convergence theory.