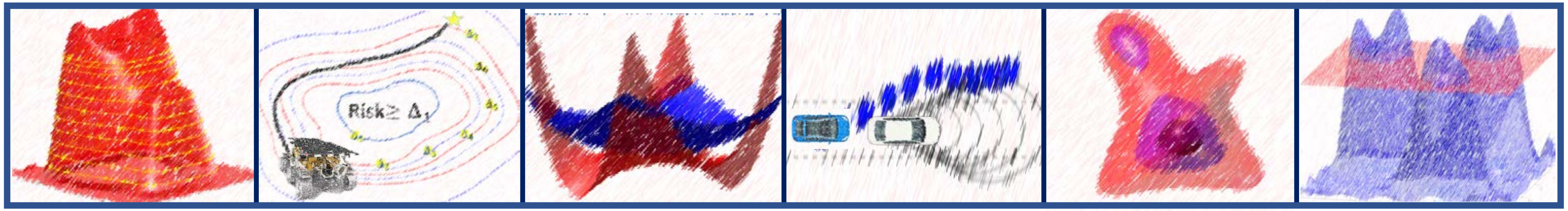
Lecture 11

Risk Aware Planning and Control Of Probabilistic Nonlinear Dynamical Systems

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

Ashkan Jasour



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➢ Topics:

- ➢ Risk Bounded Trajectory Planning in Uncertain Environments
- ➢ Control of Probabilistic Nonlinear Systems
  - Nonlinear State Feedback Control
  - Receding Horizon Control
- ➢ Flow-Tube based Control of Probabilistic Nonlinear Systems
- ➢ Chance Constrained Backward Reachability Set
- ➢ Continuous-Time Path Planning

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- We will leverage on the results of "Lecture 7:Nonlinear Chance Constrained and Chance Optimization".

General Schematics:

➢ In general, we can formulate safe control of probabilistic dynamical system as the following optimization problems:

Chance Optimization

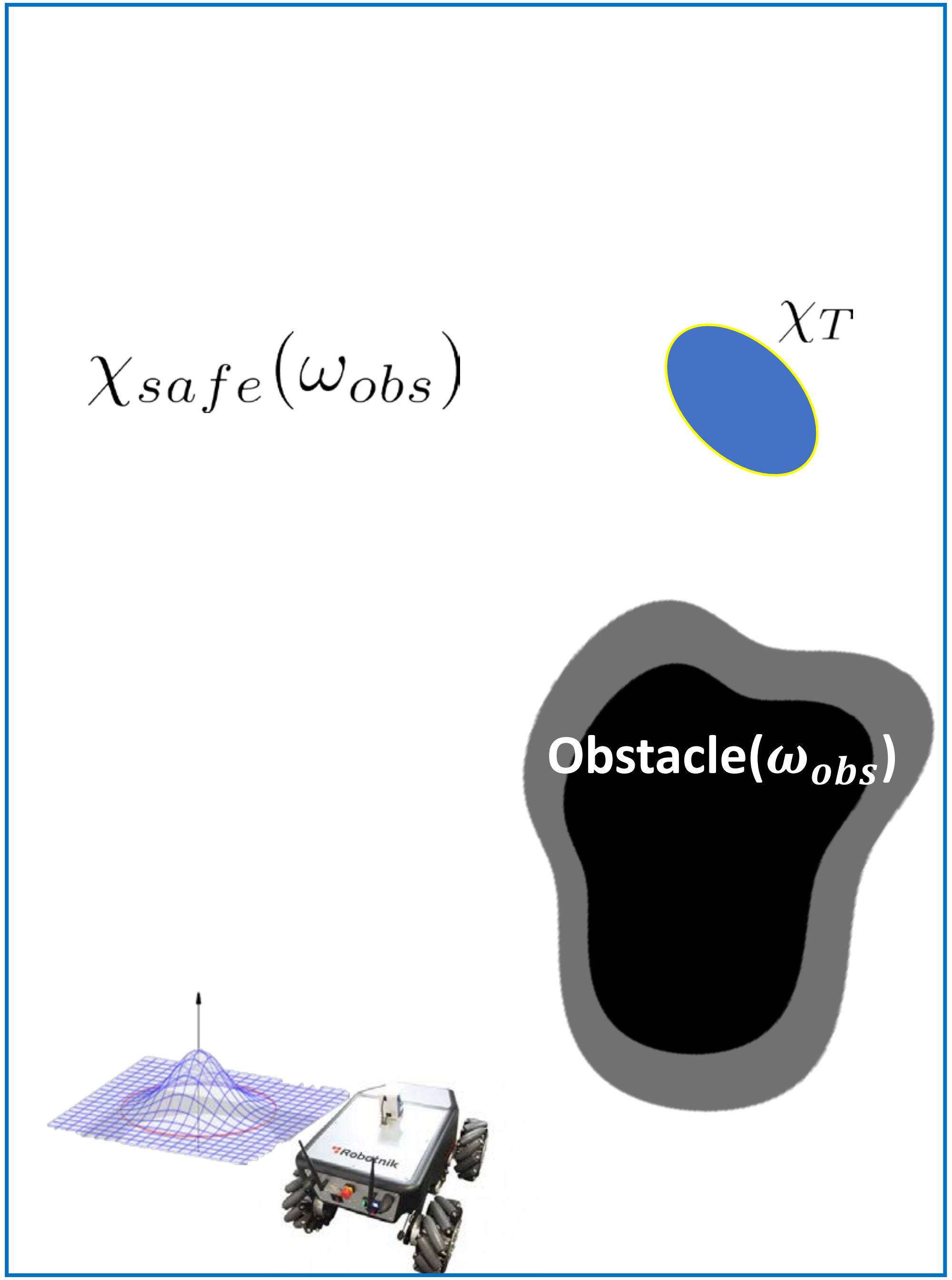
$$\begin{aligned} \underset{\text{design parameters}}{\text{maximize}} \quad & \text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \\ \text{subject to} \quad & \text{Constraints}(\text{design parameters}) \end{aligned}$$

Chance Constrained Optimization

$$\begin{aligned} \underset{\text{design parameters}}{\text{minimize}} \quad & \text{Objective Function}(\text{design parameters}) \\ \text{subject to} \quad & \text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \geq 1 - \Delta \end{aligned}$$

Acceptable risk level

- $\boldsymbol{Success}$ =Remaining safe and achieving the control objectives

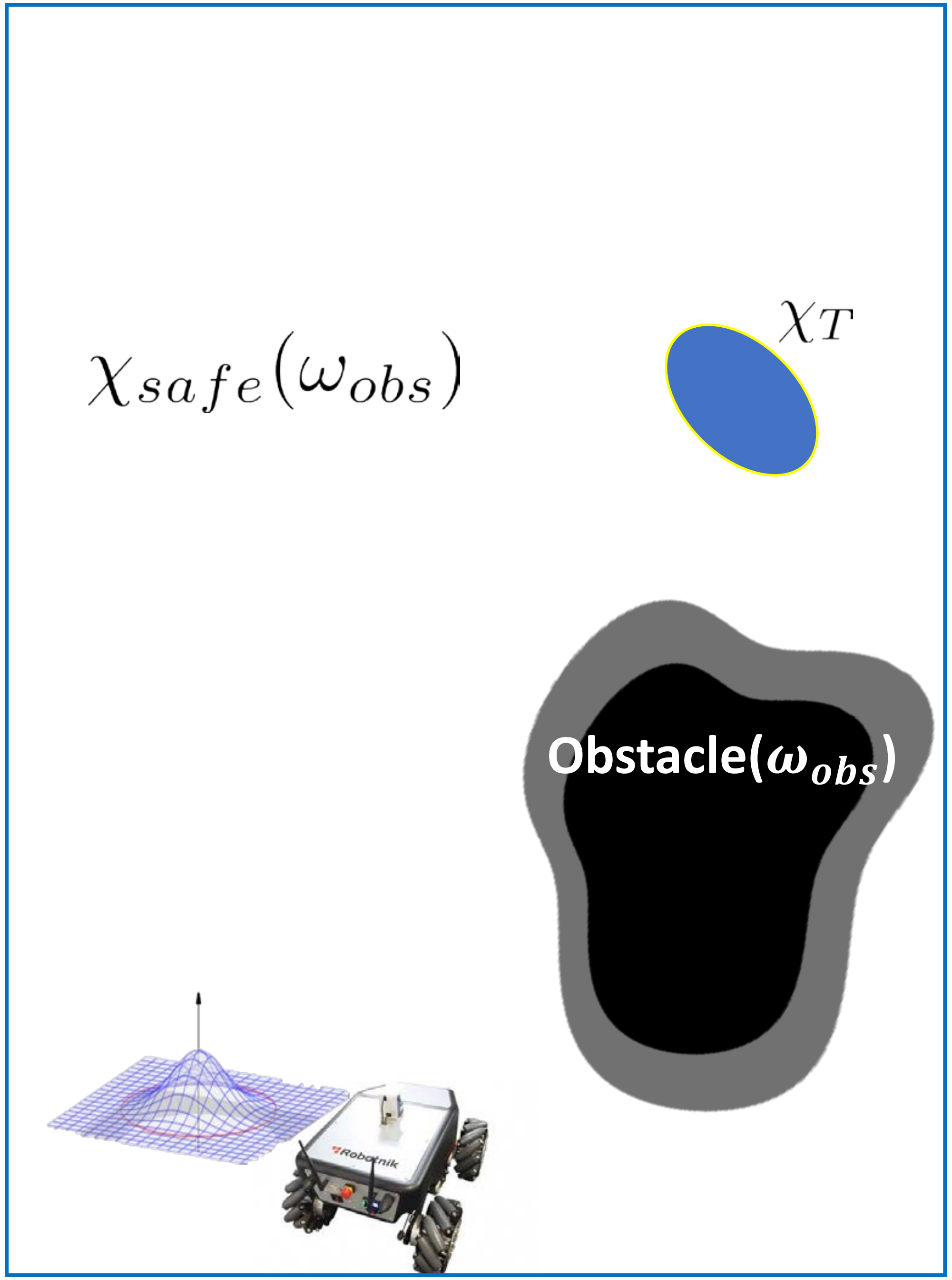

$\chi_{safe}(\omega_{obs})$
χ_T
Obstacle(ω_{obs})
Robotnik

Chance Optimization

Find a sequence of control inputs $[u_0, \dots, u_{N-1}]$

$$\begin{array}{ll} \underset{u_k|_{k=0}^{N-1}}{\text{maximize}} & \boxed{\text{Prob}(x_k \in \chi_{safe}(\omega_{obs}), x_T \in \chi_T)} \\ \text{subject to} & x_{k+1} = f(x_k, u_k, \omega_k) \\ & u_k \in \mathcal{U} \end{array}$$

- ***Success*** =Remaining safe and reaching the goal
- For state feedback control, we look for the feedback gains.



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- ***Success*** =Remaining safe and reaching the goal
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Chance Constrained Optimization

Find a sequence of control inputs $[u_0, \dots, u_{N-1}]$

$$\begin{aligned}
\underset{u_k|_{k=0}^{N-1}}{\text{minimize}} \quad & \sum_{k=0}^{N-1} u^2(k) \\
\text{subject to} \quad & \boxed{\mathrm{E}[x_T] = x_T^* \quad \text{or} \quad \mathrm{Prob}(x_T \in \chi_T) \geq 1 - \Delta} \\
& x_{k+1} = f(x_k, u_k, \omega_k) \\
& \boxed{\begin{array}{l} \mathrm{Prob}(x_k \in \chi_{safe}(\omega_{obs})) \geq 1 - \Delta \\ x_0 \sim \mathrm{pr}(x), \ \omega_k \sim \mathrm{pr}(\omega_k) \end{array}} \\
& u_k \in \mathcal{U}
\end{aligned}$$

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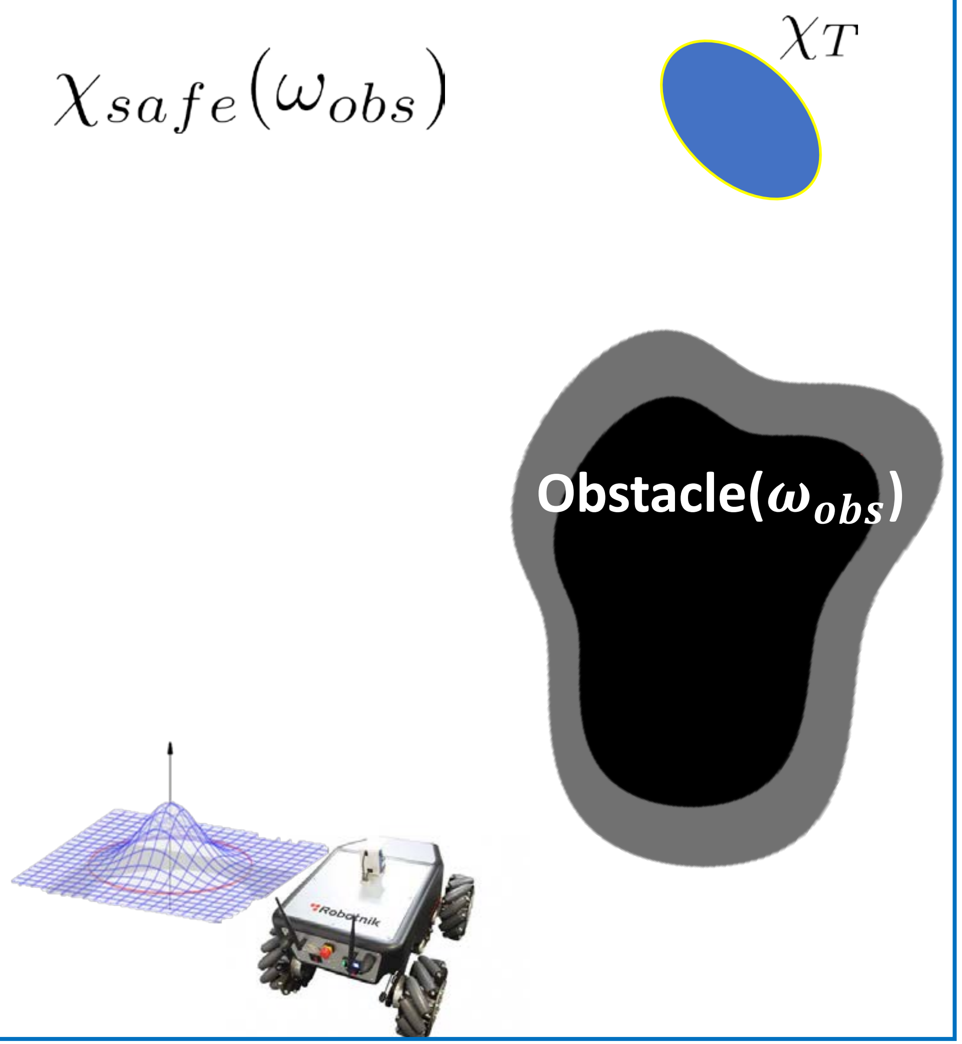

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Using the results of
Lecture 7:Nonlinear Chance Constrained and Chance Optimization:

Moment SDP

- Semialgebraic set approximation of chance constraints:

$$\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$$

$$\Downarrow$$

SOS SDP

$$\chi_{cc} = \{x \in \mathbb{R}^n : \mathcal{P}(x) \geq 1 - \Delta\}$$

$$\Downarrow$$

Deterministic optimization

In this lecture, we mainly focus on particular class of problems as follows:

- Topics:
  - Risk Bounded Trajectory Planning in Uncertain Environments →
    - Deals with environment uncertainties
    - Identifies risky regions in the environment
    - Chance constrained formulation
  - Control of Probabilistic Nonlinear Systems
    - Nonlinear State Feedback Control →
      - Chance optimization formulation
    - Receding Horizon Control →
      - Chance constrained formulation
  - Flow-Tube based Control of Probabilistic Nonlinear Systems →
    - Sequential Chance optimization formulation
  - Chance Constrained Backward Reachability Set →
    - Chance constrained formulation
  - Continuous-Time Path Planning →
    - Generates trajectories for the robot
    - SOS and Chance optimization formulation

Topics:

- ➢ Introduction
- ➢ Polynomial Representation of Obstacles and Dynamical Systems
- ➢ Risk Bounded Trajectory Planning in Uncertain Environments
- ➢ Control of Probabilistic Nonlinear Systems
  - Nonlinear State Feedback Control
  - Receding Horizon Control
- ➢ Flow-Tube based Control of Probabilistic Nonlinear Systems
- ➢ Chance Constrained Backward Reachability Set
- ➢ Continuous-Time Path Planning

Polynomial Representation of Obstacles

Problem Statement:

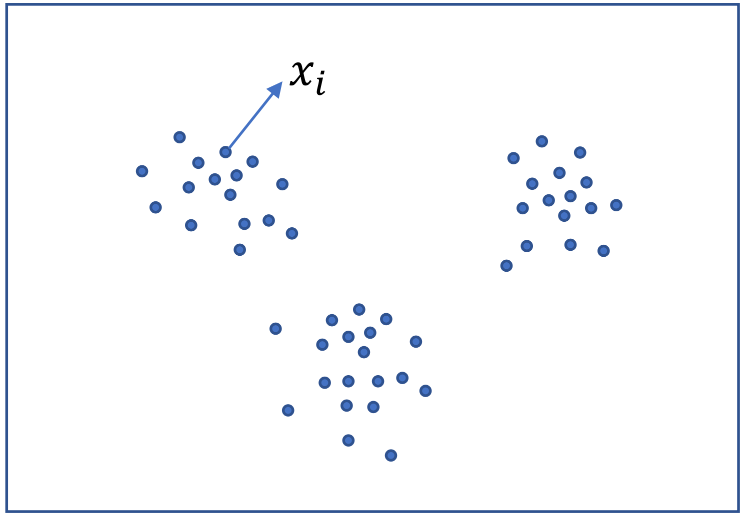
- Given a set of N points in n dimensional space:

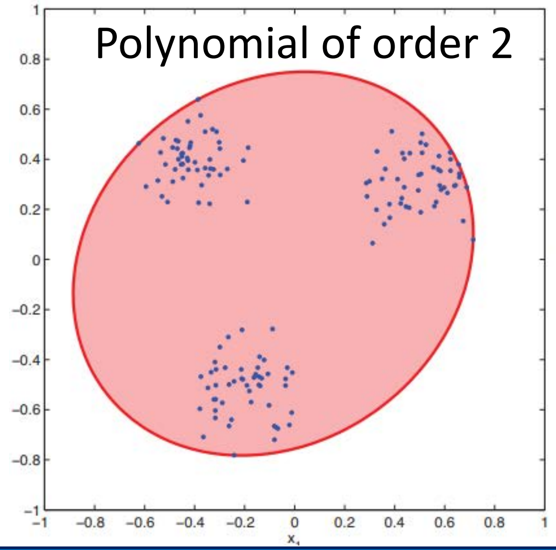
$$x_i \in \mathbb{R}^n, \ i = 1, ..., N$$

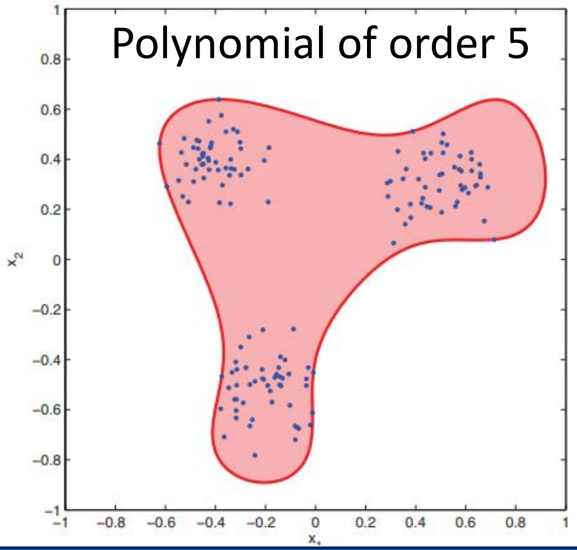
- Find the smallest semialgebraic set that contains the given data

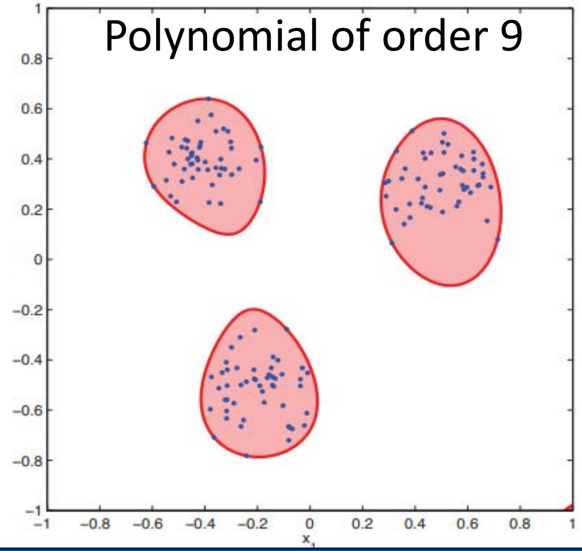
$$\chi = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

Polynomial level set







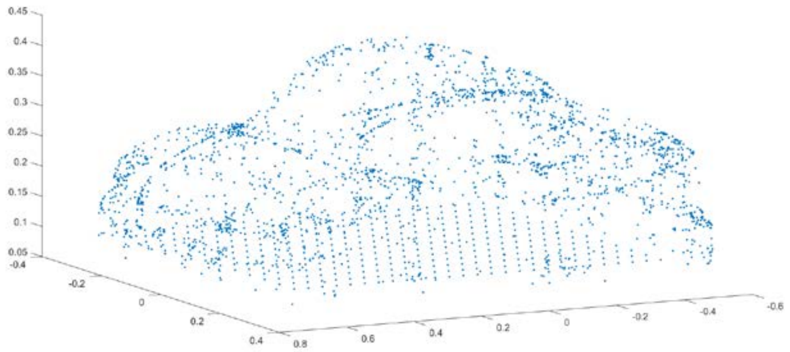


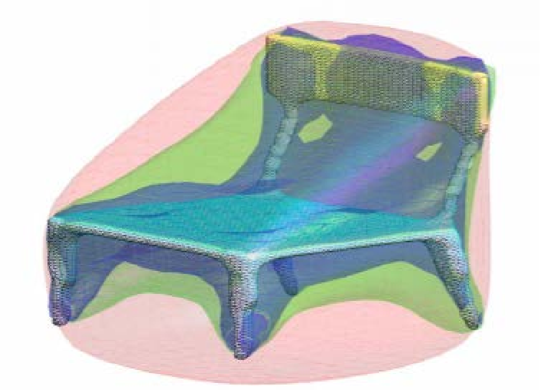
Applications

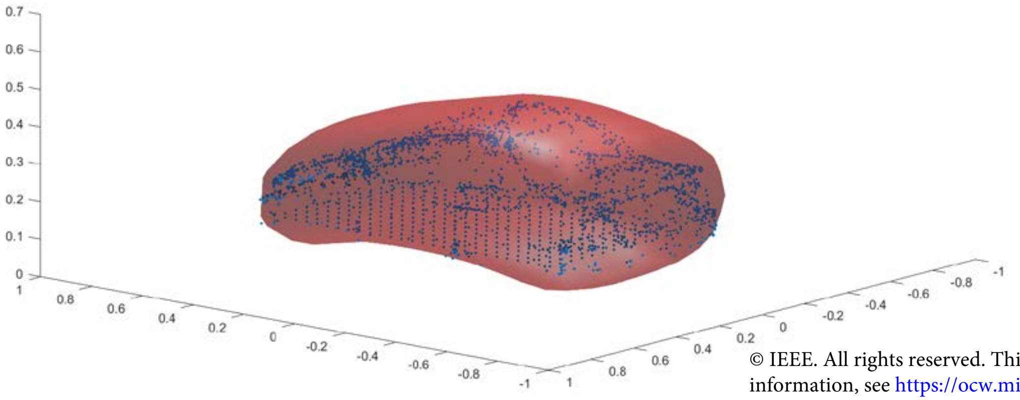
1) Geometrical representation of objects form point cloud data (sensory data)

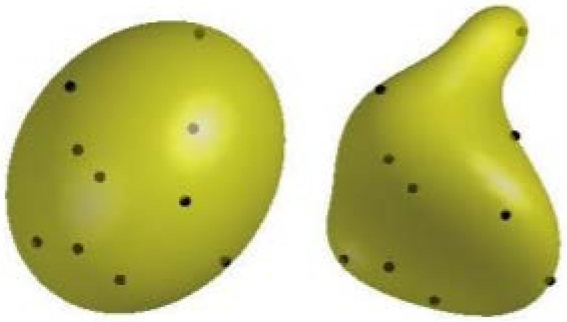
2) Uncertainty set constriction from the samples

3) Data clustering









- Fabrizio Dabbene and Didier Henrion, "Set approximation via minimum-volume polynomial sublevel sets", European Control Conference (ECC), pp 1114-1119, 2013
- F. Dabbene, D. Henrion, C. M.Lagoa "Simple approximations of semialgebraic sets and their applications to control", Automatica Volume 78, pp. 110-118, 2017.
- A. A. Ahmadi, G. Hall, A. Makadia, and V. Sindhwani, "Sum of Squares Polynomials and Geometry of 3D Environments" Robotics: Science and Systems, 2017.

- Given a set of N points in n dimensional space: $x_i \in \mathbb{R}^n, \ i = 1, ..., N$

- Find the smallest semialgebraic set that contains the given data

$$\begin{array}{ll} \underset{\chi \subset \mathbb{R}^n}{\text{minimize}} & \text{vol}(\chi) \\ \text{subject to} & x_i \in \chi, \ i = 1, ..., N \end{array}$$

Where, $\text{vol}(\chi) = \int_{\chi} dx$ is the volume of the set.

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Indicator function based representation:

- Volume of the set in terms of the indicator function: $\text{vol}(\chi) = \int_\chi dx = \int \mathbf{I}_\chi dx$

$$\mathbf{I}_\chi = \begin{cases} 1 & \forall x \in \chi; \\ 0 & \forall x \notin \chi \end{cases}$$

- Set described by its indicator function: $\chi = \{x \in \mathbb{R}^n : \mathbf{I}_\chi(x) = 1\}$

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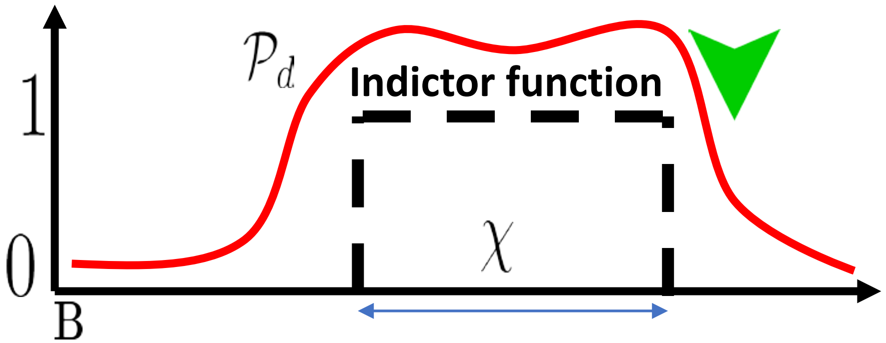
- Upper bound Polynomial approximation of the indictor function:

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x) \in \mathbb{R}_d[x]}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x) dx \\ \text{subject to} \quad & \mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \chi \\ & \mathcal{P}_d(x) \geq 0 \end{aligned}$$



$$\int_{\mathbf{B}} \mathcal{P}_d(x) dx \geq \text{vol}(\chi) = \int \mathbf{I}_{\chi} dx$$

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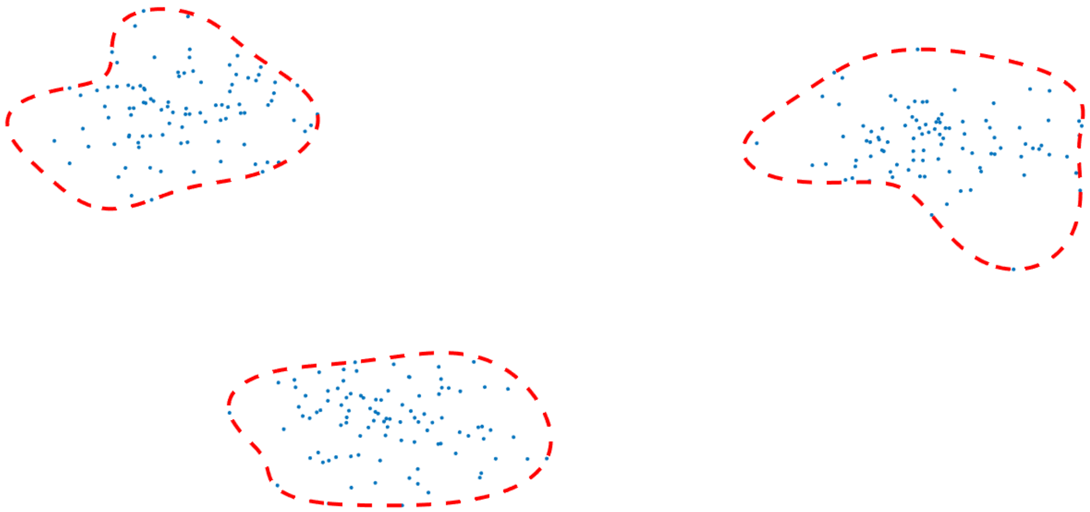
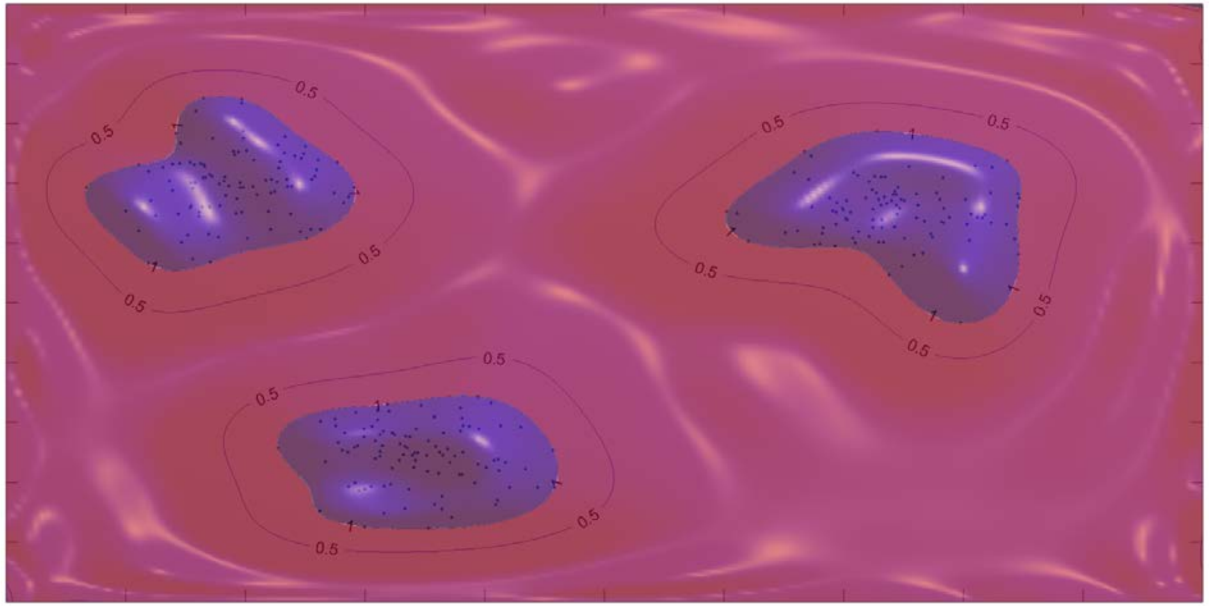
$$\begin{array}{llll} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x) \in \mathbb{R}_d[x]}{\text{minimize}} & \int_{\mathbf{B}} \mathcal{P}_d(x)dx & \xrightarrow{\hspace{3em}} \text{Linear} & \\ \text{subject to} & \mathcal{P}_d(x) - 1 \geq 0 \quad \forall x_i, i = 1, \ldots, N & \xrightarrow{\text{Linear}} & \mathcal{P}_d(x_i) - 1 \geq 0, \ i = 1, \ldots, N \\ & \mathcal{P}_d(x) \geq 0 & \xrightarrow{\text{LMI}} & \text{SOS}(\mathcal{P}_d(x)) \end{array}$$

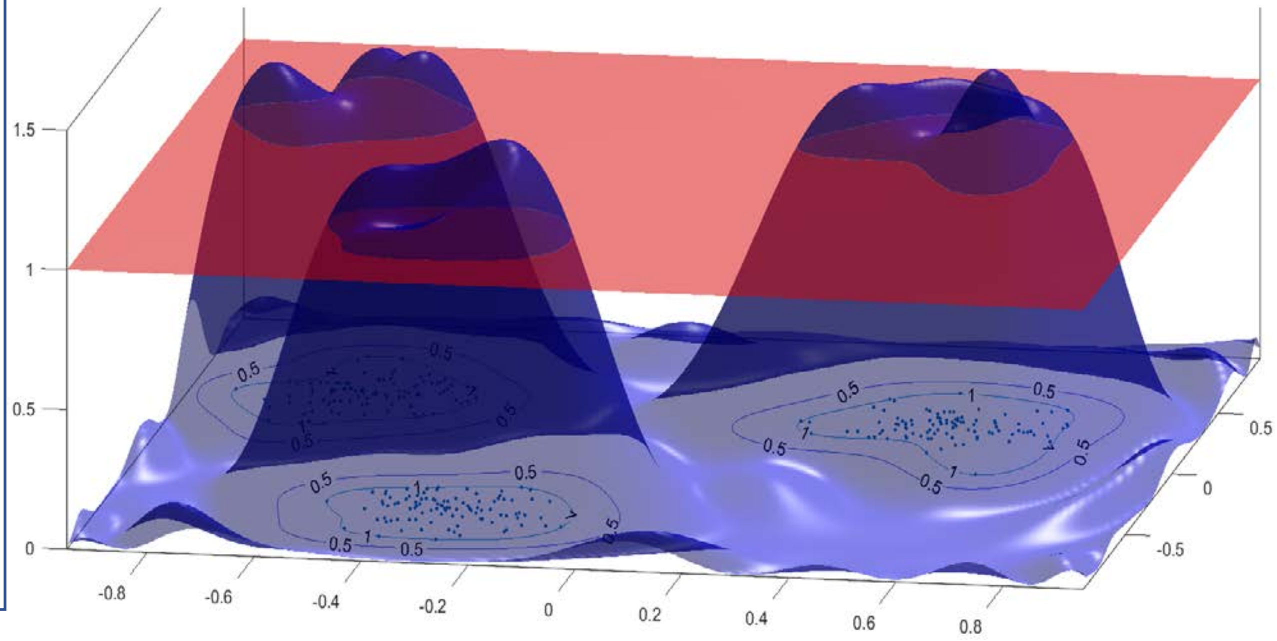
Semialgebraic set representation of data: $\chi = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$

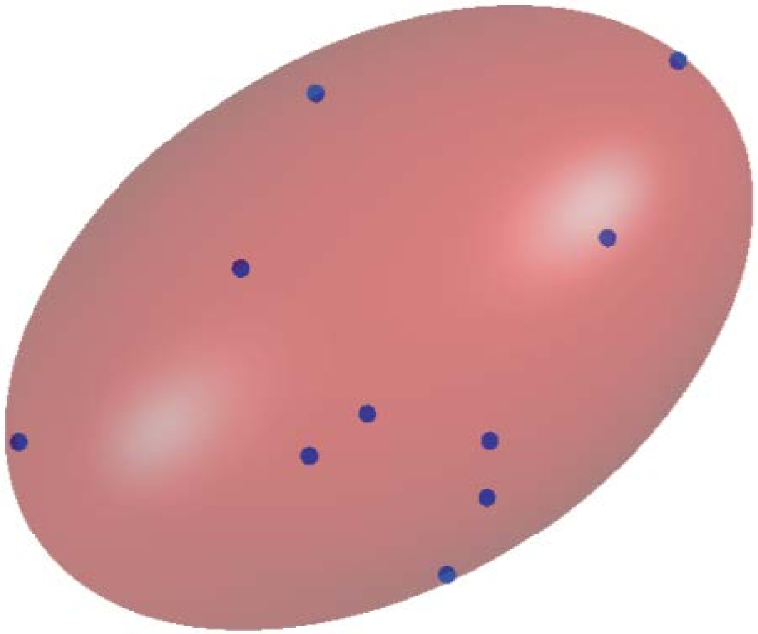


$$\chi = \{x \in \mathbb{R}^n : \mathcal{P}_{d=20}(x) \geq 1\}$$

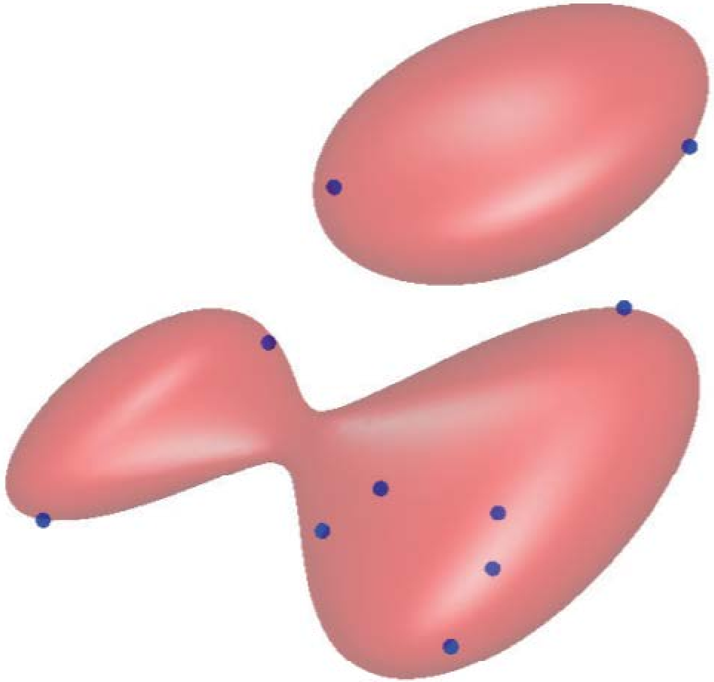
https://github.com/jasour/rarnop19/tree/master/Lecture11_Probabilistic_Nonlinear_Control/Data_Polynomial_Representation





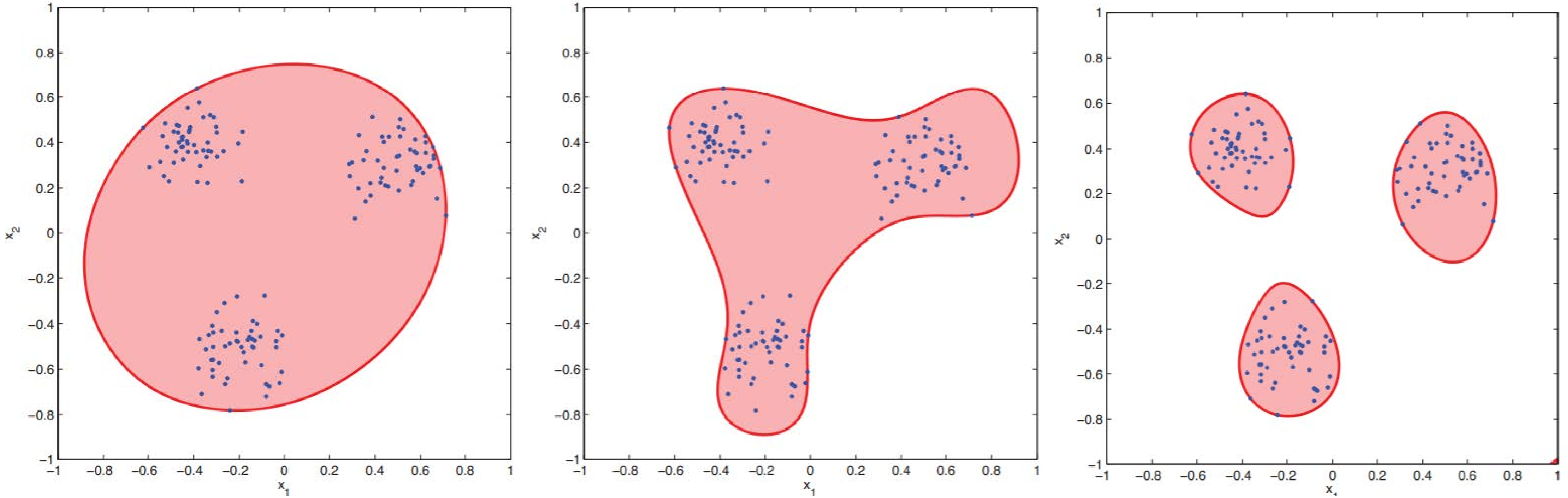


$$\chi = \{x \in \mathbb{R}^n : \mathcal{P}_{d=4}(x) \geq 1\}$$



$$\chi = \{x \in \mathbb{R}^n : \mathcal{P}_{d=10}(x) \geq 1\}$$

- F. Dabbene, D. Henrion, C. M.Lagoa "Simple approximations of semialgebraic sets and their applications to control", Automatica Volume 78, pp. 110-118, 2017.



$$\chi = \{x \in \mathbb{R}^n : \mathcal{P}_{d=2}(x) \geq 1\} \qquad \chi = \{x \in \mathbb{R}^n : \mathcal{P}_{d=5}(x) \geq 1\} \qquad \chi = \{x \in \mathbb{R}^n : \mathcal{P}_{d=9}(x) \geq 1\}$$

- F. Dabbene, D. Henrion, C. M.Lagoa “Simple approximations of semialgebraic sets and their applications to control”, Automatica Volume 78, pp. 110-118, 2017.

Convex Set Representation

$$\begin{aligned} \underset{\chi \subset \mathbb{R}^n}{\text{minimize}} \quad & \text{vol}(\chi) \\ \text{subject to} \quad & x_i \in \chi, \;\; i = 1, \ldots, N \\ & \chi : \text{convex set} \end{aligned}$$

Set representation of data: Level sets of polynomial $\mathcal{P}_d(x)$

➢ Second order convexity condition: *Hessian matrix* $\nabla^2 \mathcal{P}_d(x) \geq 0$

$$\nabla^2 \mathcal{P}_d(x) = \underbrace{\begin{bmatrix} \frac{\partial^2 \mathcal{P}_d}{\partial x_1^2} & \frac{\partial^2 \mathcal{P}_d}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 \mathcal{P}_d}{\partial x_1 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 \mathcal{P}_d}{\partial x_n \partial x_1} & \frac{\partial^2 \mathcal{P}_d}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 \mathcal{P}_d}{\partial x_n^2} \end{bmatrix}}_{\text{Polynomial matrix}}$$

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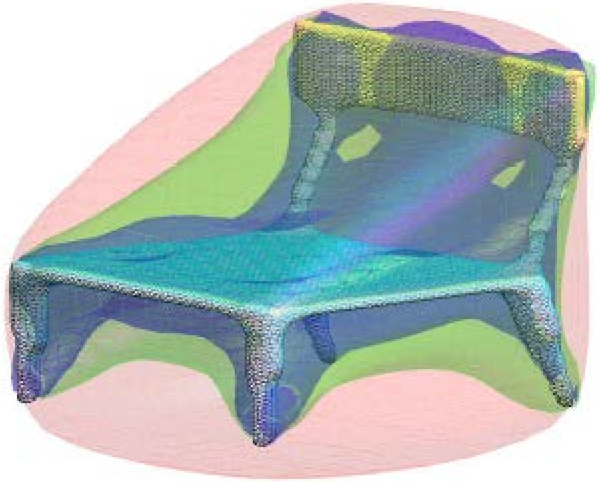
- From the definition of the PSD matrix

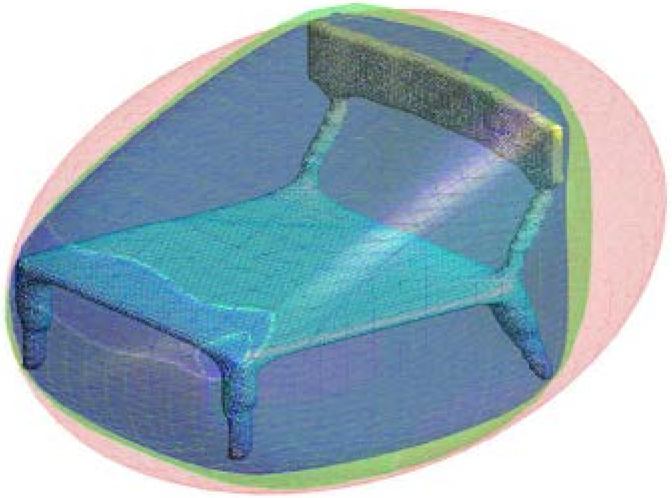
Polynomial: $\underbrace{y^T \nabla^2 \mathcal{P}_d(x)}_{\text{matrix}} \underbrace{y}_{\text{vector}} \geq 0 \;\; \forall \text{vector } y \longrightarrow \text{SOS}(y^T \nabla^2 \mathcal{P}_d(x) y)$

SOS Polynomial in x and y

$\chi : \text{convex set} \longrightarrow \text{SOS}(y^T \nabla^2 \mathcal{P}_d(x) y)$ SOS-Convexity Condition



Level sets of **SOS** polynomials of increasing degree



Level sets of **SOS-Convex** polynomials of increasing degree

- A. A. Ahmadi, G. Hall, A. Makadia, and V. Sindhwani, "Sum of Squares Polynomials and Geometry of 3D Environments" Robotics: Science and Systems, 2017.

Polynomial Representation of Dynamical Systems

1) Taylor expansion

Taylor expand the dynamics about the point (equilibrium point, way-point).

- Taylor expansion of function $f(x)$ at point a:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

1) Taylor expansion

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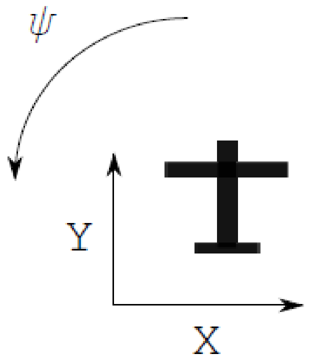
$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Example: to describe motion we need "Trigonometric functions".

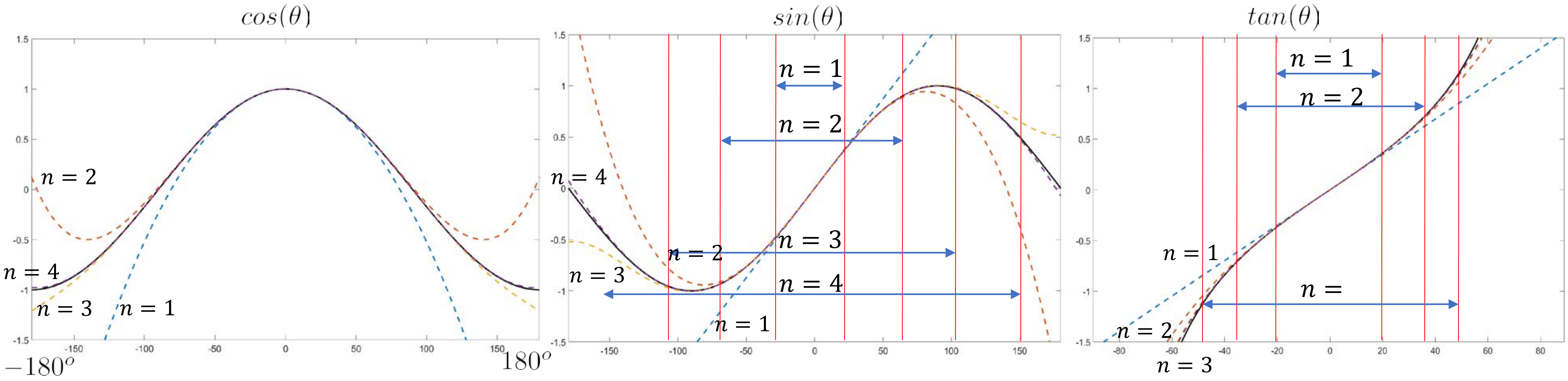


$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}, \qquad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -v \sin\psi + w \\ v \cos\psi \\ u \end{bmatrix}$$

- Polynomial dynamics: Taylor expansion of trigonometric functions to degree 3 at point $\psi = 0$

A. Ahmadi, A. Majumdary, "Some applications of polynomial optimization in operations research and real-time decision making", Optimization Letters, Volume 10, Issue 4, pp 709–729, 2016.

Trigonometric functions - Taylor expansion of order n



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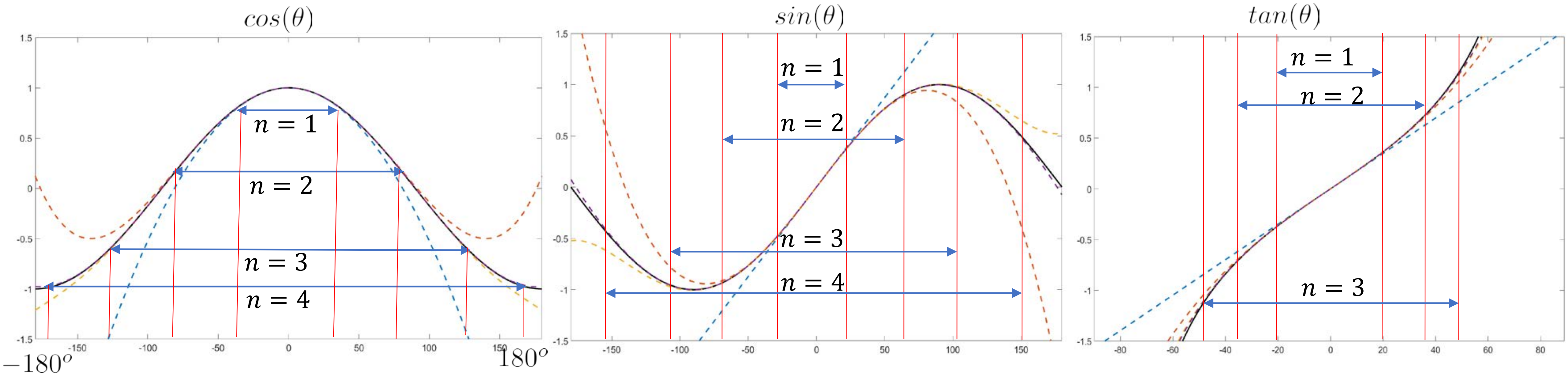
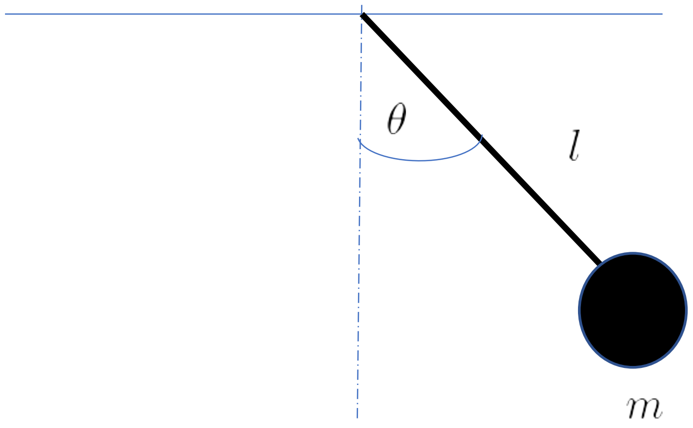


2) New state variables

Example 1:

States: $x = [\theta, \dot{\theta}]^T$

Dynamics: $\dot{x}_2 = \frac{1}{ml^2}(-mglsinx_1 - bx_2)$



$$ml^2\ddot{\theta} + mglsin\theta = -b\dot{\theta}$$

Define: $s \equiv sin\theta$ $c \equiv cos\theta$

States: $x = [s, c, \dot{\theta}]^T$

Polynomial Dynamics:

$$\dot{s} = c\dot{\theta}$$
$$\dot{c} = -s\dot{\theta}$$
$$\ddot{\theta} = \frac{1}{ml^2}(-mgls - b\dot{\theta})$$

$\Rightarrow$

$$\dot{x_1} = x_2x_3$$
$$\dot{x_2} = -x_1x_3$$
$$\dot{x_3} = \frac{1}{ml^2}(-mglx_1 - bx_3)$$

Chapter 10: Underactuated Robotics, Algorithms for Walking, Running, Swimming, Flying, and Manipulation, Russ Tedrake.

3) Change of Coordinates

Dubin's Car Model:

$$\dot{x} = v cos(\theta)$$
$$\dot{y} = v sin(\theta)$$
$$\dot{\theta} = \omega$$

Coordinate and input transform

$$z_1 = \theta \qquad z_2 = x cos(\theta) + y sin(\theta) \qquad z_3 = x sin(\theta) - y cos(\theta)$$
$$u_1 = \omega \qquad u_2 = v - z_3 u_1$$

Polynomial model:

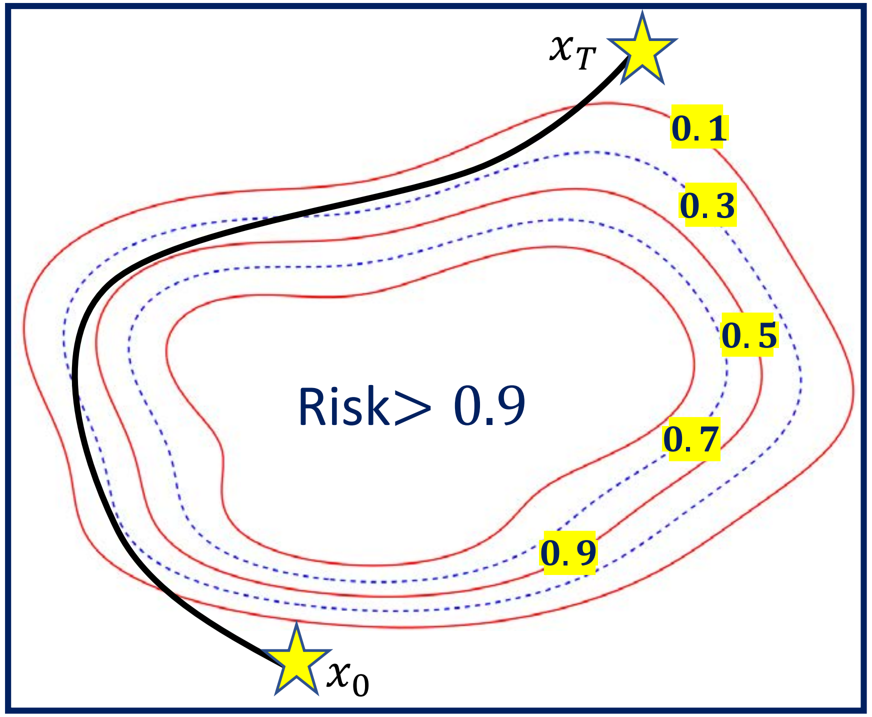
$$\dot{z}_1 = u_1$$
$$\dot{z}_2 = u_2$$
$$\dot{z}_3 = z_2 u_1$$

- D. DeVon, T. Bretl, "Kinematic and dynamic control of a wheeled mobile robot", IEEE International Conference on Intelligent Robots and Systems, 2007.

Topics:

- ➢ Introduction
- ➢ Polynomial Representation of Obstacles and Dynamical Systems
- ➢ Risk Bounded Trajectory Planning in Uncertain Environments
- ➢ Control of Probabilistic Nonlinear Systems
  - Nonlinear State Feedback Control
  - Receding Horizon Control
- ➢ Flow-Tube based Control of Probabilistic Nonlinear Systems
- ➢ Chance Constrained Backward Reachability Set
- ➢ Continuous-Time Path Planning

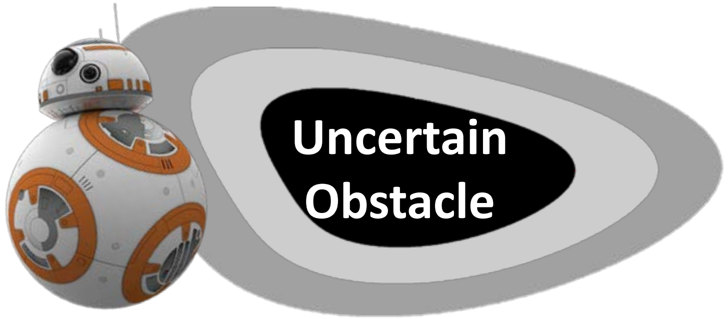
Risk Bounded Trajectory Planning in Uncertain Environments



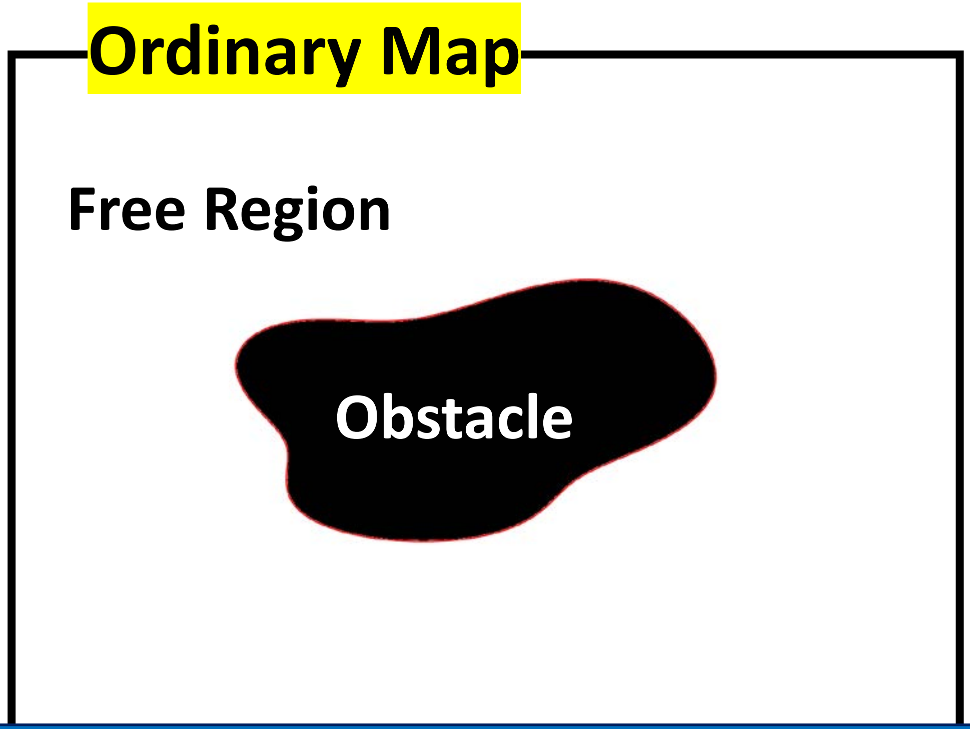
Goal: Risk Bounded Trajectory Planning in presence of perception uncertainties

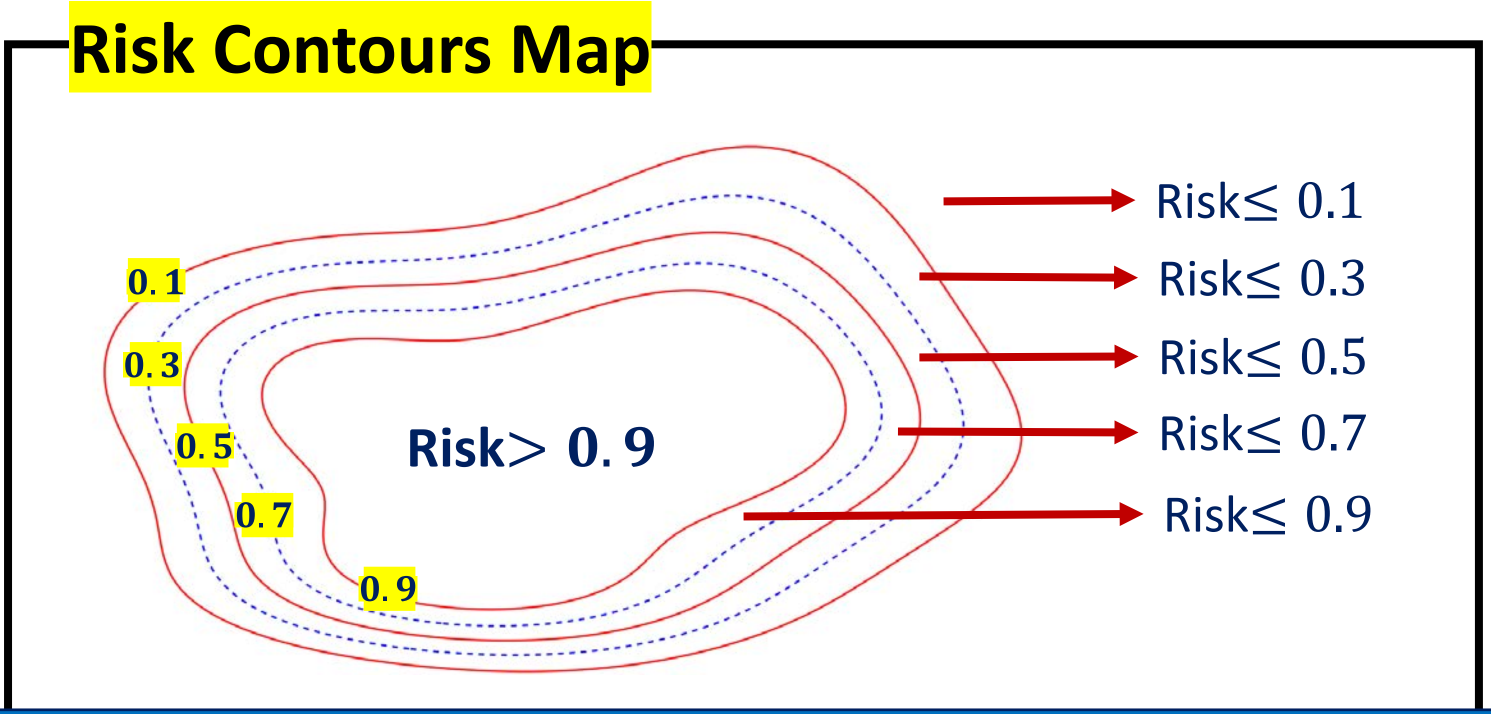
Perception Uncertainties:

Probabilistic uncertainties in location, size, and geometry of obstacles



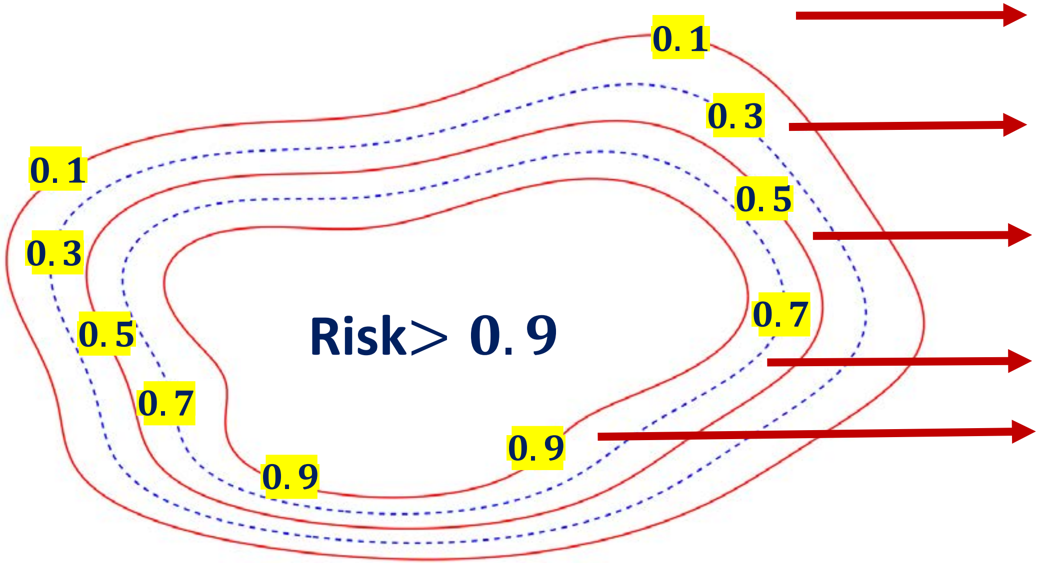
Risk: Probability of collision of robot with obstacles in presence of probabilistic uncertainties.





Δ-risk contour: $\mathcal{C}_r^{\Delta} =$ {All points whose "risk" is less than or equal to **Δ**}

Risk Contours Map



Δ-risk contours

$\mathcal{C}_r^{\Delta=0.1}$: **Risk≤ 0.1**

$\mathcal{C}_r^{\Delta=0.3}$: **Risk≤ 0.3**

$\mathcal{C}_r^{\Delta=0.5}$: **Risk≤ 0.5**

$\mathcal{C}_r^{\Delta=0.7}$: **Risk≤ 0.7**

$\mathcal{C}_r^{\Delta=0.9}$: **Risk≤ 0.9**

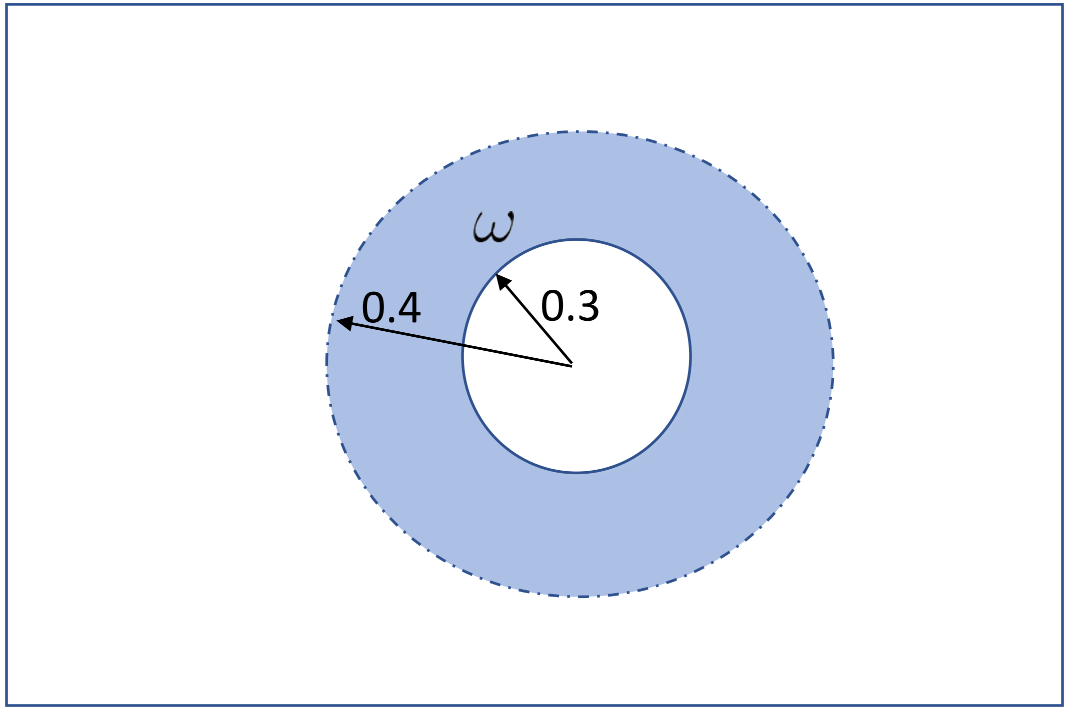
Risk Contours Map: Collection of **Δ-risk contours**   Risk levels: $\Delta_i \in [0,1]$

Illustrative Example:

Circle shaped obstacle with probabilistic size

$$\chi_{obs}(\omega) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - \omega^2 \leq 0\}$$

$$\omega \sim \text{Uniform}[0.3, 0.4]$$



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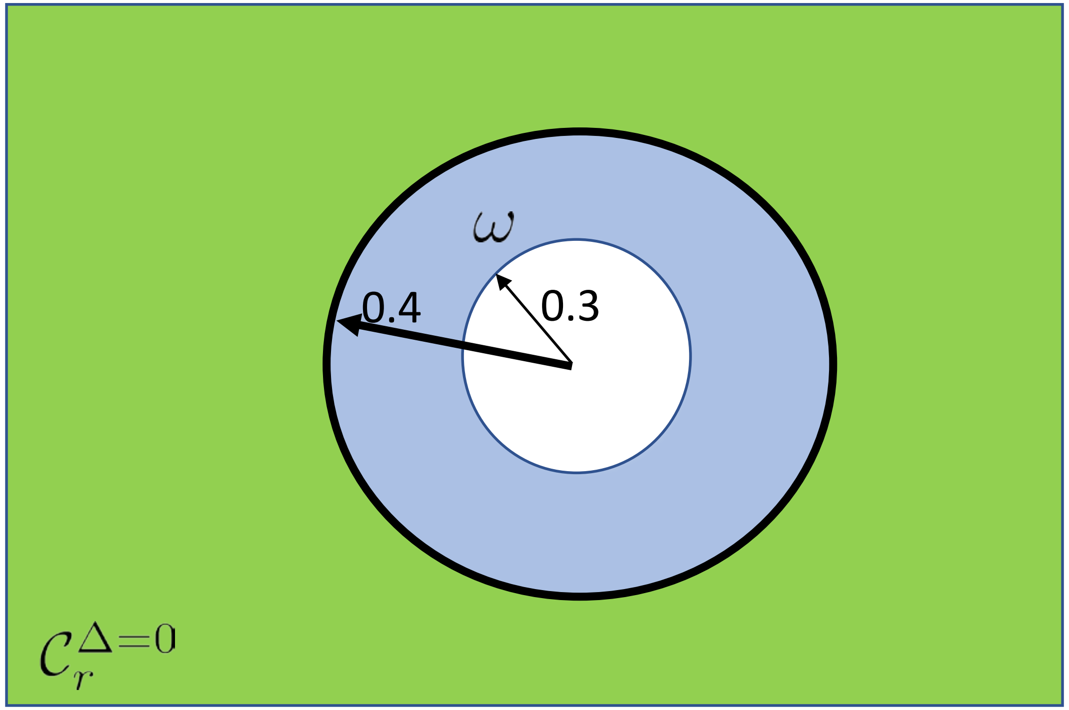
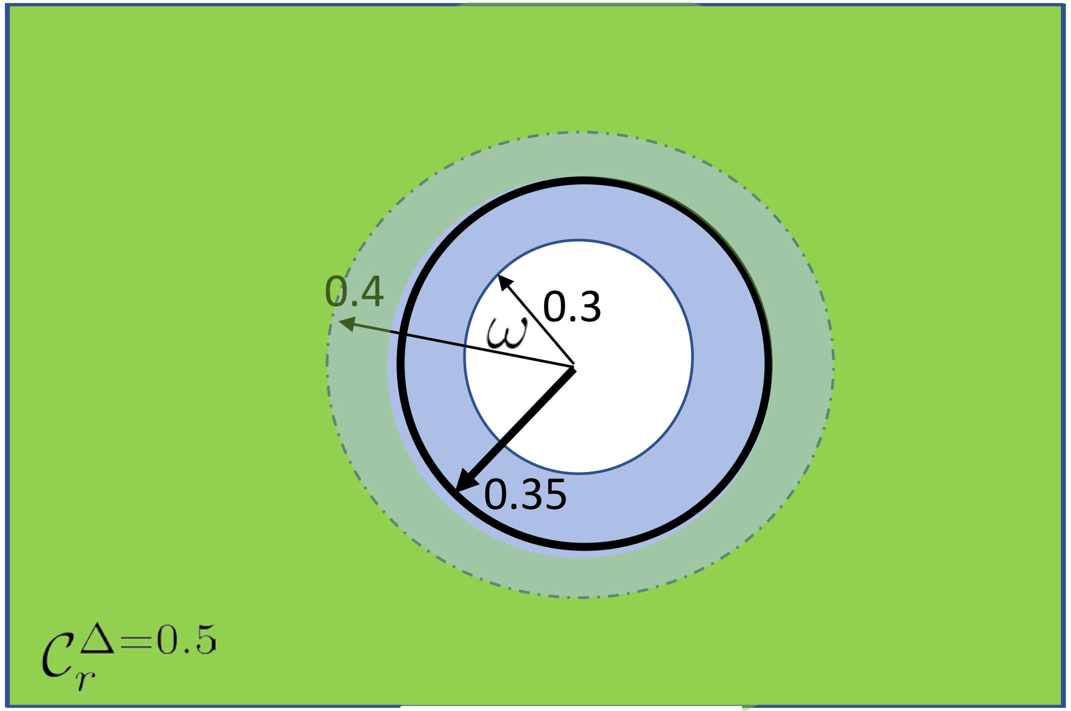


$\mathcal{C}_r^{\Delta=0}$ = {All points whose "risk" is less than or equal to **Δ**} $= \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 0.4^2 \ge 0\}$

$$\text{Probability}(\omega \ge 0.4) = 0$$

Illustrative Example:

Circle shaped obstacle with probabilistic size



$$\chi_{obs}(\omega) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - \omega^2 \leq 0\}$$

$$\omega \sim \text{Uniform}[0.3, 0.4]$$

$\mathcal{C}_r^{\Delta=0}$ = {All points whose "risk" is less than or equal to **Δ**} $= \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 0.4^2 \geq 0\}$

$$\text{Probability}(\omega \geq 0.4) = 0$$

$\mathcal{C}_r^{\Delta=0.5}$ = {All points whose "risk" is less than or equal to **Δ**} $= \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 0.35^2 \geq 0\}$

$$\text{Probability}(\omega \geq 0.35) = 0.5$$

Application: Risk Bounded Motion Planning

Find trajectory $P(t)$ such that:

i) Boundary Conditions: $P(0) = x_0,\ P(T) = x_T$

ii) Chance Constraints:

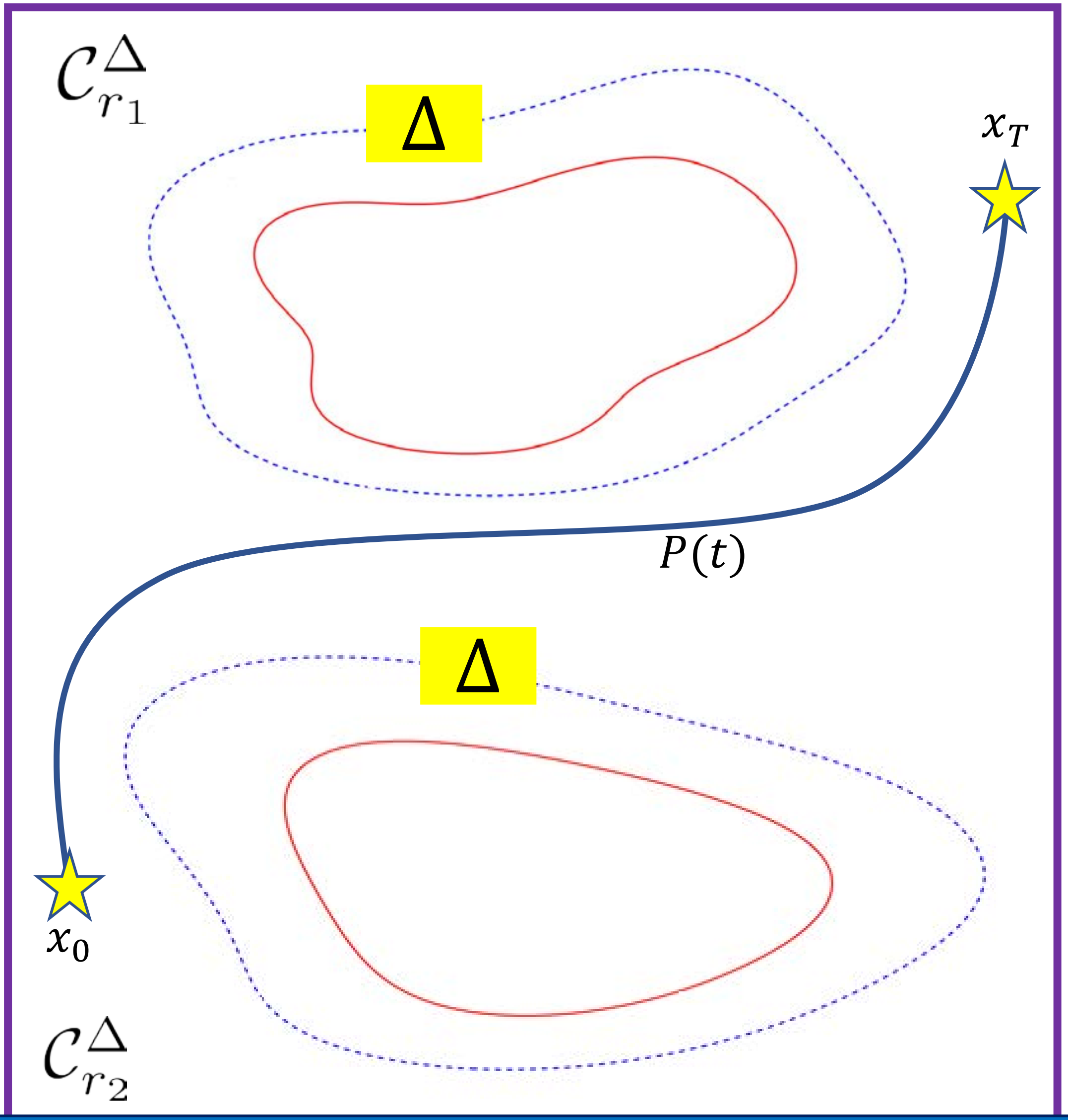
$$\mathrm{Prob}\{P(t) \in \chi_{obs_i}(\omega_i)\} \leq \Delta,\ i = 1, \dots, o,\ \forall t \in [0, T]$$

Uncertain Obstacle Acceptable Risk Level Time

Deterministic Constraints in terms of Δ-risk contours:

$$P(t) \in \{x \in \chi : \cap_{i=1}^{o} \mathcal{C}_{r_i}^{\Delta}\},\ \forall t \in [0, T]$$

Δ-risk contour of obstacle i



Risk Contours Construction

Δ-risk contour: $\mathcal{C}_r^{\Delta}$ $=$ {All points whose "risk" is less than or equal to **Δ**}

$$= \underbrace{\{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta \}}_{\text{Chance Constrained Set}}$$

Main Idea: Polynomial approximation of the probabilistic constraint.

Risk Contours Construction

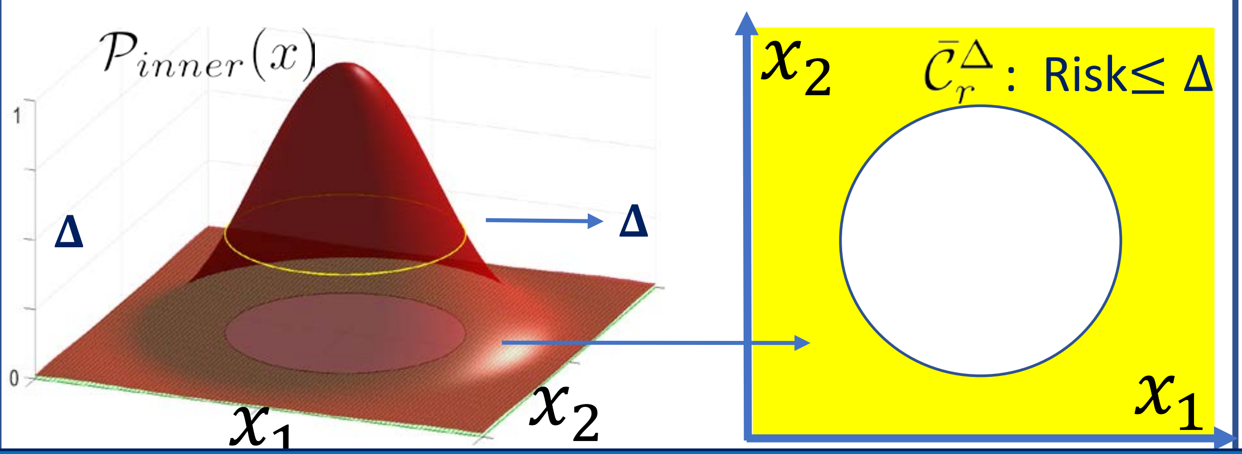
Δ-risk contour: $\mathcal{C}_r^{\Delta} =$ {All points whose "risk" is less than or equal to **Δ**}

$$= \underbrace{\{\, x \in \chi : \ \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta \}}_{\text{Chance Constrained Set}}$$

Main Idea: Polynomial approximation of the probabilistic constraint.

Inner approximation $\bar{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{inner}(x) \leq \Delta\}$

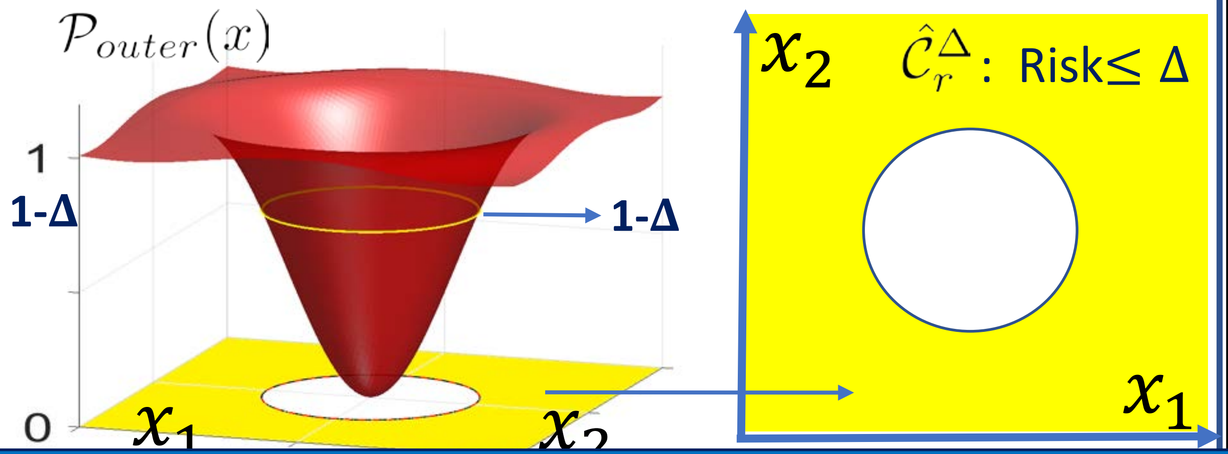
Δ Sublevel set of polynomial $\mathcal{P}_{inner}(x)$



Outer approximation

$\hat{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{outer}(x) \geq 1 - \Delta\}$

(1 − Δ) Superlevel set of polynomial $\mathcal{P}_{outer}(x)$



Outer approximation:

Δ-risk contour: $\mathcal{C}_r^{\Delta}$ $=$ {All points whose "risk" is less than or equal to **Δ**}

$$= \{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$$

Outer approximation

Δ-risk contour: $\mathcal{C}_r^{\Delta} = \{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

$$= \{\, x \in \chi : \ \underbrace{\text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{safe}(\omega))}\geq 1 - \Delta\}$$

$$\Downarrow$$

Outer approximation: **(1 − Δ)** Superlevel set of polynomial $\mathcal{P}_{outer}(x)$

$$\hat{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{outer}(x) \geq 1 - \Delta\}$$

Outer approximation:

Δ-risk contour: $\mathcal{C}_r^{\Delta} = \{\ x \in \chi:\ \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}\ = \{\ x \in \chi:\ \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{safe}(\omega)) \geq 1-\Delta\}$

Sets: $\chi_{obs}(\omega) = \{x \in \mathbb{R}^n : g(x,\omega) \leq 0\}$

Complement set: $\chi_{safe}(\omega) = \chi - \chi_{obs}(\omega)$

$\mathcal{K}_{obs} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ g(x,\omega) \leq 0\ \}$

State space uncertainty space

Complement set: $\mathcal{K}_{safe} = \chi \times \Omega - \mathcal{K}_{obs}$

Outer approximation:

Δ-risk contour: $\mathcal{C}_r^{\Delta} = \{\, x \in \chi : \ \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\} \ = \{\, x \in \chi : \ \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{safe}(\omega)) \geq 1 - \Delta\}$

Sets: $\chi_{obs}(\omega) = \{x \in \mathbb{R}^n : g(x,\omega) \leq 0\}$

Complement set: $\chi_{safe}(\omega) = \chi - \chi_{obs}(\omega)$

$\mathcal{K}_{obs} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \ g(x,\omega) \leq 0 \ \}$

Complement set: $\mathcal{K}_{safe} = \underset{\text{State space}}{\chi} \times \underset{\text{uncertainty space}}{\Omega} - \mathcal{K}_{obs}$

➤ To obtain risk contour set, we need to find polynomial approximation of **indicator function of set** $\mathcal{K}_{safe}$.

$$\begin{aligned} \mathbf{P}^{*d}_{\mathbf{sos}} = \underset{\mathcal{P}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \mathcal{P}(x,\omega)d\mu_\omega dx \\ \text{subject to} \quad & \mathcal{P}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}_{safe} \\ & \mathcal{P}(x,\omega) \geq 0 \end{aligned}$$

polynomial $\mathcal{P}(x,\omega) =$ Upper approximation of indicator function $\mathbf{I}_{\mathcal{K}_{safe}}$

$$\mathbf{I}_{\mathcal{K}_{obs}} = \begin{cases} 1 & \forall (x,\omega) \in \mathcal{K}_{safe}, \\ 0 & \forall (x,\omega) \notin \mathcal{K}_{safe} \end{cases}$$

$$\mathcal{P}_{outer}(x) = \mathrm{E}_{\mu_\omega}[\mathcal{P}(x,\omega)] = \int \mathcal{P}(x,\omega)d\mu_\omega$$

For any $x^* \in \chi$, polynomial $\mathcal{P}_{outer}(x^*)$ is an upper bound on the probability that $x^* \in \chi_{safe}$

Outer approximation : $\hat{\mathcal{C}}_r^{\Delta} = \{x \in \mathbb{R}^n : \mathcal{P}_{outer}(x) \geq 1 - \Delta\}$

Inner approximation:

Δ-risk contour: $$\mathcal{C}_r^{\Delta} = \{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$$

$$\chi = \underbrace{\{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}}_{\mathcal{C}_r^{\Delta}} + \{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) > \Delta\}$$

- In the previous slide, we obtained the outer approximation of the set $\{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{safe}(\omega)) \geq 1 - \Delta\}$
- We can apply the same methodology to obtain the outer approximation of the set $\{\, x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) > \Delta\}$

Inner approximation:

Δ-risk contour: $\mathcal{C}_r^{\Delta} = \{\ x \in \chi:\ \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

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Inner approximation $\Downarrow$

$$\bar{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{inner}(x) \leq \Delta\}$$

Outer approximation $\Downarrow$

$$\{x \in \chi : \mathcal{P}_{inner}(x) \geq \Delta\}$$

Inner approximation:

Δ-risk contour: $\mathcal{C}_r^\Delta = \{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

$$\chi = \underbrace{\{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}}_{\mathcal{C}_r^\Delta} + \{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) > \Delta\}$$

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Inner approximation $\Downarrow$

$$\bar{\mathcal{C}}_r^\Delta = \{x \in \chi : \mathcal{P}_{inner}(x) \leq \Delta\}$$

Outer approximation $\Downarrow$

$$\{x \in \chi : \mathcal{P}_{inner}(x) \geq \Delta\}$$

Δ Sublevel set of polynomial $\mathcal{P}_{inner}(x)$

Inner approximation: $\bar{\mathcal{C}}_r^\Delta = \{x \in \chi : \mathcal{P}_{inner}(x) \leq \Delta\}$

Outer approximation:

Δ-risk contour: $\mathcal{C}_r^{\Delta} = \{\ x \in \chi :\ \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

Sets: $\chi_{obs}(\omega) = \{x \in \mathbb{R}^n : g(x,\omega) \leq 0\}$

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Outer approximation:

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Outer approximation of the set $\{\ x \in \chi :\ \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) > \Delta\}$

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{P}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \bar{\mathcal{P}}(x,\omega)d\mu_\omega dx \\ \text{subject to} \quad & \bar{\mathcal{P}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}_{obs} \\ & \bar{\mathcal{P}}(x,\omega) \geq 0 \end{aligned}$$

$\Rightarrow$ polynomial $\bar{\mathcal{P}}(x,\omega) =$ Upper approximation of indicator function $\mathbf{I}_{\mathcal{K}_{obs}}$

$$\mathcal{P}_{inner}(x) = \mathrm{E}_{\mu_\omega}[\mathcal{P}(x,\omega)] = \int \bar{\mathcal{P}}(x,\omega)d\mu_\omega$$

Outer approximation:

Δ-risk contour: $\mathcal{C}_r^{\Delta} = \{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

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Outer approximation of the set $\{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) > \Delta\}$

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Outer approximation:

Δ-risk contour: $\mathcal{C}_r^{\Delta} = \{\ x \in \chi:\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

Sets: $\chi_{obs}(\omega) = \{x \in \mathbb{R}^n : g(x,\omega) \leq 0\}$ $\quad$ $\mathcal{K}_{obs} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ g(x,\omega) \leq 0\ \}$

Outer approximation of the set $\{\ x \in \chi:\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) > \Delta\}$

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Inner approximation : $\bar{\mathcal{C}}_r^{\Delta} = \{x \in \mathbb{R}^n : \mathcal{P}_{inner}(x) \leq \Delta\}$

Uncertain Obstacles $\chi_{obs}(\omega) = \{x \in \mathbb{R}^n : g(x,\omega) \leq 0\}$

Δ-risk contour $\mathcal{C}_r^{\Delta} = \{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

Inner approximation :

- Define set $\mathcal{K}_{obs} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ g(x,\omega) \leq 0\ \}$
- SOS Program ($\geq \rightarrow SOS$)

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{P}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \bar{\mathcal{P}}(x,\omega) d\mu_\omega dx \\ \text{subject to} \quad & \bar{\mathcal{P}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}_{obs} \\ & \bar{\mathcal{P}}(x,\omega) \geq 0 \end{aligned}$$

- Polynomial

$$\mathcal{P}_{inner}(x) = \text{E}_{\mu_\omega}[\mathcal{P}(x,\omega)] = \int \bar{\mathcal{P}}(x,\omega) d\mu_\omega$$

Inner approximation :

$$\bar{\mathcal{C}}_r^{\Delta} = \{x \in \mathbb{R}^n : \mathcal{P}_{inner}(x) \leq \Delta\}$$

Outer approximation :

- Define set $\mathcal{K}_{safe} = \chi \times \Omega - \mathcal{K}_{obs}$
- SOS Program ($\geq \rightarrow SOS$)

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \mathcal{P}(x,\omega) d\mu_\omega dx \\ \text{subject to} \quad & \mathcal{P}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}_{safe} \\ & \mathcal{P}(x,\omega) \geq 0 \end{aligned}$$

- Polynomial

$$\mathcal{P}_{outer}(x) = \text{E}_{\mu_\omega}[\mathcal{P}(x,\omega)] = \int \mathcal{P}(x,\omega) d\mu_\omega$$

Outer approximation :

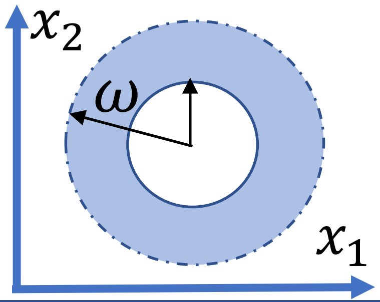
$$\hat{\mathcal{C}}_r^{\Delta} = \{x \in \mathbb{R}^n : \mathcal{P}_{outer}(x) \geq 1 - \Delta\}$$

- A. Jasour, Brian C. Williams, "Risk Contours Map for Risk Bounded Motion Planning under Perception Uncertainties", Robotics: Science and System (RSS), Germany, 2019.

Example 1: Uncertain Obstacle

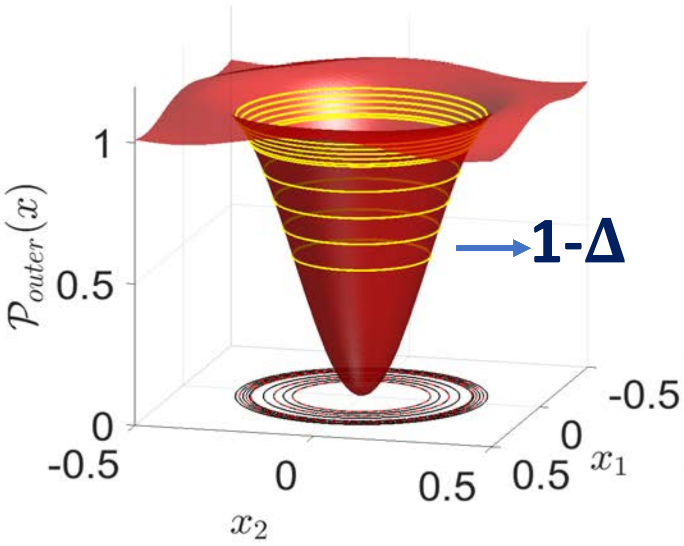
$$\chi_{obs}(\omega) = \{(x_1, x_2) : x_1^2 + x_2^2 \leq \omega^2 \}$$

$\omega \sim$ Uniform $[0.3, 0.4]$

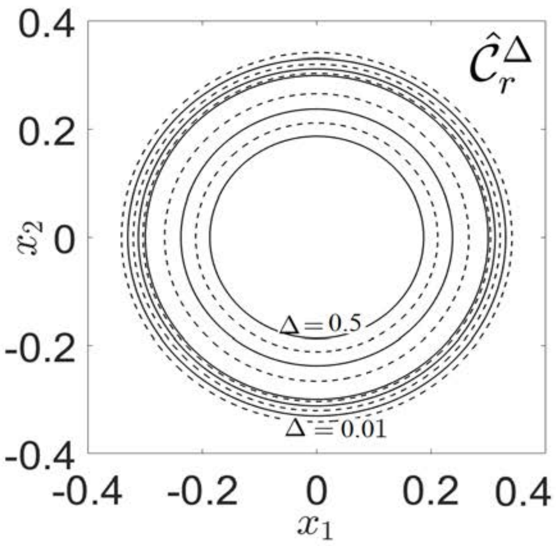


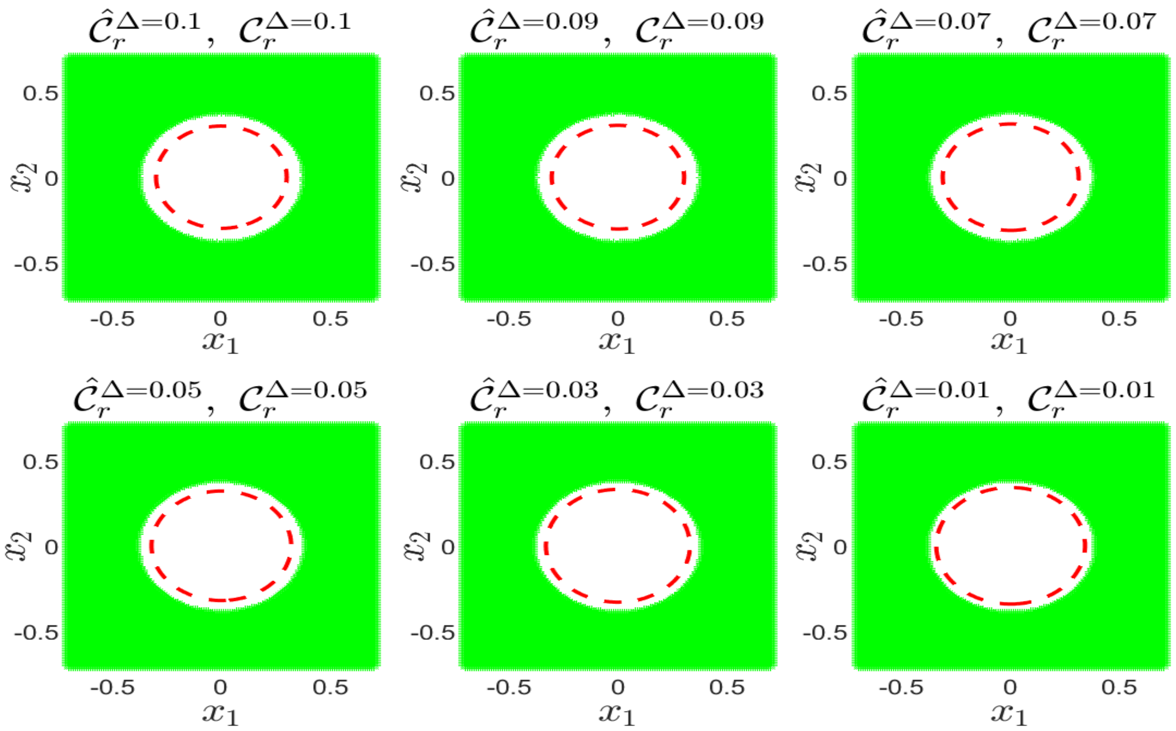
Outer Approximation of Δ-risk contour $\quad \mathcal{C}_r^{\Delta} = \{ \, x \in \chi : \; \mathrm{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

- Polynomial order $d = 20$



Risk Contours Map





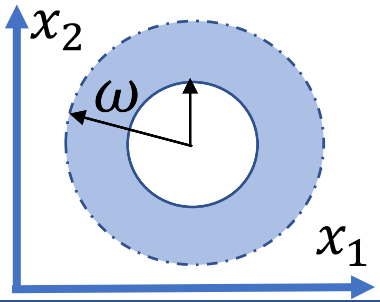
$$\hat{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{outer}(x) \geq 1 - \Delta\}$$

- Outer approximation: Outside of the dashed line
- True *Δ*-risk contour: green set

Example 1: Uncertain Obstacle

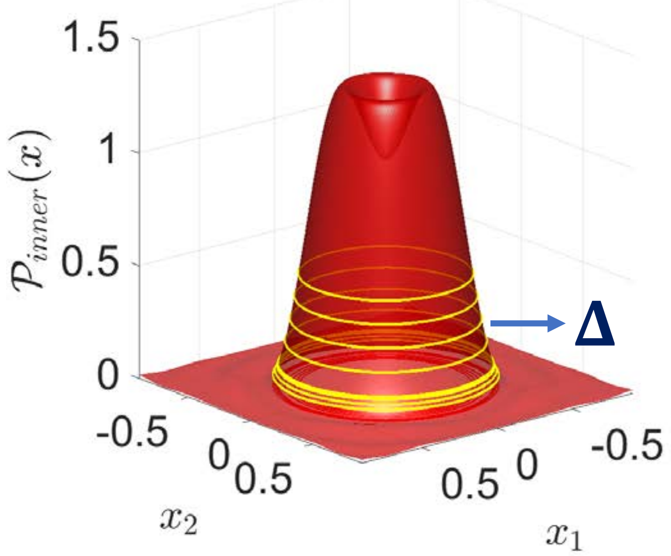
$$\chi_{obs}(\omega) = \{(x_1, x_2) : x_1^2 + x_2^2 \le \omega^2 \}$$

$\omega \sim$ Uniform $[0.3, 0.4]$

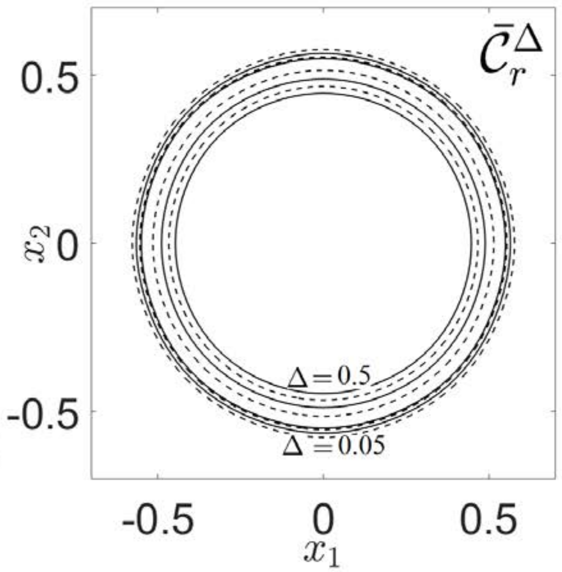


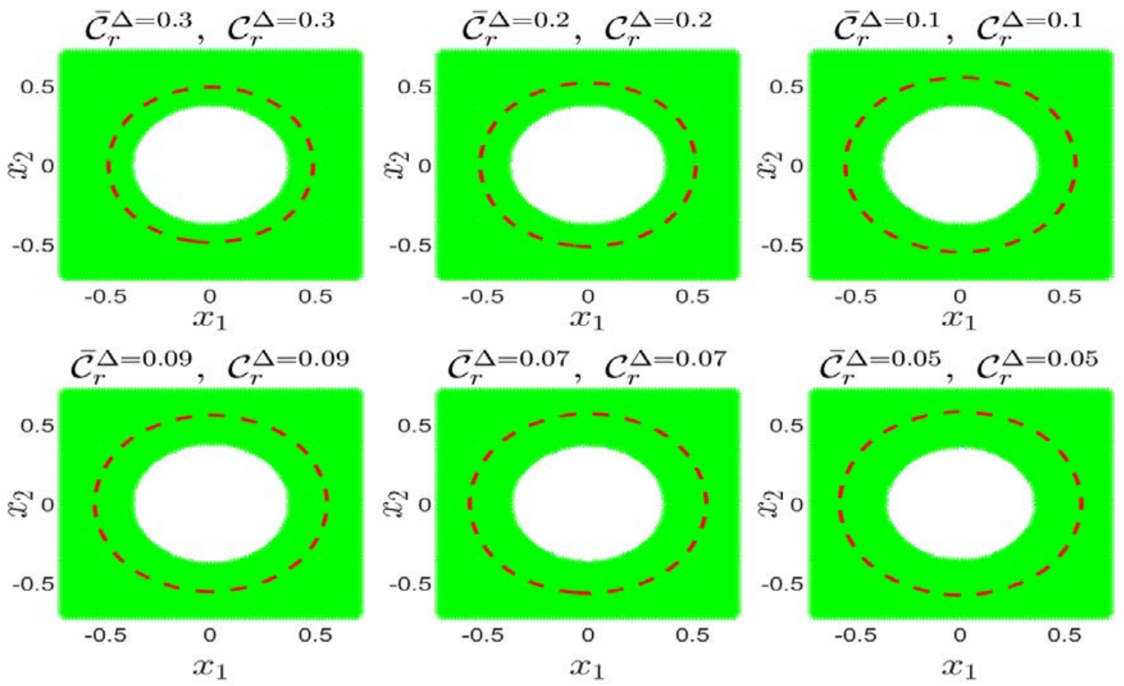
Inner Approximation of Δ-risk contour $\mathcal{C}_r^{\Delta} = \{ x \in \chi : \ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \le \Delta\}$

- Polynomial order $d = 20$



Risk Contours Map





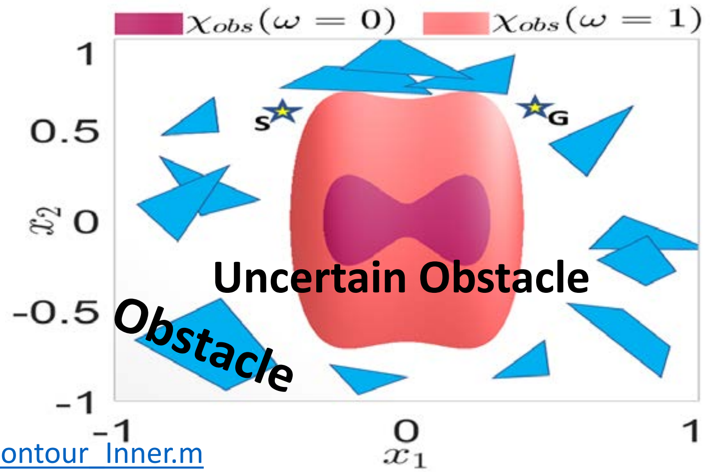
$$\bar{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{inner}(x) \le \Delta\}$$

- Inner approximation: Outside of the dashed line
- True *Δ*-risk contour: green set

Example 2: Uncertain Unsafe Region

$$\chi_{obs}(\omega) = \{(x_1, x_2) \in \chi : -39.0625x_1^4 + 3.125x_1^2 - 2.25x_2^2 + 0.01 + 0.5\omega \leq 0\}$$

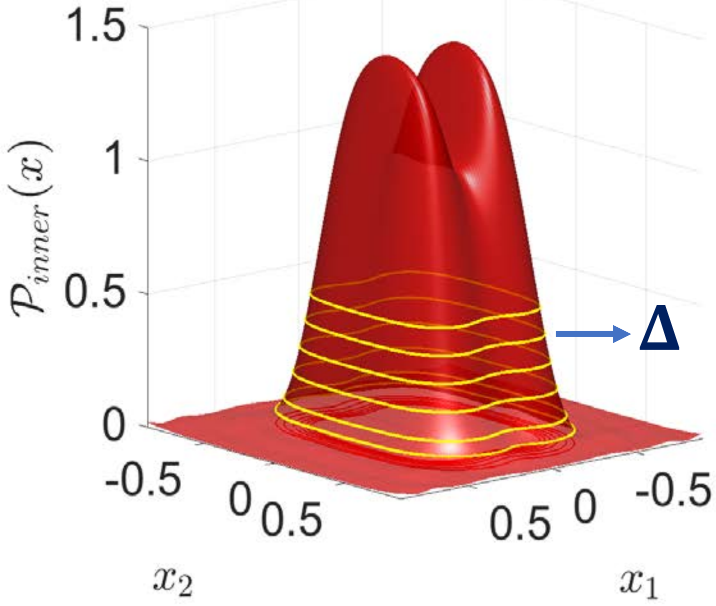
$\omega \in [0\ 1] \sim$ Beta $(1.1,5)$

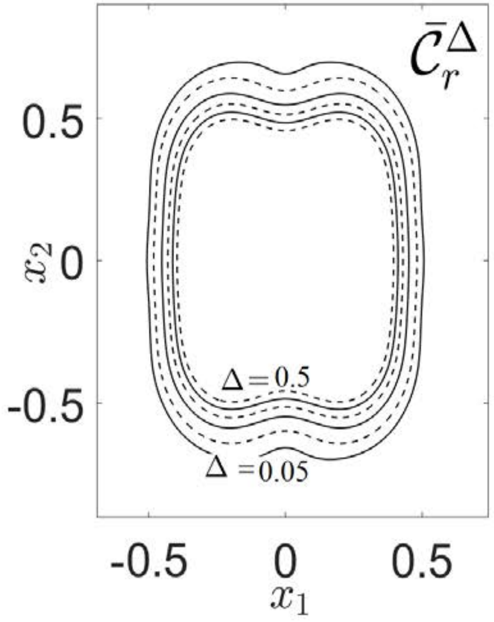


https://github.com/jasour/rarnop19/blob/master/Lecture11_Probabilistic_Nonlinear_Control/Risk_Contours_Map/Example_1_RiskContour_Inner.m

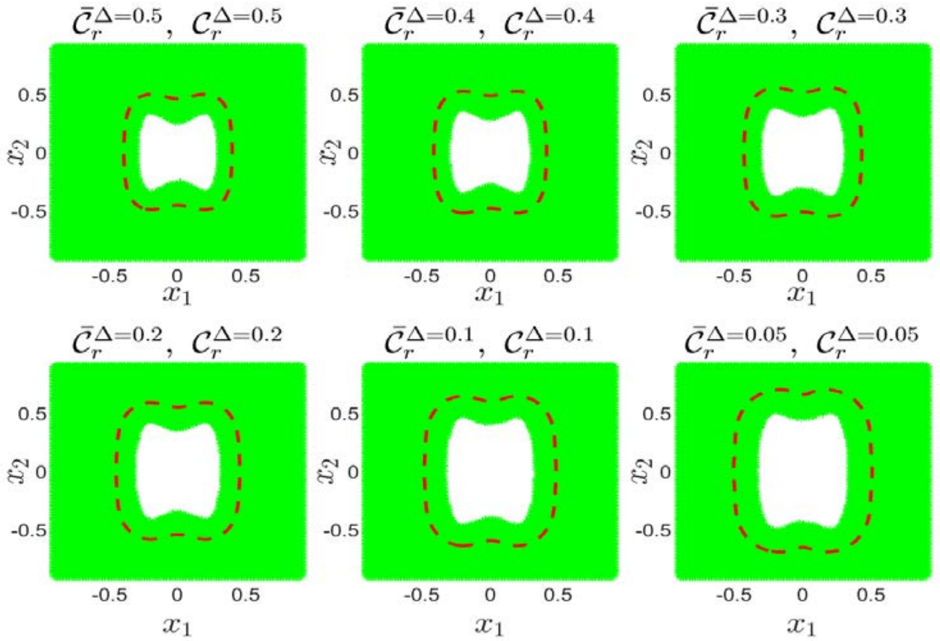
Inner Approximation of Δ-risk contour

$$\mathcal{C}_r^{\Delta} = \{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$$





$$\bar{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{inner}(x) \leq \Delta\}$$

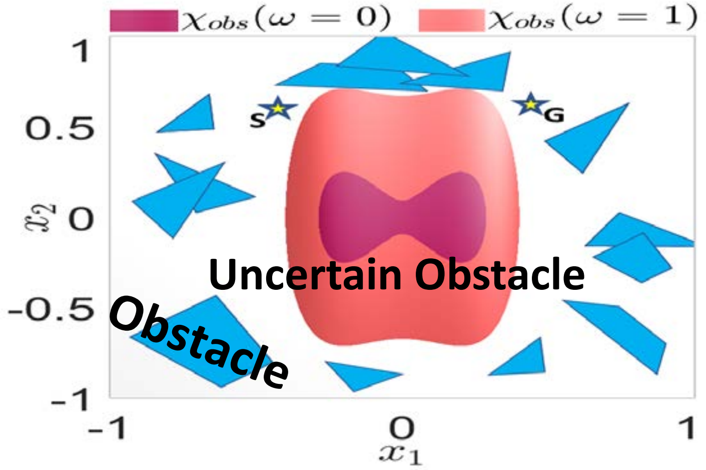


- Inner approximation: Outside of the dashed line
- True *Δ*-risk contour: green set

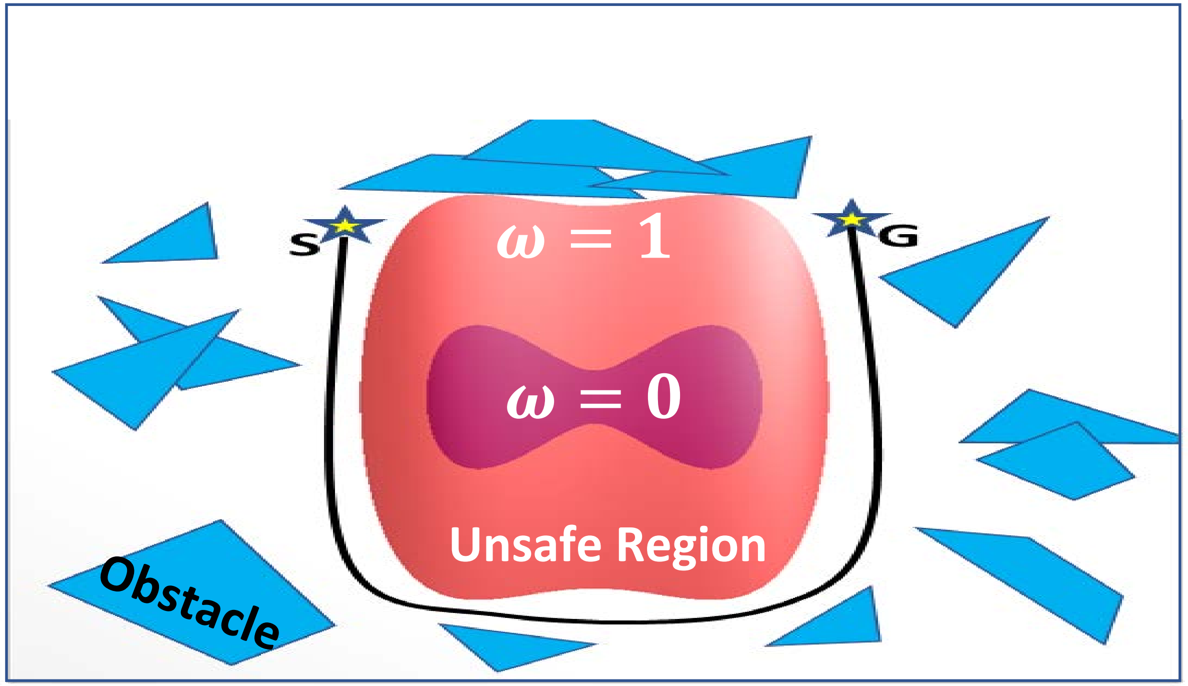
Example 2: Uncertain Unsafe Region

$$\chi_{obs}(\omega) = \{(x_1, x_2) \in \chi : -39.0625x_1^4 + 3.125x_1^2 - 2.25x_2^2 + 0.01 + 0.5\omega \leq 0\}$$

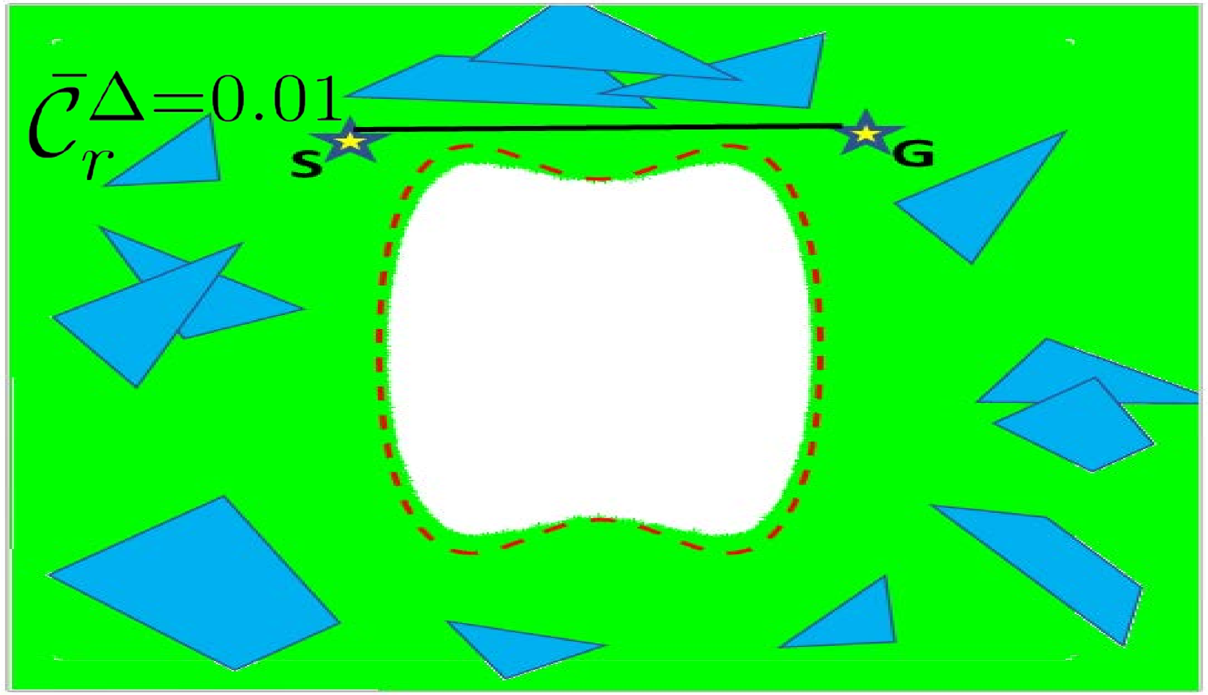
$$\omega \in [0\ 1] \sim \text{Beta}\ (1.1,5)$$



Robust Planning:



Risk Bounded Planning:

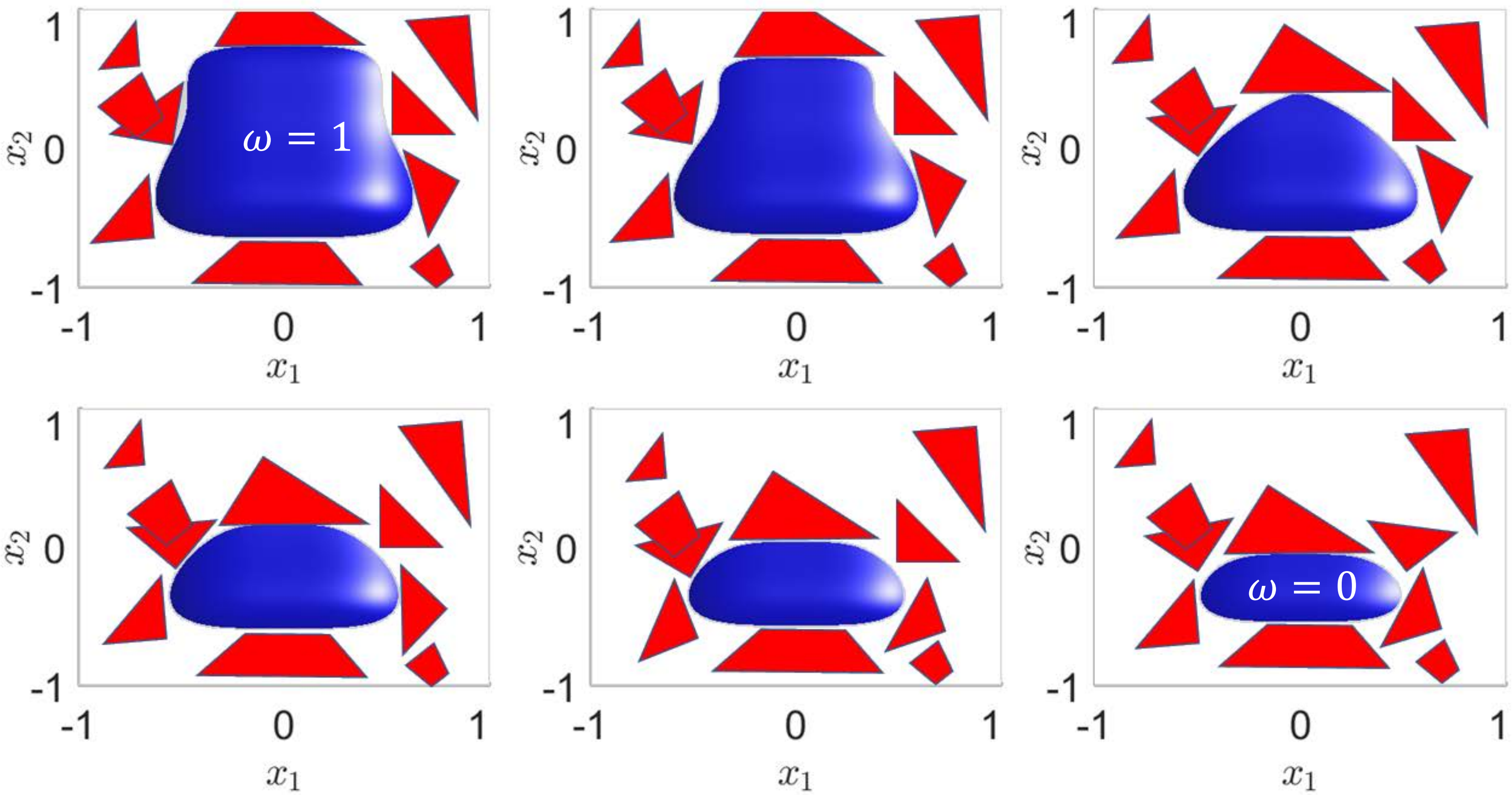


Example 3: Uncertain Safe Region Between Moving Obstacles

$$\chi_{safe}(\omega) = \{(x_1, x_2) \in \chi : -(x_1^4 + (x_2 - 0.4)^4 + (x_2 - 0.4)^3 - 0.1(\omega - 0.5)) \geq 0\}$$

$\omega \in [0\ 1] \sim$ Triangular probability

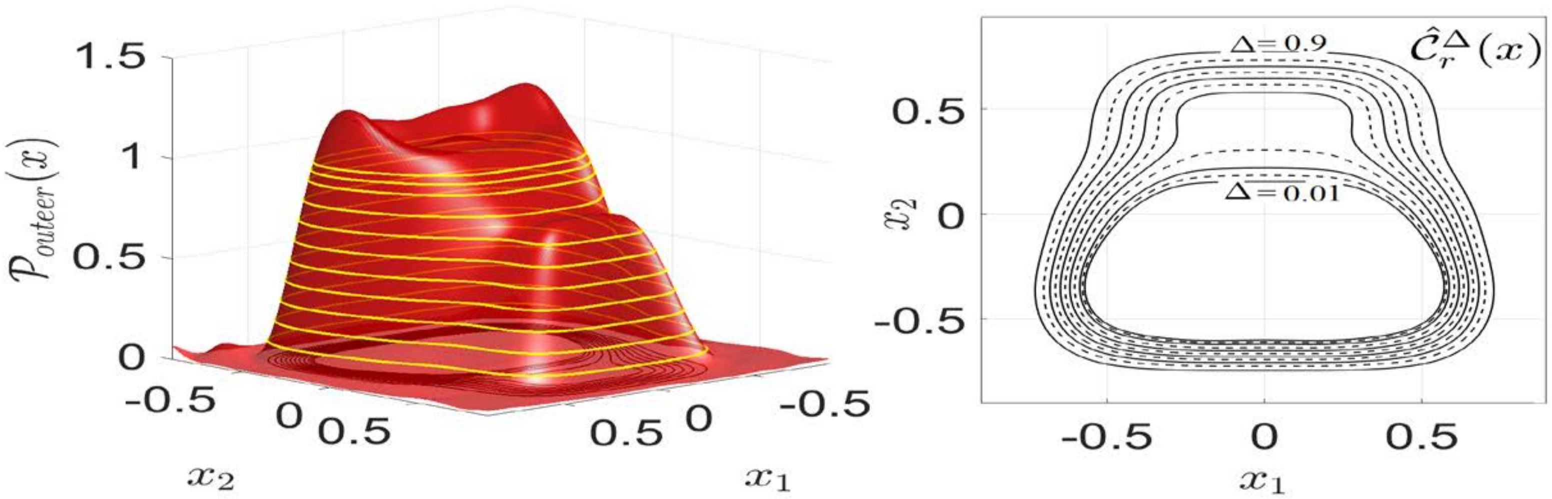
Moving Obstacles (red)   Safe region(blue)



Δ-risk contour $\mathcal{C}_r^{\Delta} = \{\ x \in \chi :\ \text{Prob}_{\mu_\omega(\omega)}(x \in \chi_{obs}(\omega)) \leq \Delta\}$

Outer Approximation of Δ-risk contour

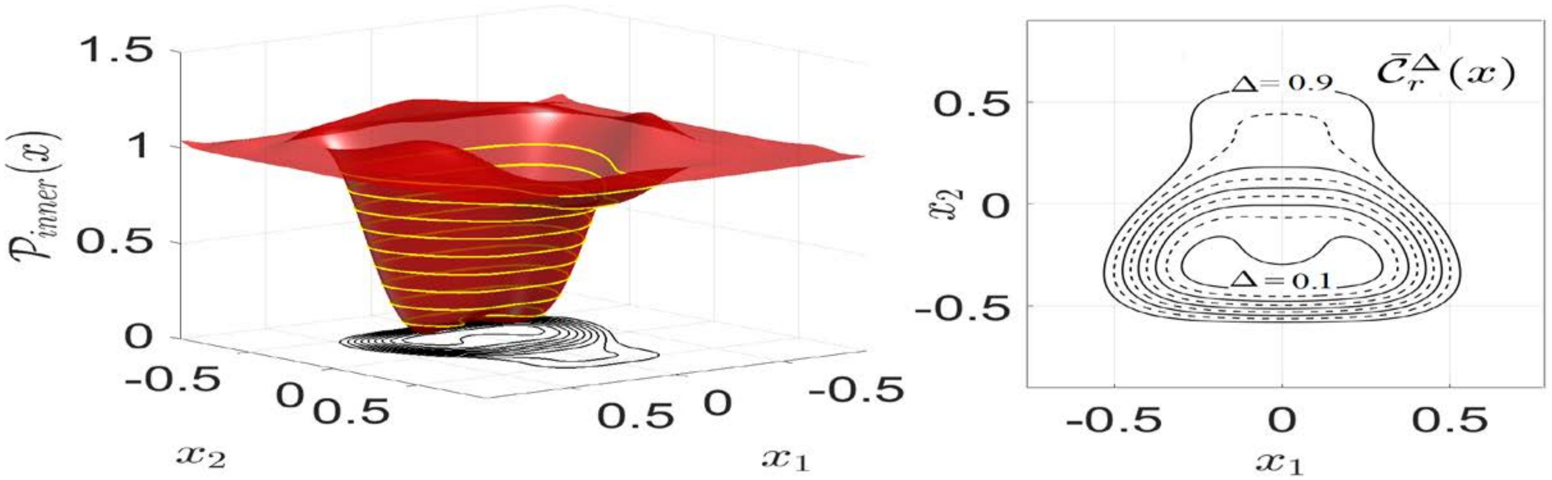
$$\hat{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{outer}(x) \geq 1 - \Delta\}$$



- Inside of the contours, risk is less or equal Δ

Inner Approximation of Δ-risk contour

$$\bar{\mathcal{C}}_r^{\Delta} = \{x \in \chi : \mathcal{P}_{inner}(x) \leq \Delta\}$$

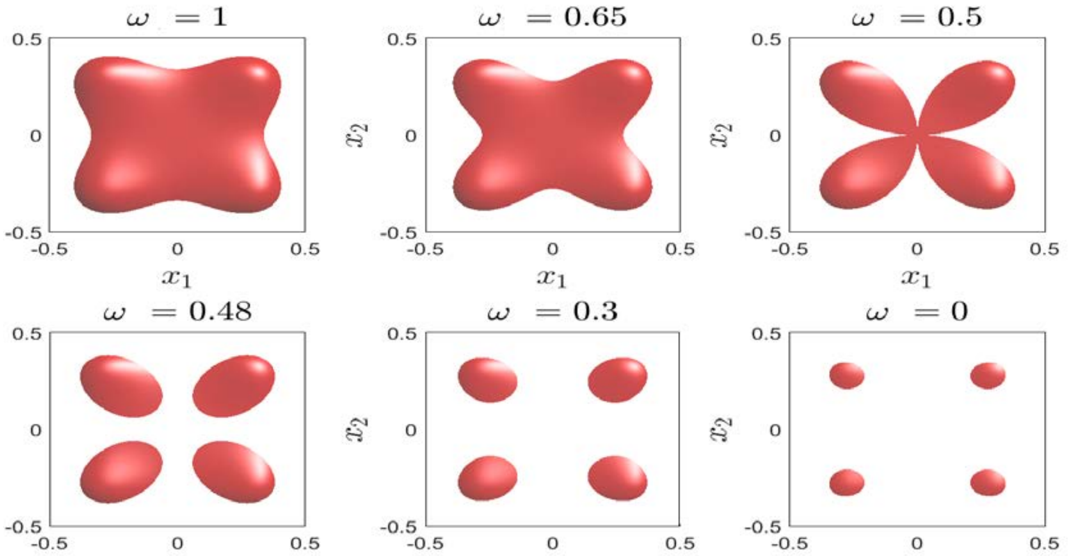


- Inside of the contours, risk is less or equal Δ

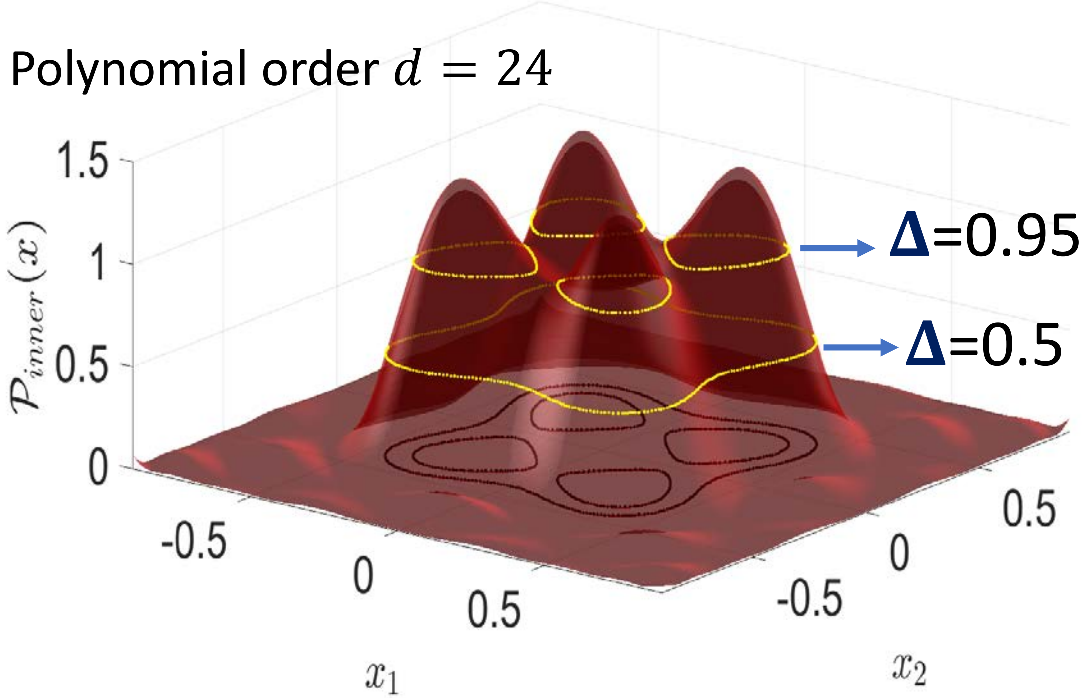
Example 4: Random Crowd Pattern

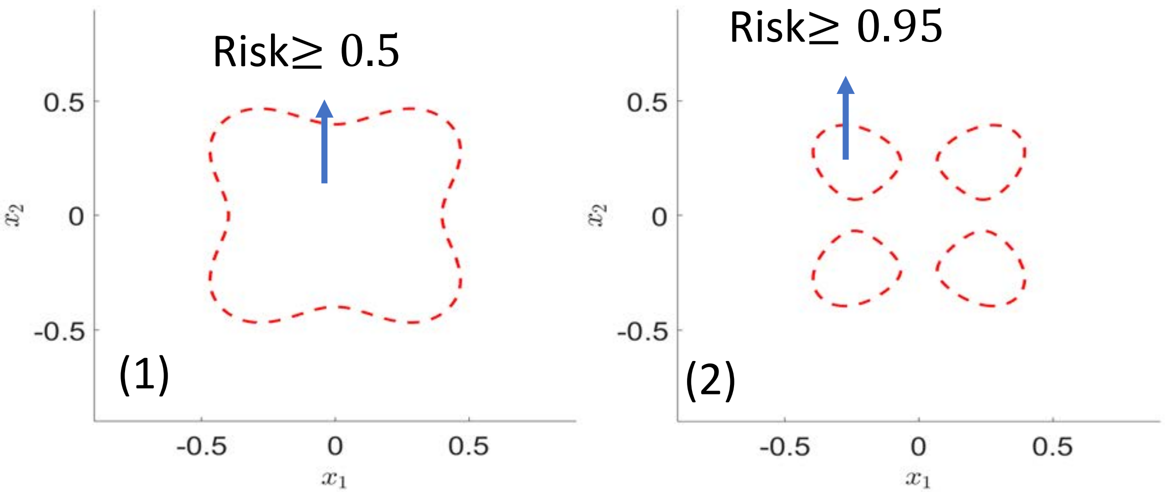
$\omega \in [0\ 1] \sim$ Beta $(1.1, 1.2)$

$$\chi_{obs}(\omega) = \{x \in \mathbb{R}^2 : -(4x_1^2 + 4x_2^2)^3 + 64x_1^2x_2^2 + 0.2\omega - 0.1 \leq 0\}$$



- Polynomial order $d = 24$





- Outer approximation of the set of all points whose probability of collision is greater than Δ.
- Probability of observing patterns 1 and 2 are greater or equal to 0.5 and 0.95, respectively.

Topics:

- Introduction
- Polynomial Representation of Obstacles and Dynamical Systems
- Risk Bounded Trajectory Planning in Uncertain Environments
- Control of Probabilistic Nonlinear Systems
  - Nonlinear State Feedback Control
  - Receding Horizon Control
- Flow-Tube based Control of Probabilistic Nonlinear Systems
- Chance Constrained Backward Reachability Set
- Continuous-Time Path Planning

- Control of Probabilistic Nonlinear Systems

  Chance Optimization for Nonlinear State Feedback

Nonlinear Probabilistic Systems

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

$\mathbf{x}(k) = [x_1(k), ..., x_n(k)]^T \in \chi \subset \mathbb{R}^n$ → States

$\mathbf{u}(k) = [u_1(k), ..., u_m(k)]^T \in \mathcal{U} \subset \mathbb{R}^m$ → Control Inputs

$\omega(k) = [\omega_1(k), ..., \omega_l(k)]^T \in \Omega \subset \mathbb{R}^l$ → Probabilistic uncertainty $\sim pr(\omega_k)$

- Uncertain Safety Constraint: $\chi(\omega_g) := \{x \in \mathbb{R}^n : g_j(x, \omega_g) \geq 0, j = 1, \ldots, n_g\}$

e.g., uncertain safe set, uncertain obstacle set, state constraints

- Final state Constraints $\chi_T := \{x \in \mathbb{R}^n : p_j(x) \geq 0, j = 1, \ldots, n_p\}$

- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$, $\omega_g \sim pr(\omega_g)$

- Design a closed-loop controller to:

i) Drive the robot to the goal region
ii) Satisfy safety constraints (e.g. avoid the obstacles)

in the presence of system and environment uncertainties.

- Closed-loop controller in the form of "*Polynomial State Feedback*", i.e., $u(x_k) = \sum_\alpha \underbrace{p_\alpha}_{\text{Feedback gains}} x_k^\alpha$

Chance Optimization

$$\begin{aligned} \underset{\text{Feedback gains}}{\text{maximize}} \quad & \text{Probability}(\text{Reaching the Target Set \& Satisfying Safety Constraints Over Planning Horizon}) \\ \text{subject to} \quad & \text{Constraints}(\text{design parameters}) \end{aligned}$$

Ashkan Jasour, C. Lagoa, "Convex Relaxations of a Probabilistically Robust Control Design Problem", 52st IEEE Conference on Decision and Control, Florence, Italy, 2013

Example 1:

- Dynamical system: $$x_1(k+1) = x_2(k)$$ $$x_2(k+1) = x_1(k)x_2(k) + \omega(k) + u(k)$$

- Uncertainties: $x(0) \sim \text{Uniform}[-5,5]^2, \quad \omega(k) \sim \text{Uniform}[-0.5, 0.5]$

- Target set: $\chi_T = [-1,1]^2$ $\quad T = 2$

- Feedback Control: $u(x(k)) = b_1 x_1^2(k) + b_2 x_1(k) x_2(k) + b_3 x_2^2(k)$ $\quad (b_1, b_2, b_3) \in [-1,1]^3$

- **Goal:** $x(2) \in \chi_T = [-1,1]^2$

- **Chance Optimization:**

$$\begin{aligned} \underset{b_1,b_2,b_3}{\text{maximize}} \quad & \text{Probability}(x(2) \in \chi_T = [-1,1]^2) \\ \text{subject to} \quad & (b_1, b_2, b_3) \in [-1,1]^3 \end{aligned}$$

- Dynamical system:

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_1(k)x_2(k) + \omega(k) + u(k)$$

- Feedback Control:

$$u(x(k)) = b_1 x_1^2(k) + b_2 x_1(k)x_2(k) + b_3 x_2^2(k)$$

- By recursion:

$$\begin{cases} x_1(2) = x_2(1) \\ x_2(2) = x_1(1)x_2(1) + \omega(1) + u(1) \end{cases}$$

$$\begin{cases} x_1(1) = x_2(0) \\ x_2(1) = x_1(0)x_2(0) + \omega(0) + u(0) \end{cases}$$

- Dynamical system:

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_1(k)x_2(k) + \omega(k) + u(k)$$

- Feedback Control:

$$u(x(k)) = b_1 x_1^2(k) + b_2 x_1(k)x_2(k) + b_3 x_2^2(k)$$

- By recursion:

$$\begin{cases} x_1(2) = x_2(1) \\ x_2(2) = x_1(1)x_2(1) + \omega(1) + u(1) \end{cases}$$

$$\begin{cases} x_1(1) = x_2(0) \\ x_2(1) = x_1(0)x_2(0) + \omega(0) + u(0) \end{cases}$$

- $x_1(2), x_2(2)$ in terms of uncertain parameters $x_1(0), x_2(0), \omega(0), \omega(1)$ and design parameters b_1, b_2, b_3:

$$\begin{cases} x_1(2) = x_1(0)x_2(0) + \omega(0) + b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0) \\ x_2(2) = x_2(0)\left(x_1(0)x_2(0) + \omega(0) + b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0)\right) + \omega(1) \\ \quad + b_1 (x_2(0))^2 + b_2 x_2(0)\left(x_1(0)x_2(0) + \omega(0) + \left(b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0)\right)\right) + \\ \quad b_3\left(x_1(0)x_2(0) + \omega(0) + \left(b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0)\right)\right)^2 \end{cases}$$

- Dynamical system:

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_1(k)x_2(k) + \omega(k) + u(k)$$

- Feedback Control:

$$u(x(k)) = b_1 x_1^2(k) + b_2 x_1(k)x_2(k) + b_3 x_2^2(k)$$

- By recursion:

$$\begin{cases} x_1(2) = x_2(1) \\ x_2(2) = x_1(1)x_2(1) + \omega(1) + u(1) \end{cases}$$

$$\begin{cases} x_1(1) = x_2(0) \\ x_2(1) = x_1(0)x_2(0) + \omega(0) + u(0) \end{cases}$$

- $x_1(2), x_2(2)$ in terms of uncertain parameters $x_1(0), x_2(0), \omega(0), \omega(1)$ and design parameters b_1, b_2, b_3:

$$\begin{cases} x_1(2) = \underbrace{x_1(0)x_2(0) + \omega(0) + b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0)}_{p_{x_1}(x(0),\omega(0),b_1,b_2,b_3)} \\ x_2(2) = x_2(0)\left(x_1(0)x_2(0) + \omega(0) + b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0)\right) + \omega(1) \\ \qquad \underbrace{\begin{aligned} &+ b_1 (x_2(0))^2 + b_2 x_2(0)\left(x_1(0)x_2(0) + \omega(0) + \left(b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0)\right)\right) + \\ &b_3\left(x_1(0)x_2(0) + \omega(0) + \left(b_1 x_1^2(0) + b_2 x_1(0)x_2(0) + b_3 x_2^2(0)\right)\right)^2 \end{aligned}}_{p_{x_2}(x(0),\omega(0),\omega(1),b_1,b_2,b_3)} \end{cases}$$

- **Chance Optimization:**

$$\underset{b_1,b_2,b_3}{\text{maximize}} \quad \text{Probability}(-1 \leq \underbrace{p_{x_1}(x(0),\omega(0),b_1,b_2,b_3)}_{x(1)} \leq 1, -1 \leq \underbrace{p_{x_2}(x(0),\omega(0),\omega(1),b_1,b_2,b_3)}_{x(2)} \leq 1)$$

- **Convex Chance Optimization in measures:**

$$\mathbf{P}_\mu^* := \text{maximize}_{\mu_b,\mu} \int d\mu,$$

$$\begin{aligned} \text{s.t.} \quad & \mu \preccurlyeq \mu_b \times (\mu_{x_0} \times \mu_{\omega_0} \times \mu_{\omega_1}) \\ & \mu_b \text{ is a probability measure} \\ & supp(\mu_b) \subset \{(b_1,b_2,b_3) \in [-1,1]^3\}, \\ & supp(\mu) = \{-1 \leq p_{x_1}(x(0),\omega(0),b_1,b_2,b_3) \leq 1, -1 \leq p_{x_2}(x(0),\omega(0),\omega(1),b_1,b_2,b_3) \leq 1\} \end{aligned}$$

- **Moment SDP with relaxation order 6:** $u(x(k)) = -0.98x_1^2(k) - 0.94x_1(k)x_2(k) - 0.98x_2^2(k)$

$$\text{Probability} = 1$$

Ashkan Jasour, C. Lagoa, "Convex Relaxations of a Probabilistically Robust Control Design Problem", 52st IEEE Conference on Decision and Control, Florence, Italy, 2013

Example 2:

- Dynamical system:

$$x_1(k+1) = \delta x_2(k)$$
$$x_2(k+1) = x_1(k) x_3(k)$$
$$x_2(k+1) = x_1(k) - x_2(k) + x_3(k) + u(k)$$

- Uncertainties: $x(0) \sim \text{Uniform}[-1,1]^2, \quad \delta \sim \text{Uniform}[-0.2, 0.2]$
- Target set: $\chi_T = [-0.2, 0.2]^3 \qquad T = 3$
- Obstacle: $\chi_{obs} = \{(x_1, x_2, x_3) : 0.3^2 - (x_1 + 0.5)^2 - (x_2 + 0.5)^2 - (x_3)^2 \geq 0\}$
- Feedback Control: $u(x(k)) = b_1 x_1(k) + b_2 x_2(k) + b_3 x_3(k) \qquad (b_1, b_2, b_3) \in [-1, 1]^3$

- **Goal:** $x(3) \in \chi_T = [-0.2, 0.2]^3 \qquad x(1), x(2) \in \chi_{safe}$
- **Chance Optimization:**

$$\begin{aligned} \underset{b_1, b_2, b_3}{\text{maximize}} \quad & \text{Probability}(x(3) \in \chi_T = [-0.2, 0.2]^2, x(1), x(2) \in \chi_{safe}) \\ \text{subject to} \quad & (b_1, b_2, b_3) \in [-1, 1]^3 \end{aligned}$$

- **Goal:** $x(3) \in \chi_T = [-0.2, 0.2]^3 \qquad x(1), x(2) \in \chi_{safe}$

- **Chance Optimization:**

$$\begin{aligned} \underset{b_1, b_2, b_3}{\text{maximize}} \quad & \text{Probability}(x(3) \in \chi_T = [-0.2, 0.2]^2, x(1), x(2) \in \chi_{safe}) \\ \text{subject to} \quad & (b_1, b_2, b_3) \in [-1, 1]^3 \end{aligned}$$

- By recursion, we describe states $x(1)$, $x(2)$, and $x(3)$ in terms of uncertain parameters $x_1(0), x_2(0), x_3(0), \delta$ and design parameters b_1, b_2, b_3.

- **Goal:** $x(3) \in \chi_T = [-0.2, 0.2]^3 \qquad x(1), x(2) \in \chi_{safe}$

- **Chance Optimization:**

$$\begin{aligned} \underset{b_1, b_2, b_3}{\text{maximize}} \quad & \text{Probability}(x(3) \in \chi_T = [-0.2, 0.2]^2, x(1), x(2) \in \chi_{safe}) \\ \text{subject to} \quad & (b_1, b_2, b_3) \in [-1, 1]^3 \end{aligned}$$

- By recursion, we describe states $x(1)$, $x(2)$, and $x(3)$ in terms of uncertain parameters $x_1(0), x_2(0), x_3(0), \delta$ and design parameters b_1, b_2, b_3.

  - Solution obtained by Moment SDP with relaxation order 8:

$$u(x(k)) = -0.3x_1(k) + 0.47x_2(k) - 0.86x_3(k)$$

$$\text{Probability} = 0.99$$

- **Goal:** $x(3) \in \chi_T = [-0.2, 0.2]^3 \qquad x(1), x(2) \in \chi_{safe}$

- **Chance Optimization:**

$$\begin{aligned} \underset{b_1, b_2, b_3}{\text{maximize}} \quad & \text{Probability}(x(3) \in \chi_T = [-0.2, 0.2]^2, x(1), x(2) \in \chi_{safe}) \\ \text{subject to} \quad & (b_1, b_2, b_3) \in [-1, 1]^3 \end{aligned}$$

- By recursion, we describe states $x(1)$, $x(2)$, and $x(3)$ in terms of uncertain parameters $x_1(0), x_2(0), x_3(0), \delta$ and design parameters b_1, b_2, b_3.

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$$u(x(k)) = -0.3x_1(k) + 0.47x_2(k) - 0.86x_3(k)$$

$$\text{Probability } = 0.99$$

- To improve the estimation of the probability of achieving control goals for the designed controller
  1) We can solve moment SDP provided in the lecture 10 (risk estimation) or 2) Monte-Carlo

- Estimated probability of reaching the target set is 0.95 and estimated probability remaining safe over planning horizon is 1.

Chance Optimization

$$\begin{array}{ll} \underset{\text{Feedback gains}}{\text{maximize}} & \text{Probability}(\text{Reaching the Target Set \& Satisfying Safety Constraints Over Planning Horizon}) \\ \text{subject to} & \text{Constraints}(\text{design parameters}) \end{array}$$

- In the chance optimization based controller design, size of the SDP increases as the planning horizon increase.

Chance Optimization

$$\begin{array}{ll} \underset{\text{Feedback gains}}{\text{maximize}} & \text{Probability}(\text{Reaching the Target Set \& Satisfying Safety Constraints Over Planning Horizon}) \\ \text{subject to} & \text{Constraints}(\text{design parameters}) \end{array}$$

- In the chance optimization based controller design, size of the SDP increases as the planning horizon increase.

$$x(T) \in \chi_T \qquad x(k) \in \chi_{safe}, \ k = 1, ..., T-1$$

- As T increases, we need higher order polynomials to describe the states (by recursion) in terms of uncertain and design parameters. Hence, we need higher relaxation order to solve moment SDP.

Chance Optimization

$$\underset{\text{Feedback gains}}{\text{maximize}} \quad \text{Probability}(\text{Reaching the Target Set \& Satisfying Safety Constraints Over Planning Horizon})$$
$$\text{subject to} \quad \text{Constraints}(\text{design parameters})$$

- In the chance optimization based controller design, size of the SDP increases as the planning horizon increase.

$$x(T) \in \chi_T \qquad x(k) \in \chi_{safe}, \; k = 1, ..., T-1$$

- As T increases, we need higher order polynomials to describe the states (by recursion) in terms of uncertain and design parameters. Hence, we need higher relaxation order to solve moment SDP.

- To address problems with long planning horizons:
  1) Receding Horizon Formulation 2) Flow-Tube based control

- Control of Probabilistic Nonlinear Systems

  Chance Constrained Receding Horizon Control

Chance Constrained Receding Horizon Control

- Uncertain Dynamical Model $x_{k+1} = f(x_k, u_k, \omega_k)$
- Target Set: $\chi_T := \{x \in \mathbb{R}^n : p_T(x) \leq 0 \}$
- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$
- Control Constraints: $u_k \in \mathcal{U}_k$

Control Goals:

1) Reach the target set with high probability
2) Minimize the expected value of the given cost function in terms of states and control input, i.e. $\mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})]$

Chance Constrained Receding Horizon Control

- Uncertain Dynamical Model $x_{k+1} = f(x_k, u_k, \omega_k)$

- Target Set: $\chi_T := \{x \in \mathbb{R}^n : p_T(x) \leq 0 \}$

- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$

- Control Constraints: $u_k \in \mathcal{U}_k$

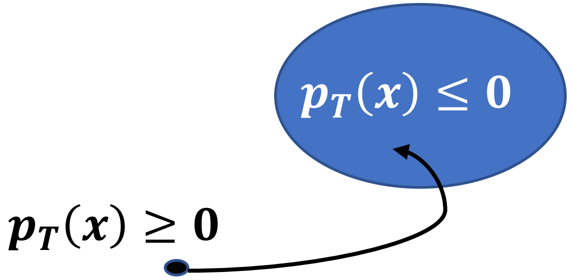
Control Goals:

1) Reach the target set with high probability
2) Minimize the expected value of the given cost function in terms of states and control input, i.e. $\mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})]$

➢ In the chance constrained receding horizon formulation, at each time step, we look for the control input such that states gets closer to the target set with some non-zero probability.

- Given the target set $\chi_T := \{x \in \mathbb{R}^n : p_T(x) \leq 0 \}$, polynomial $p_T(x)$ represent the distance to the target set.

- $p_T(x)$ decreases as the states x gets closer to the target set.



- States get closer to the target set if

$$p_T(x_{k+1}) \leq \alpha p_T(x_k)$$

Where, $0 < \alpha < 1$

Chance Constrained Optimization at time k over horizon h:

$$\begin{aligned} \underset{u_i|_{i=k}^{k+h}}{\text{maximize}} \quad & \mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})] \\ \text{subject to Probability} & \left(p_T(x_{k+1}) \leq \alpha p_T(x_k)\right) \geq 1 - \beta p_T(x_k) \\ & u_k \in \mathcal{U}_k \end{aligned}$$

Where, $0 < \alpha, \beta < 1$ $\quad$ $0 \leq \beta p_T(x) < 1$ for all $x \in \mathrm{X}$ (state space)

Chance Constrained Optimization at time k over horizon h:

$$\begin{aligned} \underset{u_i|_{i=k}^{k+h}}{\text{maximize}} \quad & \mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})] \\ \text{subject to } & \text{Probability}\left(p_T(x_{k+1}) \leq \alpha p_T(x_k)\right) \geq 1 - \beta p_T(x_k) \\ & u_k \in \mathcal{U}_k \end{aligned}$$

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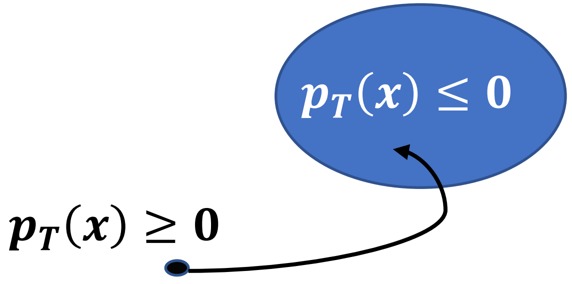

$$\text{Probability}\,(\underbrace{p_T(x_{k+1}) \leq \alpha p_T(x_k)}) \geq \underbrace{1 - \beta p_T(x_k)}$$

- With some probability states at next time step gets closer to the target set
  - This probability increases as states gets closer to the target set

Chance Constrained Optimization at time k over horizon h:

$$\underset{u_i|_{i=k}^{k+h}}{\text{maximize}} \quad \mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})]$$

$$\text{subject to Probability}\left(p_T(x_{k+1}) \leq \alpha p_T(x_k)\right) \geq 1 - \beta p_T(x_k)$$

$$u_k \in \mathcal{U}_k$$

Where, $0 < \alpha, \beta < 1$ $\quad$ $0 \leq \beta p_T(x) < 1$ for all $x \in \mathrm{X}$ (state space)

- In this formulation, chance constraints only depends on the states at next time step. Hence, results in smaller moment SDP.

- **Target Set:** $\chi_T := \{x \in \mathbb{R}^n : p_T(x) \leq 0 \}$
- **Chance Constraint:** $\text{Probability}\left(p_T(x_{k+1}) \leq \alpha p_T(x_k)\right) \geq 1 - \beta p_T(x_k)$

- Target Set: $\chi_T := \{x \in \mathbb{R}^n : p_T(x) \leq 0\ \}$
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Theorem: Given an initial state x_0 and $\epsilon > 0$ there exist a time step $\hat{k}$ and lower bound probability $\hat{P}$ such that

$$\text{Prob}\{p_T(x_k) \leq \epsilon,\ \forall k \geq \hat{k}\} \geq \hat{P}$$

where

$$\hat{k} \geq \frac{ln(\epsilon) - ln(p_T(x_0))}{ln(\alpha)}$$

$$\hat{P} = \prod_{i=0}^{\hat{k}-1}(1 - \beta\alpha^i) > 0$$

Theorem 1, Ashkan Jasour, Constantino Lagoa, "Convex Chance Constrained Model Predictive Control", IEEE 55th Conference on Decision and Control (CDC), 2016, Las Vegas, USA

- Target Set: $\chi_T := \{x \in \mathbb{R}^n : p_T(x) \leq 0 \}$
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$$\hat{k} \geq \frac{ln(\epsilon) - ln(p_T(x_0))}{ln(\alpha)}$$

$$\hat{P} = \prod_{i=0}^{\hat{k}-1} (1 - \beta\alpha^i) > 0$$

- The probability lower bound is a **convergent** product and converges to a non-zero constant.

Example: $(\alpha, \beta) = (0.8, 0.05)$ $\hat{P} \to 0.8169$ $\hat{k} \geq 36$

- The lower bound probability is a conservative bound and the actual probability of the reaching the set is greater than $\hat{P}$.

Theorem 1, Ashkan Jasour, Constantino Lagoa, "Convex Chance Constrained Model Predictive Control", IEEE 55th Conference on Decision and Control (CDC), 2016, Las Vegas, USA

- **Chance Constrained Optimization:**

$$\underset{u_i|_{i=k}^{k+h}}{\text{maximize}} \quad \mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})]$$

$$\text{subject to Probability}\left(p_T(x_{k+1}) \leq \alpha p_T(x_k)\right) \geq 1 - \beta p_T(x_k)$$

$$u_k \in \mathcal{U}_k$$

Let: $u = [u_i, i = k, ..., k+h]$

$$u \in \mathcal{U} = \{\mathcal{U}_i, i = k, ..., k+h\}$$

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$$\underset{u_i|_{i=k}^{k+h}}{\text{maximize}} \quad \mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})]$$
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- By recursion, we describe states x_k in terms of uncertain Parameters and control input. Hence

$$\mathrm{E}[p_{cost}(x_i|_{i=k}^{k+h}, u_i|_{i=k}^{k+h})] = p_E(u)$$

Polynomial in terms of u
($E[\omega^a]$ replaced by the moments y_a)

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$$u \in \mathcal{U}$$

- **Chance Constrained Optimization:**

$$\begin{aligned} &\underset{u}{\text{maximize}} \;\; p_E(u) \\ &\text{subject to Probability}\left(p_T(x_{k+1}) \leq \alpha p_T(x_k)\right) \geq 1 - \beta p_T(x_k) \\ &\qquad\qquad u \in \mathcal{U} \end{aligned}$$

Using the measure based deterministic and chance optimization:

- **Convex Chance Optimization in measures:**

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_u,\mu} \int p_E(u) d\mu_u,$$

$$\begin{aligned} \text{s.t.} \quad & \mu \preccurlyeq \mu_u \times \prod_{i=k}^{k+h} \mu_{\omega_i} \\ & \mu_u \text{ is a probability measure} \\ & \int d\mu \geq 1 - \beta p_T(x_k) \\ & supp(\mu_u) \subset \mathcal{U}, \quad supp(\mu) = \{p_T(x_{k+1}) - \alpha p_T(x_k) \leq 0\} \end{aligned}$$

➢ Moment representation to obtain Moment SDP.

Theorem 2, Ashkan Jasour, Constantino Lagoa, "Convex Chance Constrained Model Predictive Control", IEEE 55th Conference on Decision and Control (CDC), 2016, Las Vegas, USA

- Moment representation to obtain Moment SDP.

Receding Horizon Algorithm:

- $k = 0$
- Solve the moment SDP over the horizon h to obtain $u(i), i = k, .., k + h$
- Apply the obtained $u(k)$ to the system to obtain $x(k + 1)$.
- $k \leftarrow k + 1$

Ashkan Jasour, Constantino Lagoa, "Convex Chance Constrained Model Predictive Control", IEEE 55th Conference on Decision and Control (CDC), 2016, Las Vegas, USA

Example:

- Dynamical system:

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_2(k+1) = x_1(k) - x_2(k) + x_3(k) + \omega(k) + u(k)$$

- Uncertainties: $\omega(k) \sim \text{Uniform}[-0.5, 0.5]$
- Target set: $\chi_T = \{(x_1, x_2, x_3) : 0.2^2 - x_1^2 - x_2^2 - x_3^2 \geq 0\}$
- Receding horizon: $h = 3$
- $(\alpha, \beta) = (0.9, 0.2027)$
- $x_0 = (1,1,1)$

The obtained control input at each time k for the initial condition $x_0 = (1,1,1)$

$$u_k = [-0.227, -0.219, -0.325, -0.196, -0.215, -0.605, 0.550]$$

Where results in the trajectory of

$$x_1(k) = [1, 1, 1, 0.752, 0.892, 0.417, -0.101, 0.0487]$$

$$x_2(k) = [1,1,0.752,0.892,0.417, -0.101,0.0487,0.041]$$

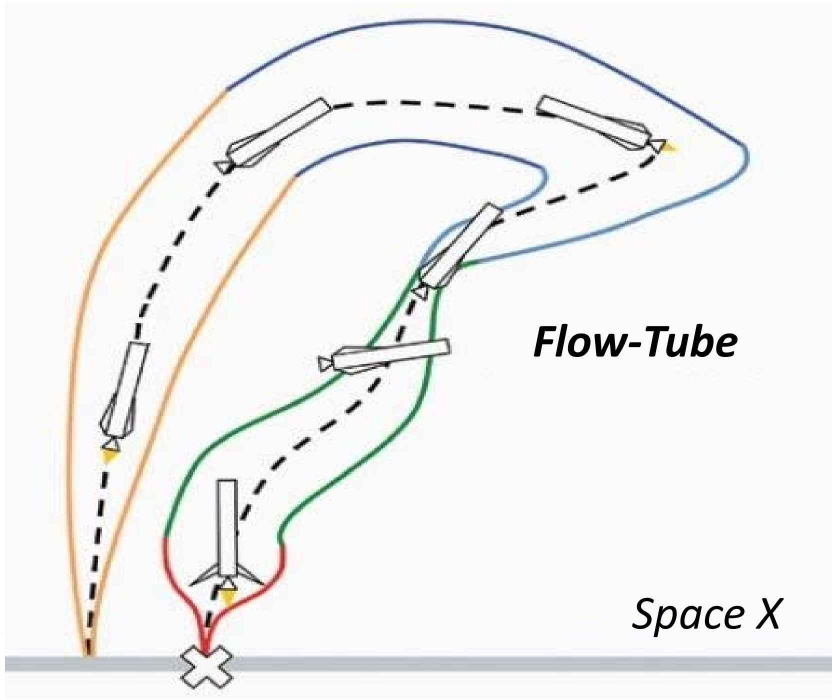
$$x_3(k) = [1, 0.752, 0.892, 0.554, -0.113, 0.116, -0.410, 0.171]$$

Hence, in $\boxed{7}$ steps the trajectory of the system under the control reaches the desired set.

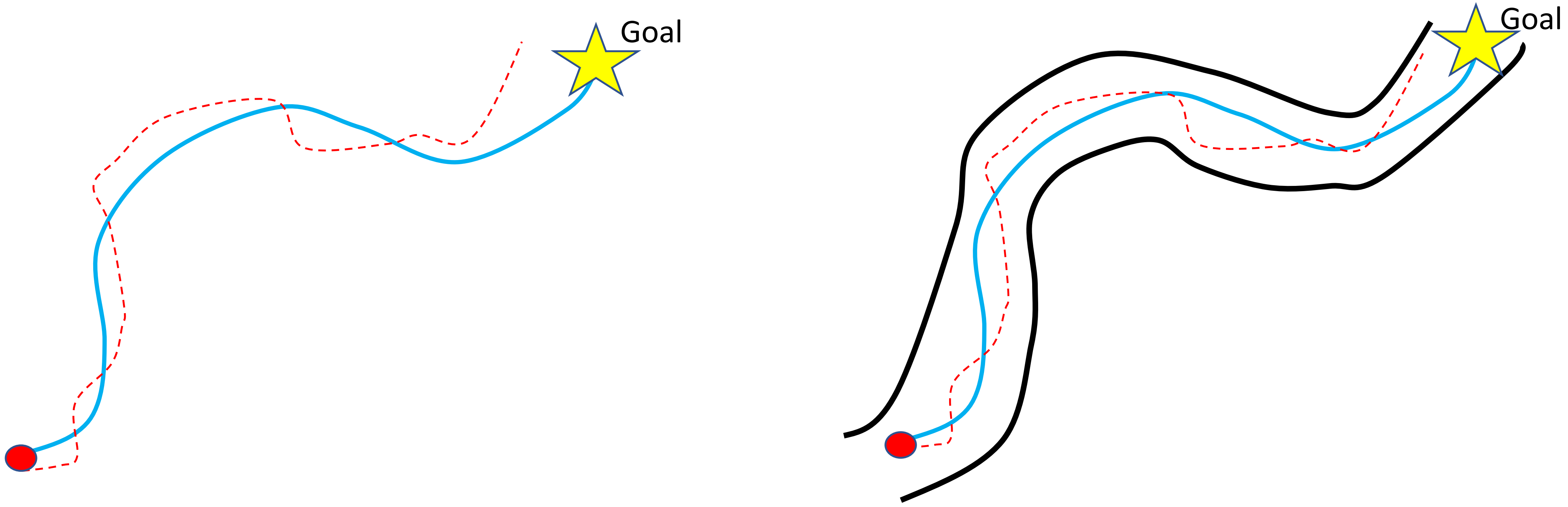
Topics:

- Introduction
- Polynomial Representation of Obstacles and Dynamical Systems
- Risk Bounded Trajectory Planning in Uncertain Environments
- Control of Probabilistic Nonlinear Systems
  - Nonlinear State Feedback Control
  - Receding Horizon Control
- Flow-Tube based Control of Probabilistic Nonlinear Systems
- Chance Constrained Backward Reachability Set
- Continuous-Time Path Planning

Flow-Tube Based Control Of Probabilistic Nonlinear Systems



2) Flow-Tube Based Control Of Probabilistic Nonlinear Systems

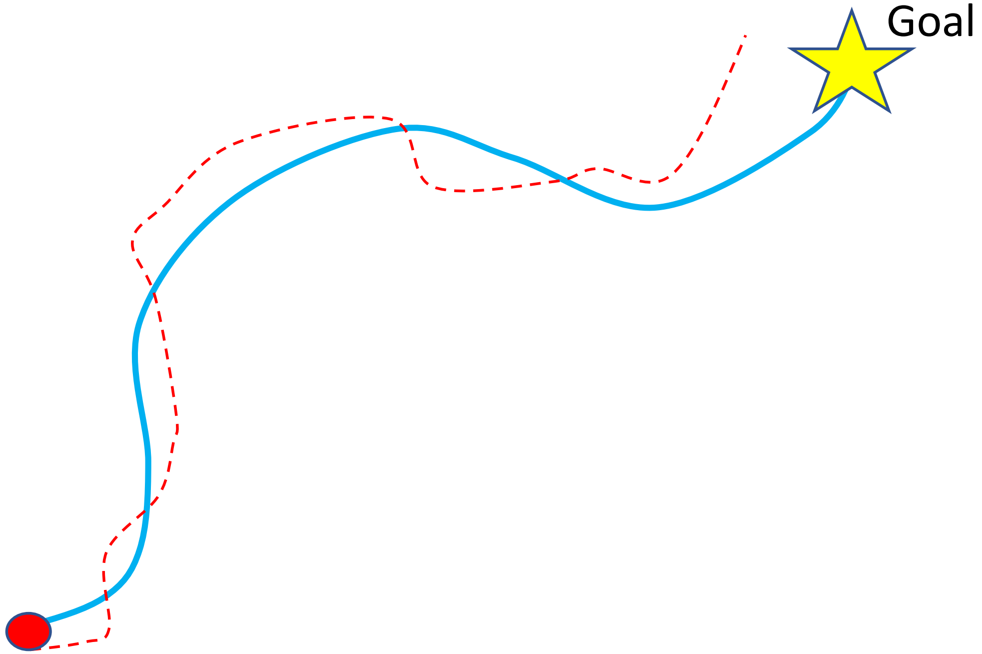


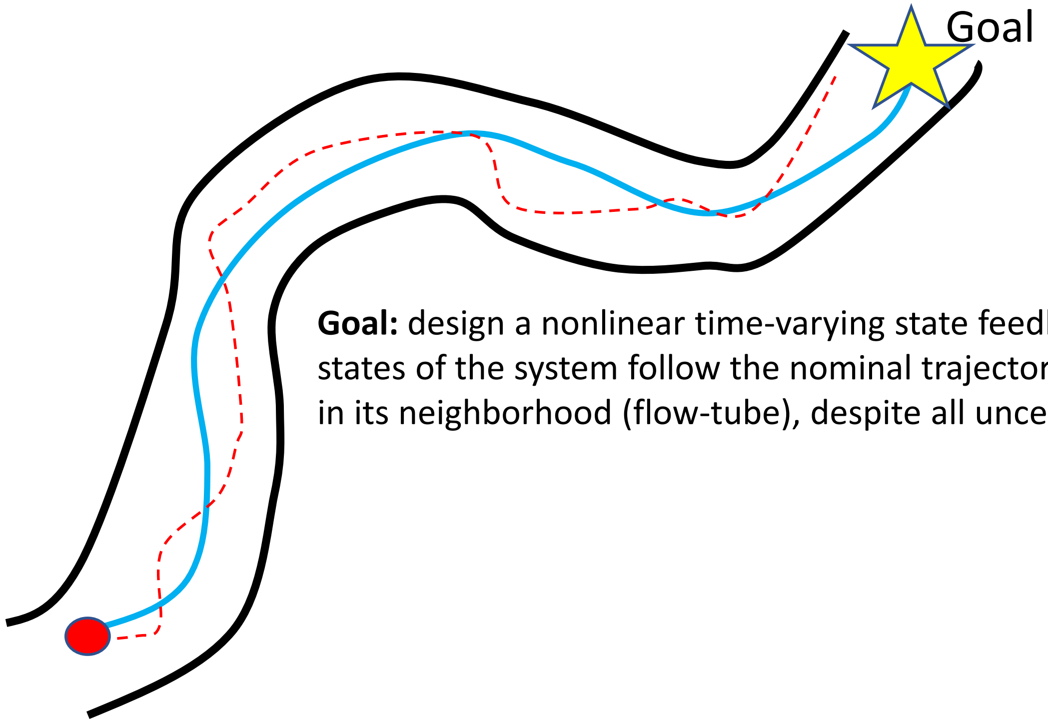
Planned Trajectory from initial pose to the goal pose.

Actual trajectory due to disturbances.

Tube around the planned trajectory

2) Flow-Tube Based Control Of Probabilistic Nonlinear Systems





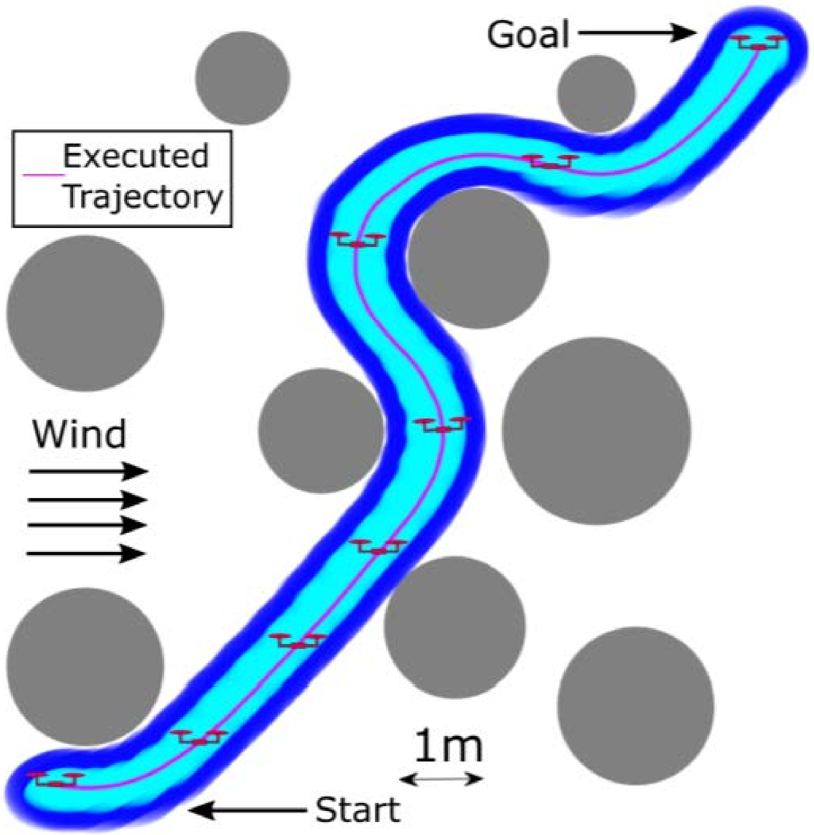
Goal: design a nonlinear time-varying state feedback such that states of the system follow the nominal trajectory and remain in its neighborhood (flow-tube), despite all uncertainties.

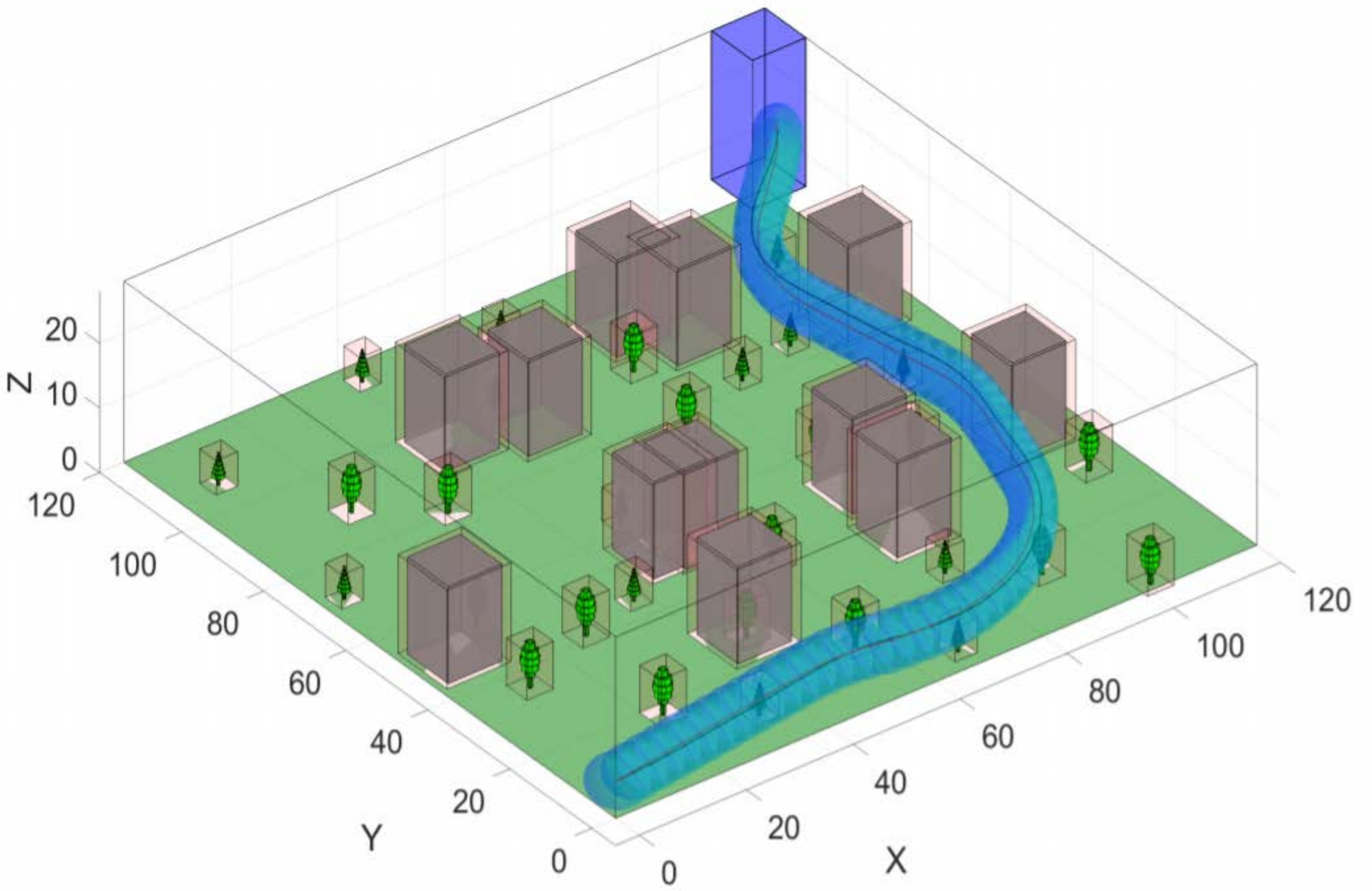
Planned Trajectory from initial pose to the goal pose.

Actual trajectory due to disturbances.

Tube around the planned trajectory

- As long as the tube is obstacle free, safety is assured.



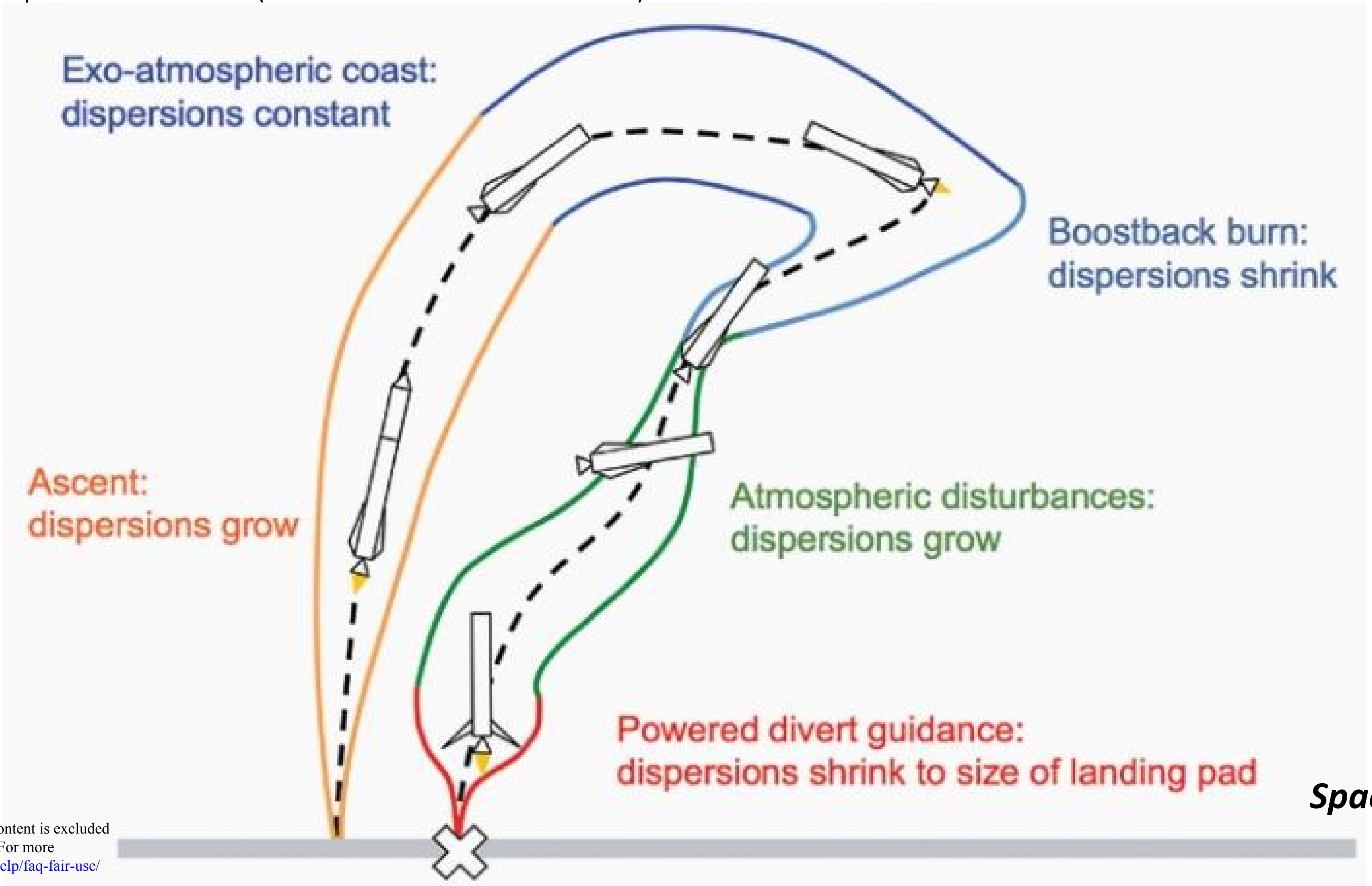


“Robust Tracking with Model Mismatch for Fast and Safe Planning: an SOS Optimization Approach’”, Sumeet Singh, Mo Chen, Sylvia L. Herbert, Claire J. Tomlin, Marco Pavone.

“Robust Online Motion Planning via Contraction Theory and Convex Optimization”, Sumeet Singh, Anirudha Majumdar, Jean-Jacques Slotine, Marco Pavone

“Funnel libraries for real-time robust feedback motion planning”, Anirudha Majumdar, Russ Tedrake

1) Design a trajectory and the associated tube for different steps of the mission (Planning Step),
2) Stick to the plan despite all uncertainties (follow and remain inside the tube)



Space X

Flow-Tube Based Control Of Probabilistic Nonlinear Systems

Continuous State space model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states inputs Uncertainty ~ p(ω_k):probability distribution

Nominal Trajectory

$$x^* = \{x^*(k), k = 1, \ldots, T\} \qquad u^* = \{u^*(k), k = 0, \ldots, T-1\}$$

Flow-Tube Based Control Of Probabilistic Nonlinear Systems

Continuous State space model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states   inputs   Uncertainty ~ p(ω_k):probability distribution

Nominal Trajectory

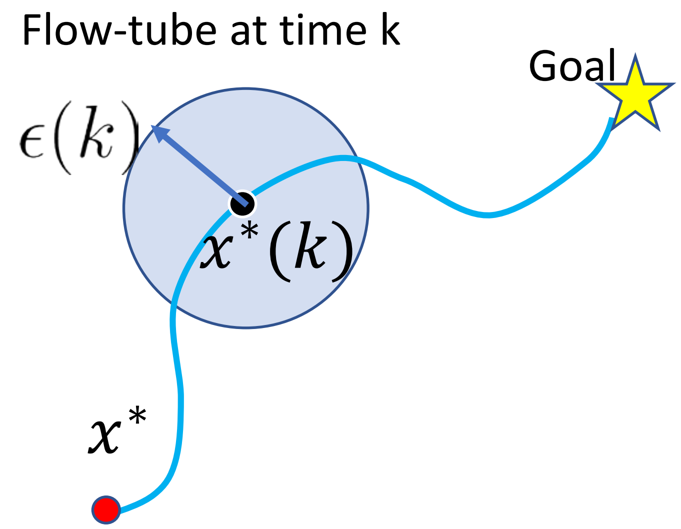
$$x^* = \{x^*(k), k = 1, \dots, T\} \qquad u^* = \{u^*(k), k = 0, \dots, T-1\}$$

Flow-tube ($\mathcal{FT}$) : a neighborhood around the nominal trajectory x^*

$$\mathcal{FT}(k) = \{\mathbf{x} : \ \mathcal{P}_{k_j}(x) \geq 0, \ j = 1, \dots, \ell\}$$

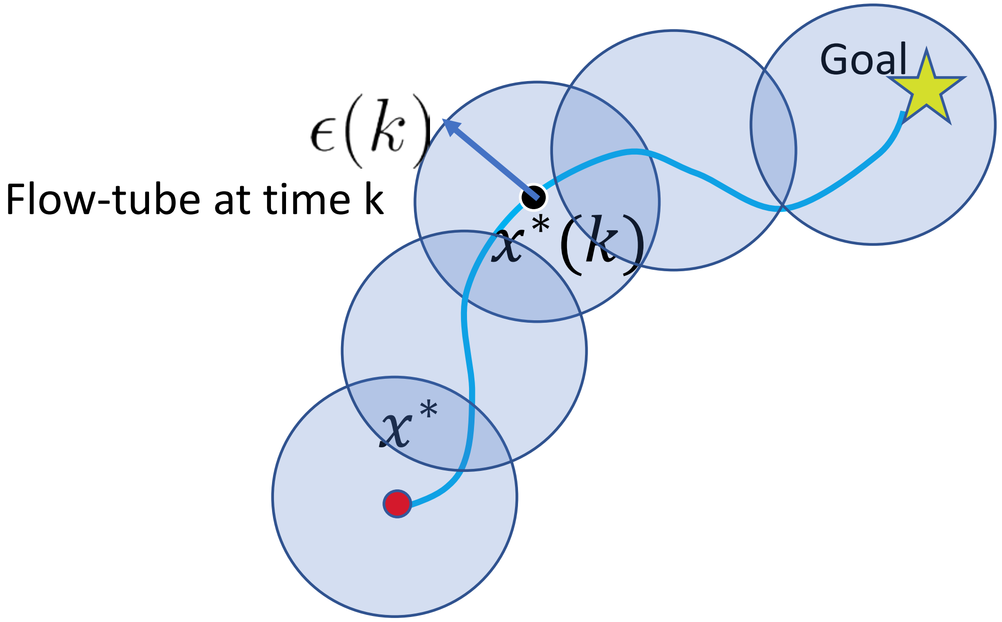
Flow-tube at time k   polynomial

Example: $\mathcal{FT}(k) = \{\mathbf{x} : \ ||\mathbf{x}(k) - \mathbf{x}^*(k)||_2^2 \leq \epsilon(k)\}$



Time-Varying Feedback Controller:

$$\mathbf{u}(k) = \bar{\mathbf{u}}(k) + \mathbf{u}^*(k)$$



Flow-tube ($\mathcal{FT}$)

Goal: design a nonlinear time-varying state feedback such that states of the system follow the nominal trajectory and remain in the given flow-tube, despite all uncertainties.

Time-Varying Feedback Controller:

$$\mathbf{u}(k) = \bar{\mathbf{u}}(k) + \mathbf{u}^*(k)$$

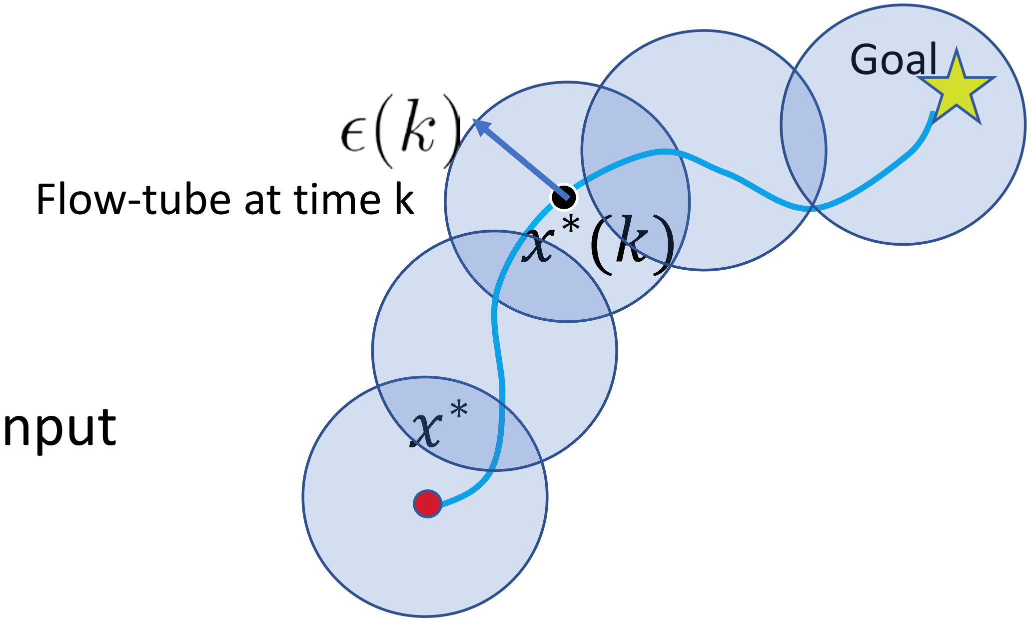
polynomial state feedback control in error state $\bar{\mathbf{x}}(k) = \mathbf{x}(k) - \mathbf{x}^*(k)$

Nominal control input

e.g., $\bar{\mathbf{u}}(k) = \sum_{\alpha \in \mathbb{N}^n} g_{\alpha i}(k)\bar{\mathbf{x}}(k)^{\alpha}$

Coefficients of polynomial

powers of polynomial



Flow-tube ($\mathcal{FT}$)

Goal: design a nonlinear time-varying state feedback such that states of the system follow the nominal trajectory and remain in the given flow-tube, despite all uncertainties.

Chance Optimization: Find design parameters to $max \ \ Prob(Success)$

- **Success:** states of the system follow the nominal trajectory and remain in the given flow-tube.
- **Design parameters:** Parameters of nonlinear time-varying state feedback i.e.,

$$\{\mathbf{u}(k), k = 0, ..., T-1\}$$

Chance Optimization:

$$\max_{\mathbf{G}_i(k)|_{i=1,k=0}^{i=m,k=T-1}} \text{Probability}_{p_{\mathbf{x}_0},p_{\omega_k}|_{k=0}^{T-1}}\left(\cap_{k=1}^{T}\{\mathbf{x}(k)\in\mathcal{FT}(k)\}\right)$$

→ Probability that states of the system follow the nominal trajectory and remain in the given flow-tube.

$$\text{s.t.}\quad \mathbf{G}_i(k) = [g_{\alpha i}(k), \alpha\in\mathbb{N}^n], i = 1,...,m$$ → Feedback gains

$$\mathbf{x}(k+1) = f(\mathbf{x}(k),\mathbf{u}(k),\omega(k))$$ → Uncertain dynamical system

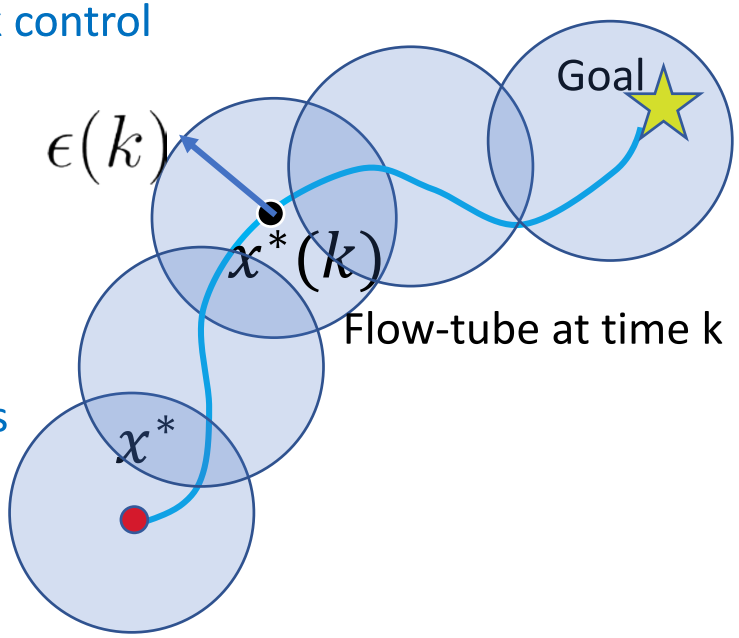
$$\left.\begin{aligned}&\mathbf{u}(k) = \bar{\mathbf{u}}(k) + \mathbf{u}^*(k)\\ &\bar{u}_i(k) = \sum_{\alpha\in\mathbb{N}^n} g_{\alpha i}(k)\bar{\mathbf{x}}(k)^{\alpha}, i = 1,...,m\\ &\bar{\mathbf{x}}(k) = \mathbf{x}(k) - \mathbf{x}^*(k)\end{aligned}\right\}$$ → Polynomial state feedback control

$$\mathbf{u}(k)\in\mathcal{U} = [a_1,b_1]\times ...\times[a_m,b_m]$$ → Control constraints

$$g_{\alpha i}(k)\in[a_{\alpha i},b_{\alpha i}],\ i = 1,...,m, \alpha\in\mathbb{N}^n$$ → Feedback gain constraints

$$x(0)\sim p_{x_0}(x), \omega(k)\sim p_{\omega_k}(\omega)$$ → Probability distributions of uncertainties

$$k = 1,...,T-1$$ → Planning horizon



Flow-tube ($\mathcal{FT}$)

Chance Optimization: Find design parameters to $max \;\; Prob(Success)$

- **Success:** states of the system follow the nominal trajectory and remain in the given flow-tube.
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$$\{\mathbf{u}(k), k = 0, ..., T-1\}$$

- Chance optimization formulation for long planning horizons T, result in a large moment SDP

Chance Optimization: Find design parameters to $max \;\; Prob(Success)$

- **Success:** states of the system follow the nominal trajectory and remain in the given flow-tube.
- **Design parameters:** Parameters of nonlinear time-varying state feedback i.e.,

$$\{\mathbf{u}(k), k = 0, \dots, T-1\}$$

Smaller SDP

Long Planning Horizon T

Sequential Chance Optimization:

- Break the original chance optimization into smaller chance optimization problems.

For $k = 0$ to T solve following chance optimization:

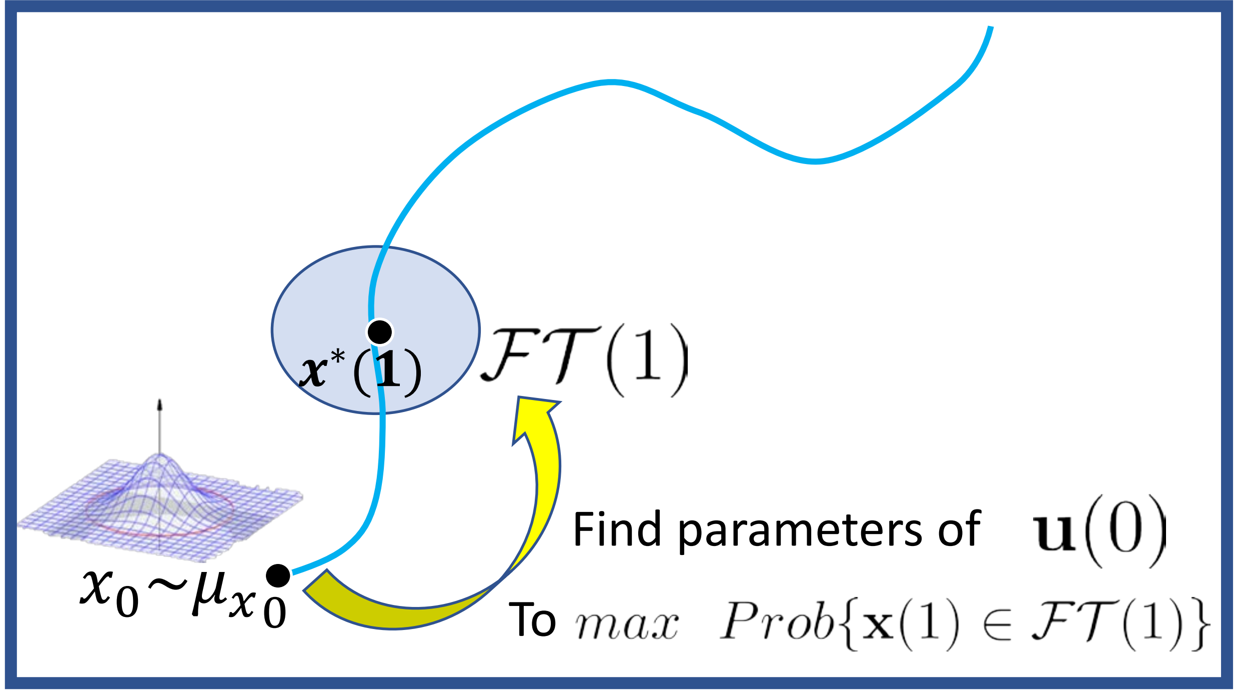
- **Design parameters:** Parameters of nonlinear time-varying state feedback at time k, i.e., $\{\mathbf{u}(k)\}$
- **Success:** $x(k+1) \in \mathcal{FT}(k+1)$

Sequential Chance Optimization:

- At $k = 0$, given:

$$x_{k+1} = f(x_k, u_k, \omega_k),\ \{x^*(1), u^*(1)\}, \mathcal{FT}(1)$$

$$x_0 \sim \mu_{x_0},\ \omega_0 \sim \mu_{\omega_0}$$



Sequential Chance Optimization:

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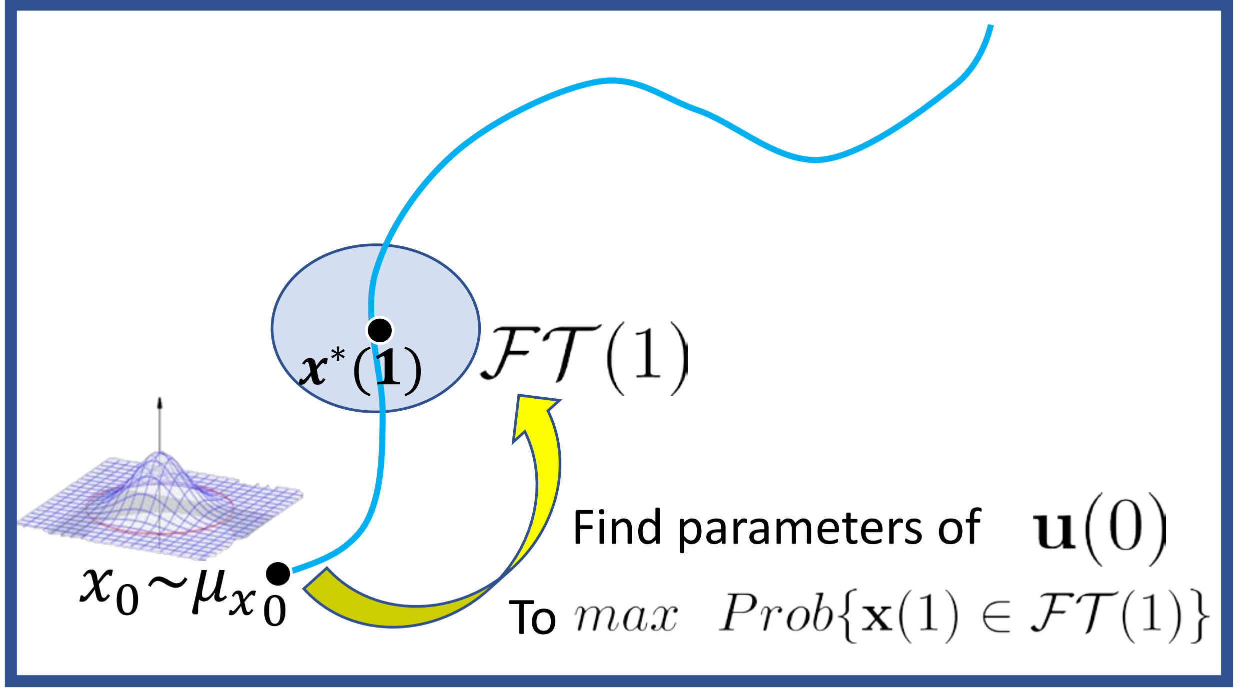


- At $k = 1$, given:

$$x_{k+1} = f(x_k, u_k, \omega_k), \{x^*(2), u^*(2)\}, \mathcal{FT}(2)$$

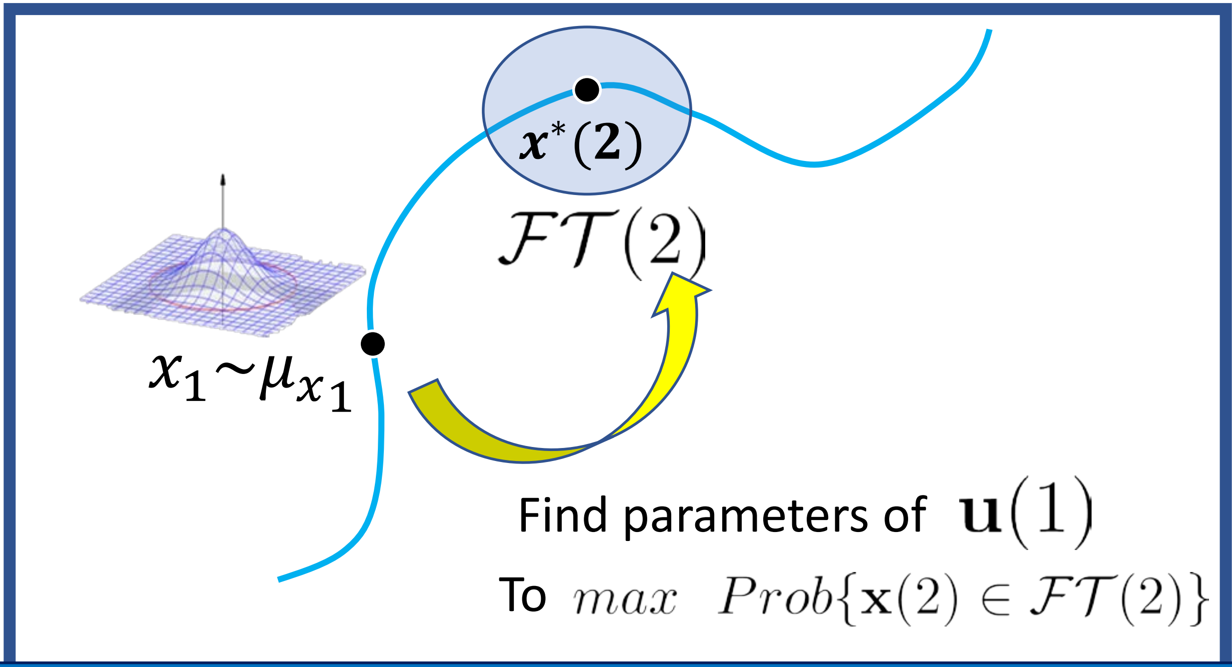
$x_1 \sim \mu_{x_1}$, $\omega_1 \sim \mu_{\omega_1}$

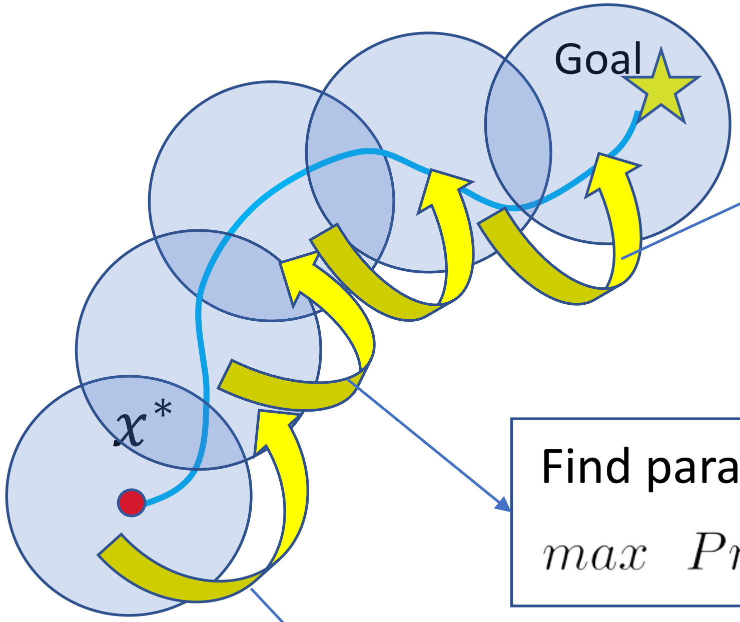
?

Uncertainty Propagation

Moments of uncertain state x_1:

$$y_\alpha = E[x_1^\alpha] = E[f^\alpha(x_0, u_0, \omega_0)]$$





Flow-tube ($\mathcal{FT}$ **)**

Find parameters of $\mathbf{u}(T-1)$

$max \quad Prob\{\mathbf{x}(T) \in \mathcal{FT}(T)\}$

Find parameters of $\mathbf{u}(1)$

$max \quad Prob\{\mathbf{x}(2) \in \mathcal{FT}(2)\}$

Find parameters of $\mathbf{u}(0)$

$max \quad Prob\{\mathbf{x}(1) \in \mathcal{FT}(1)\}$

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

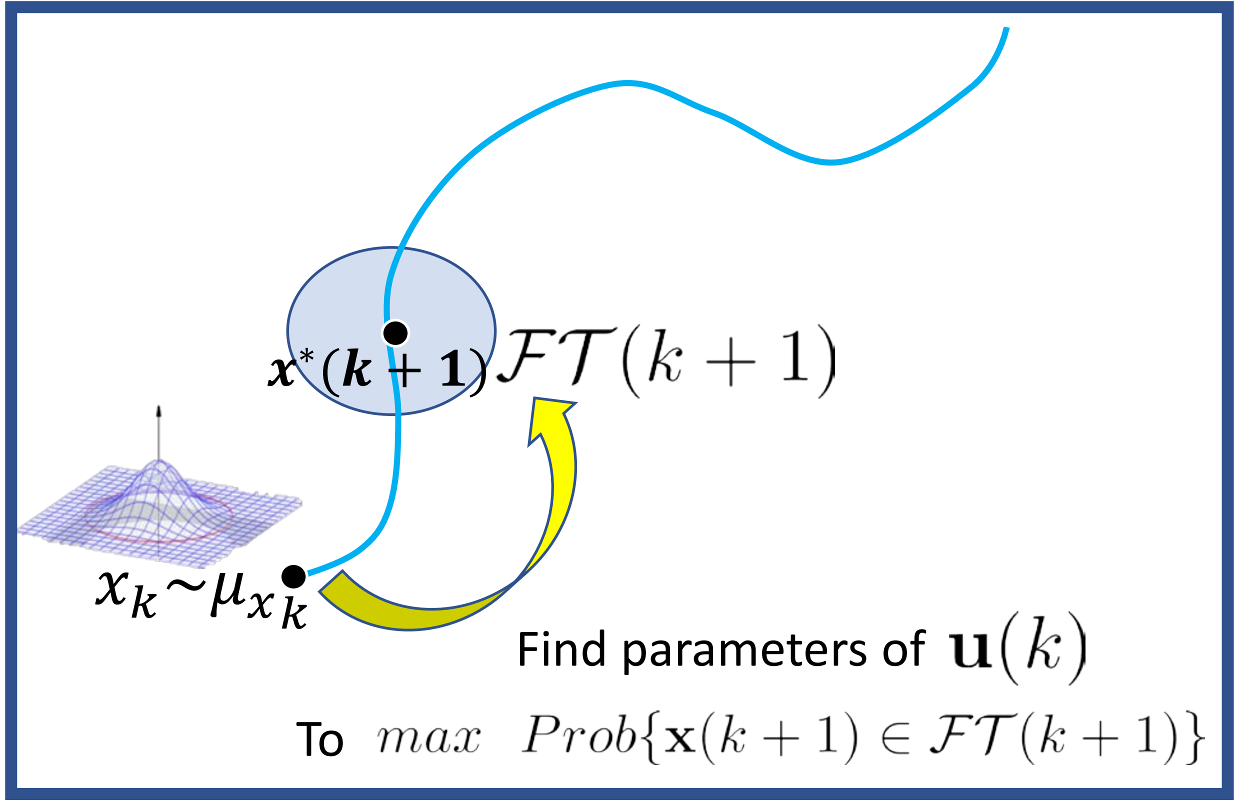
Source of uncertainties:

Initial states $(x_1(0), x_2(0), x_3(0)) \sim pr_{x_0}(x_1, x_2, x_3)$

Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$

- Suppose at time k:

$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1\,, 0.1]^3)$ $\omega(k) \sim Beta(2,5)$



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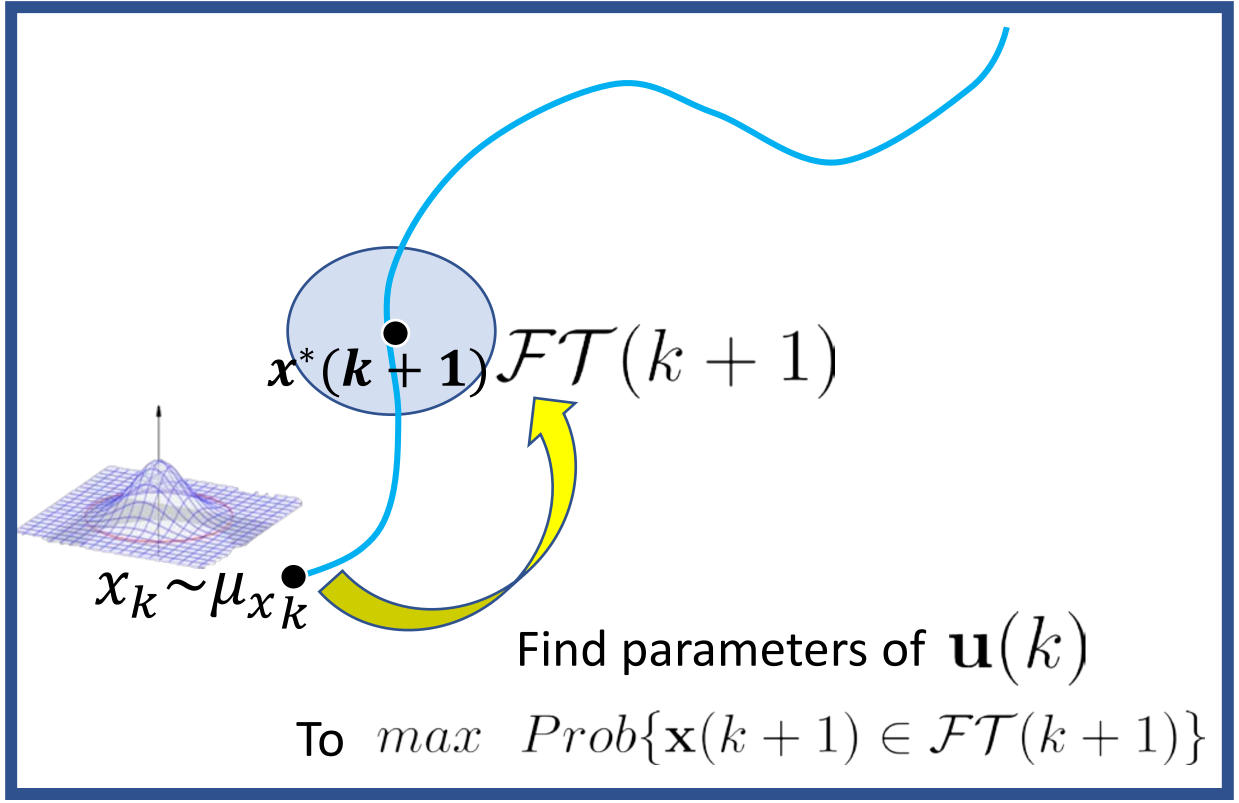
Source of uncertainties:

Initial states $(x_1(0), x_2(0), x_3(0)) \sim pr_{x_0}(x_1, x_2, x_3)$

Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$



- Suppose at time k:

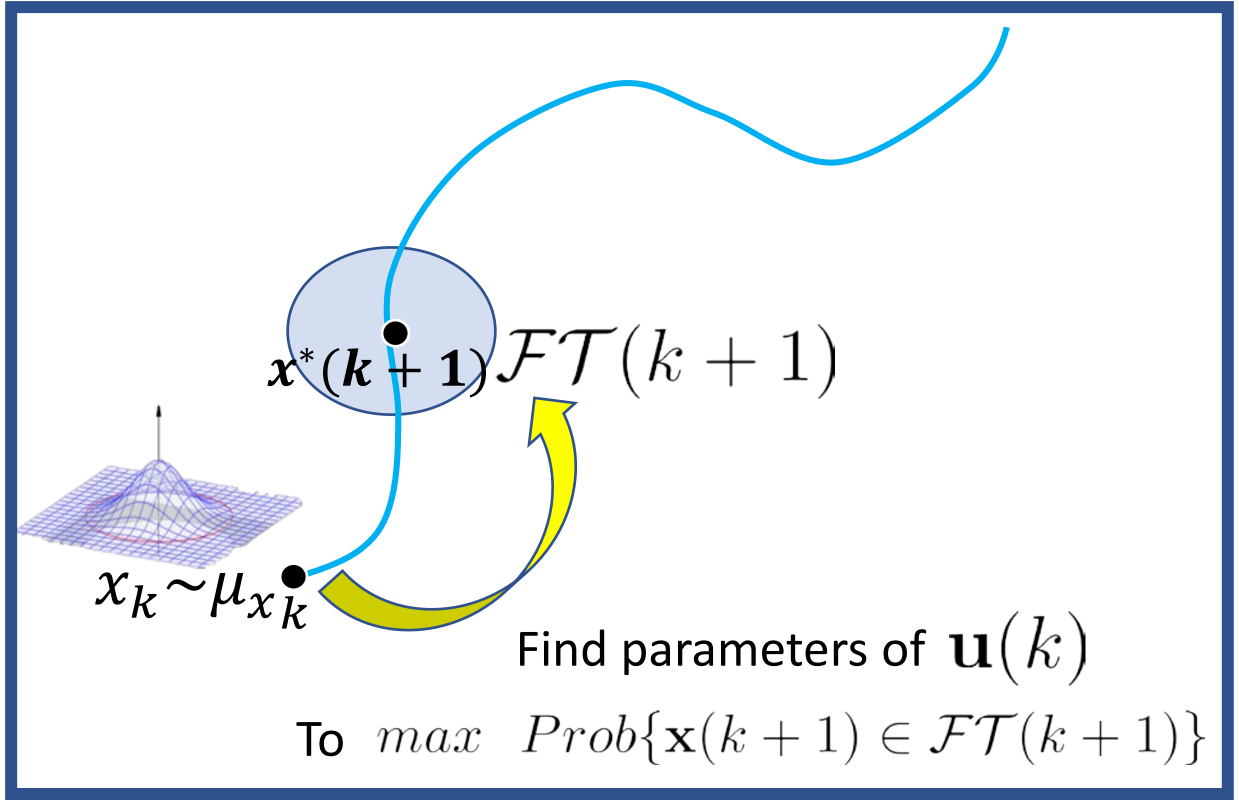
$$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1\,, 0.1]^3) \qquad \omega(k) \sim Beta(2,5)$$

- We want to find the control input at time k, i.e., $u(k)$, such that states $(x_1(k+1), x_2(k+1), x_3(k+1))$ reach the neighborhood of the given way-point $(0,0,0.9)$, i.e. a ball around the way-point $1^2 - \left(\frac{x_1-0}{0.03}\right)^2 - \left(\frac{x_2-0}{0.02}\right)^2 - \left(\frac{x_3-0.9}{0.4}\right)^3 \geq 0$, with a high probability.

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$



Source of uncertainties:

Initial states $(x_1(0), x_2(0), x_3(0)) \sim pr_{x_0}(x_1, x_2, x_3)$

Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$

- Suppose at time k:

$$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1, 0.1]^3) \qquad \omega(k) \sim Beta(2,5)$$

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$$\mathbf{P}^* = \underset{u(k)}{\text{maximize}} \quad \text{Probability}\left(1 - \left(\frac{x_1(k+1)}{0.03}\right)^2 - \left(\frac{x_2(k+1)}{0.02}\right)^2 - \left(\frac{x_3(k+1)}{0.4}\right)^2 \geq 0\right)$$
$$\text{subject to} \qquad -1 \leq u(k) \leq 1$$

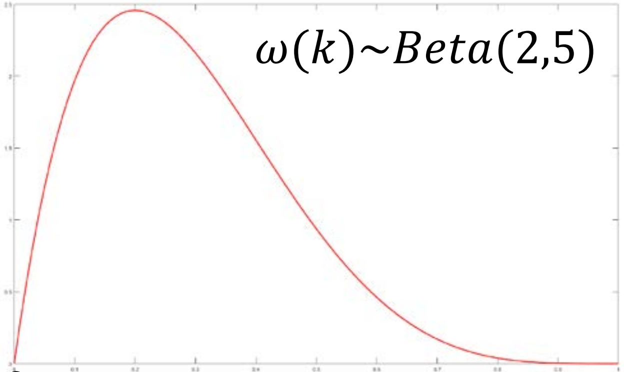
Example: Control of Uncertain Nonlinear System

- $(x_1(k), x_2(k), x_3(k)) \sim U([-0.1, 0.1]^3)$

  $i - th$ moment of $Uniform([a,b])$: $y_i = \frac{1}{b-a}\frac{b^{i+1}-a^{i+1}}{i+1}$

- $\omega_k \sim Beta(5,2)$

  $i - th$ moment of $Beta(\alpha,\beta)$: $y_i = \frac{\alpha+i-1}{\alpha+\beta+i-1} y_{i-1}, y_0 = 1$



$$\mathbf{P}^* = \underset{u(k)}{\text{maximize}} \quad \text{Probability}\left(1 - \left(\frac{\omega(k)x_2(k)}{0.03}\right)^2 - \left(\frac{x_1(k)x_3(k)}{0.02}\right)^2 - \left(\frac{1.2x_1(k) - 0.5x_2(k) + 2u(k)}{0.4}\right)^2 \geq 0\right)$$

$$\text{subject to} \qquad -1 \leq u(k) \leq 1$$

$$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1, 0.1]^3)$$

$$\omega(k) \sim Beta(5,2)$$

Example: Control of Uncertain Nonlinear System

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  $i-th$ moment of $Uniform([a,b])$: $y_i = \frac{1}{b-a}\frac{b^{i+1}-a^{i+1}}{i+1}$

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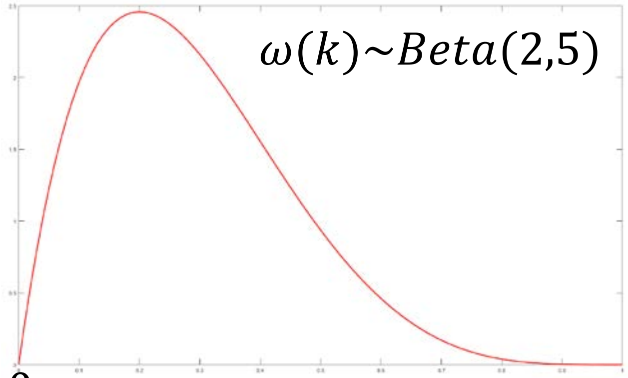


$$\mathbf{P}^* = \underset{u(k)}{\text{maximize}} \quad \text{Probability}\left(1 - \left(\frac{\omega(k)x_2(k)}{0.03}\right)^2 - \left(\frac{x_1(k)x_3(k)}{0.02}\right)^2 - \left(\frac{1.2x_1(k) - 0.5x_2(k) + 2u(k)}{0.4}\right)^2 \geq 0\right)$$

$$\text{subject to} \quad -1 \leq u(k) \leq 1$$

$$(x_1(k), x_2(k), x_3(k)) \sim U([-0.1, 0.1]^3)$$

$$\omega(k) \sim Beta(5,2)$$

d=2 $\quad y_u = [1, 0.476, 0.2601, 0.1260, 0.4934]$

Rank Test:

$$\text{Rank } M_d(y_u) = \text{Rank } M_{d-1}(y_u) \approx 1$$

eigenvalues = 0.0273, 0.3939, 1.3324 eigenvalues = 0.0273, 1.2328

$\Rightarrow$

$u(k) = y_{u_1} = 0.476$ (Instead of open loop control, we can look for the feedback gains)

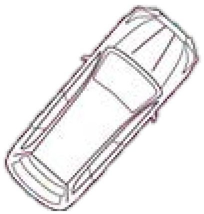
$Prob\ of\ Success = y(1) = 1$

True Prob for u=0.476 obtained by Monte-Carlo = 1

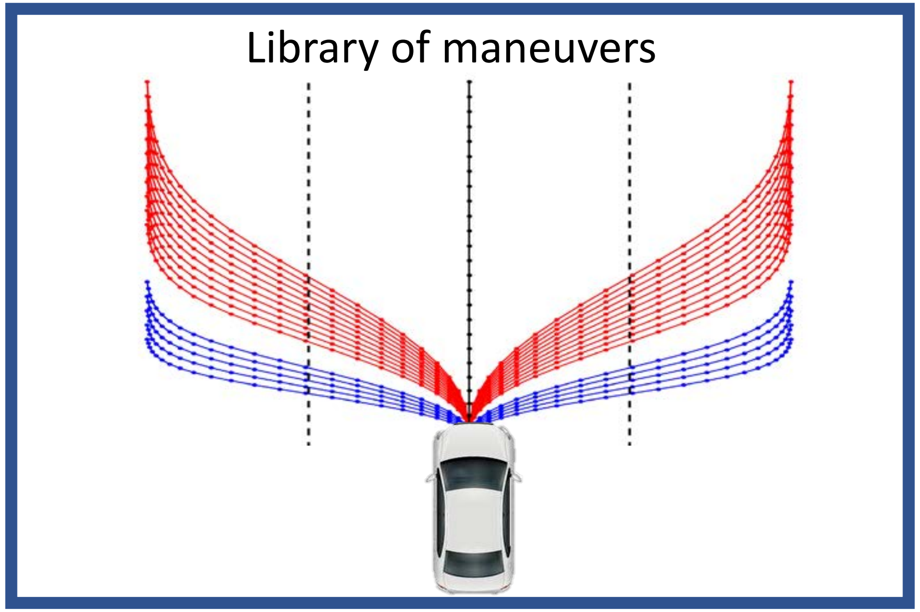
https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_3_Moment_ChanceOpt

Vehicle Control

$$x(k+1) = x(k) + \Delta T(v(k) + 0.1\tilde{v}(k) - 0.05)cos(\theta(k)))$$
$$y(k+1) = y(k) + \Delta T(v(k) + \tilde{v}(k))sin(\theta(k))$$
$$\theta(k+1) = \theta(k) + \Delta T(\psi(k) + 0.2\tilde{\psi}(k) - 0.1)$$

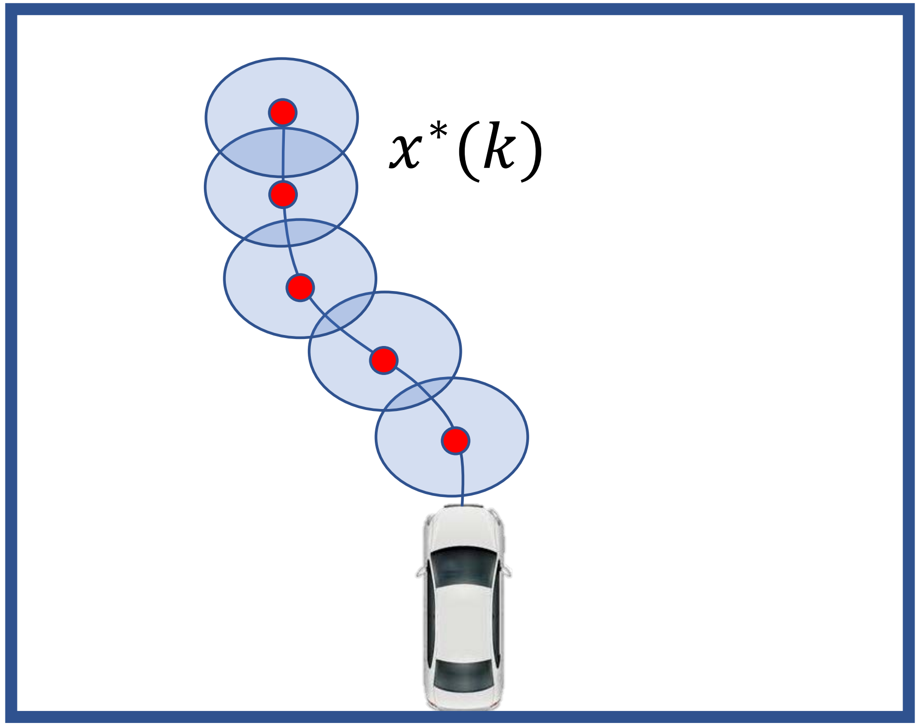


- States (x, y, θ): position and Steering angle
- Control inputs: (v, ψ) Linear and Angular Velocities
- Control Disturbances: $(\tilde{v}, \tilde{\psi})$



➢ Design the maneuvers, tubes, and nonlinear controller in the offline step.

➢ In the real-time, execute the right maneuver.



Vehicle Control

$$x(k+1) = x(k) + \Delta T(v(k) + 0.1\tilde{v}(k) - 0.05)cos(\theta(k))$$
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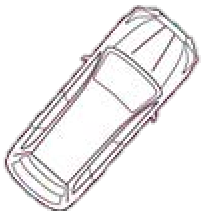
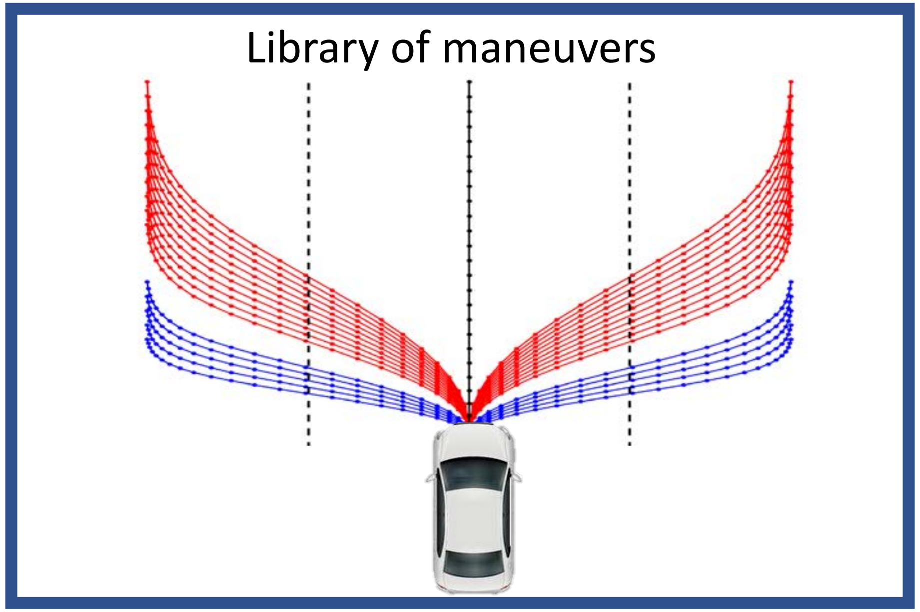


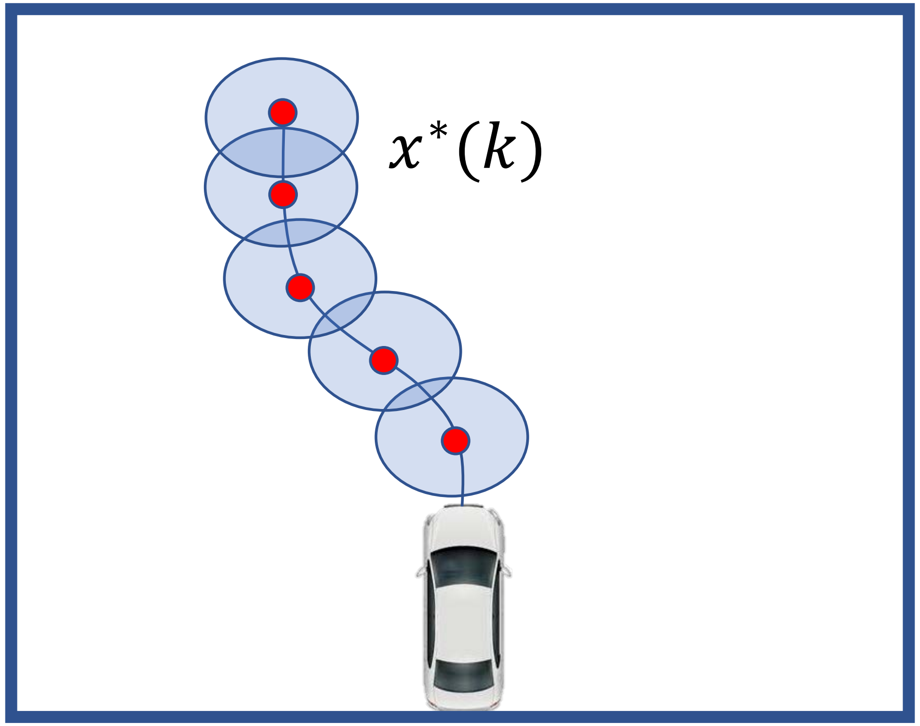
- States (x, y, θ): position and Steering angle
- Control inputs: (v, ψ) Linear and Angular Velocities
- Control Disturbances: $(\tilde{v}, \tilde{\psi})$
- Nominal trajectory for candidate maneuver

$$x^* = \{0, 0.15, 0.3, 0.44, 0.56, 0.66, 0.71, 0.72\}$$
$$y^* = \{0, 0, 0, 0.04, 0.12, 0.24, 0.38, 0.53\}$$
$$\theta^* = \{0, 0, 0.3, 0.6, 0.9, 1.2, 1.5, 1.5708\}$$
$$(v, \psi)^* = \{(1.5, 0), (1.5, 3), (1.5, 3), (1.5, 3), (1.5, 3), (1.5, 3), (1.5, 0.7)\}$$

- Flow-Tube at time step k

$$\mathcal{FT}(k) = \{(x, y) : \ x_k^* - 0.06 \leq x \leq x_k^* + 0.06, y_k^* - 0.06 \leq y \leq y_k^* + 0.06\}$$





Vehicle Control

$$x(k+1) = x(k) + \Delta T(v(k) + 0.1\tilde{v}(k) - 0.05)cos(\theta(k))$$
$$y(k+1) = y(k) + \Delta T(v(k) + \tilde{v}(k))sin(\theta(k))$$
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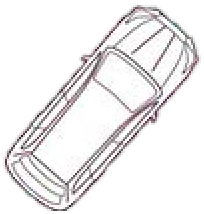

- States (x, y, θ): position and Steering angle
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$$(v, \psi)^* = \{(1.5, 0), (1.5, 3), (1.5, 3), (1.5, 3), (1.5, 3), (1.5, 3), (1.5, 0.7)\}$$

- Flow-Tube at time step k

$$\mathcal{FT}(k) = \{(x, y) : \; x_k^* - 0.06 \leq x \leq x_k^* + 0.06, y_k^* - 0.06 \leq y \leq y_k^* + 0.06\}$$

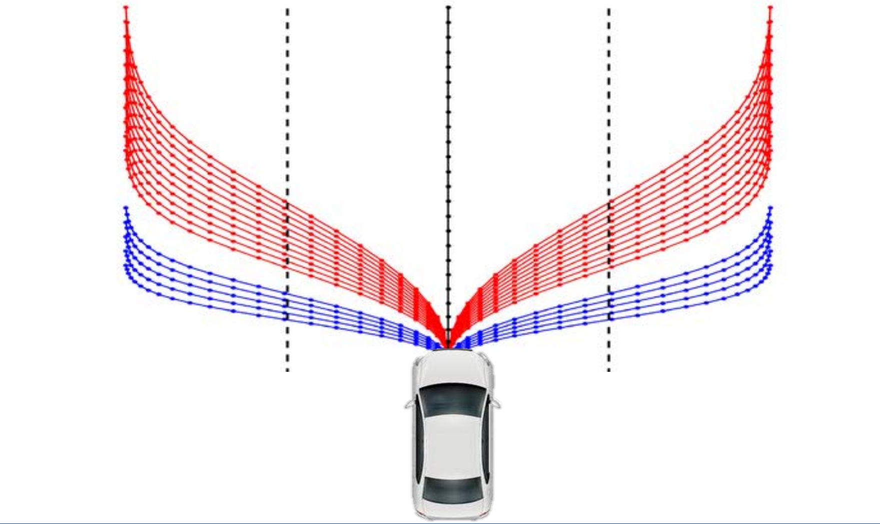
- State Feedback control

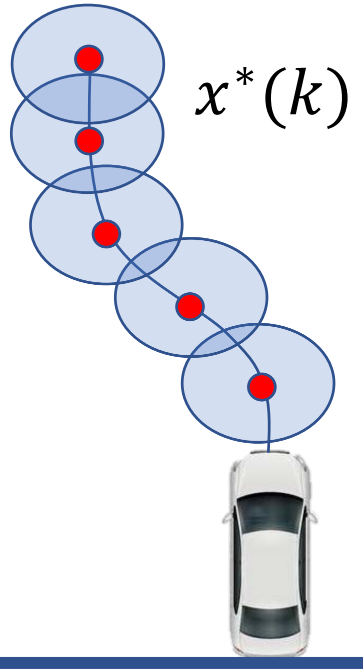
$$v(k) = g_{12}(x(k) - x^*(k)) + g_{22}(y(k) - y^*(k)) + v^*(k)$$
$$\psi(k) = g_{11}(\theta(k) - \theta^*(k)) + g_{21}(x(k) - x^*(k)) + g_{31}(y(k) - y^*(k)) + \psi^*(k)$$

- Control constraints:

$$-10 \leq g_{11}, g_{21}, g_{31}, g_{12}, g_{22} \leq 10$$
$$0 \leq v(k) \leq 2$$

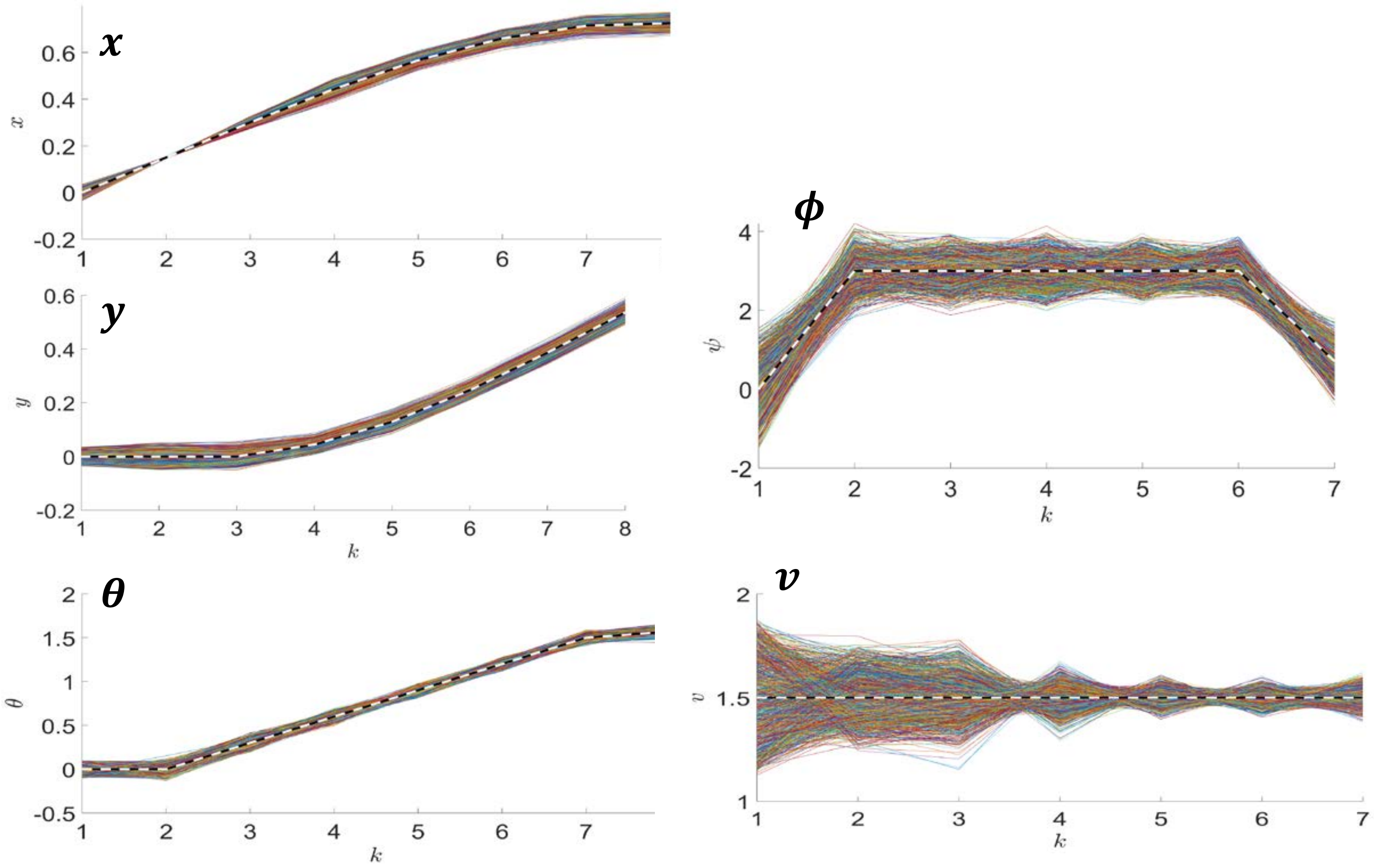
Library of maneuvers



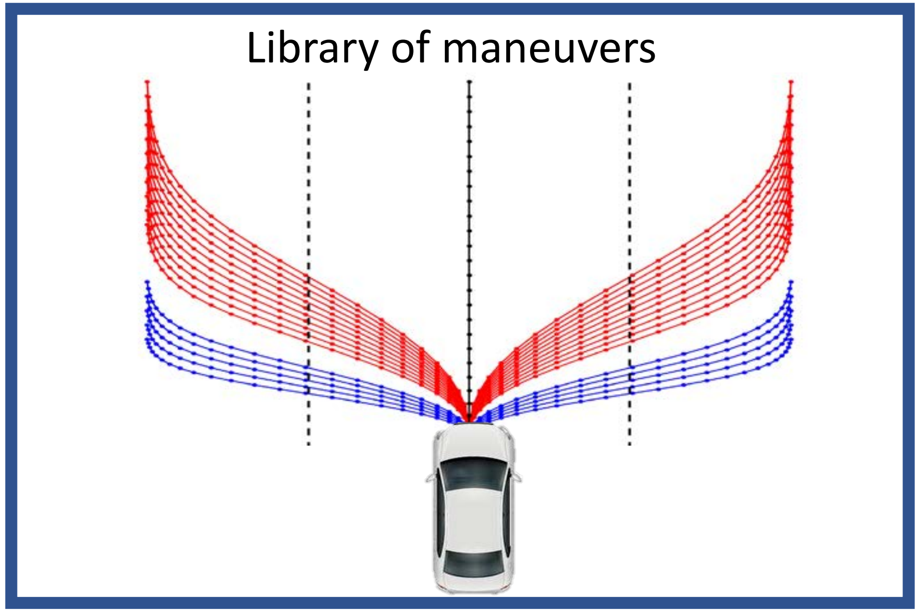


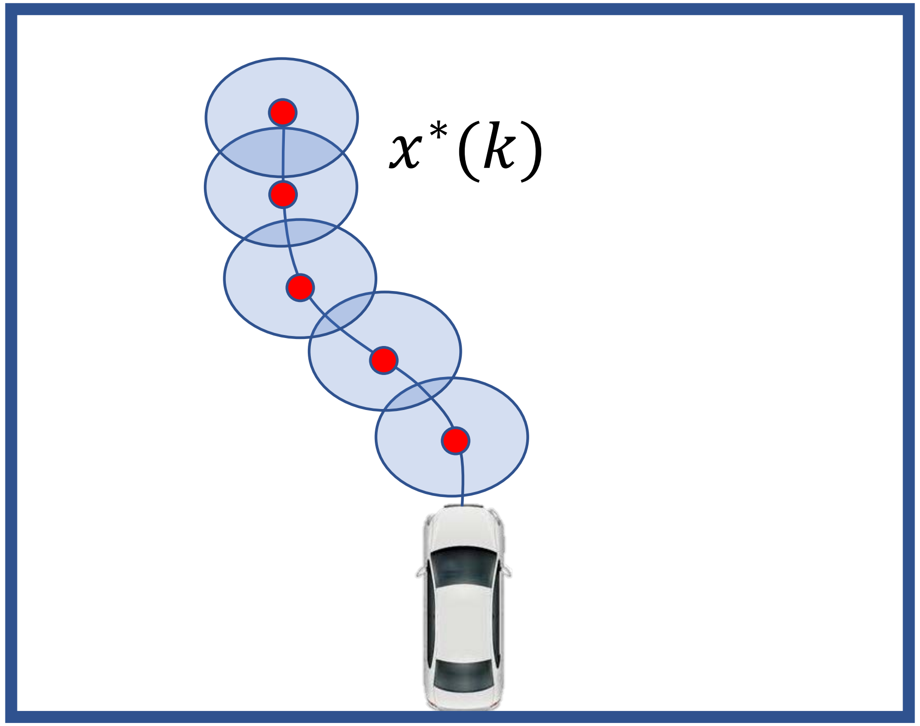
Vehicle Control

To obtain the polynomial dynamics, we compute a degree 3 Taylor expansion of the dynamics of the system around the nominal trajectory, at each time k.

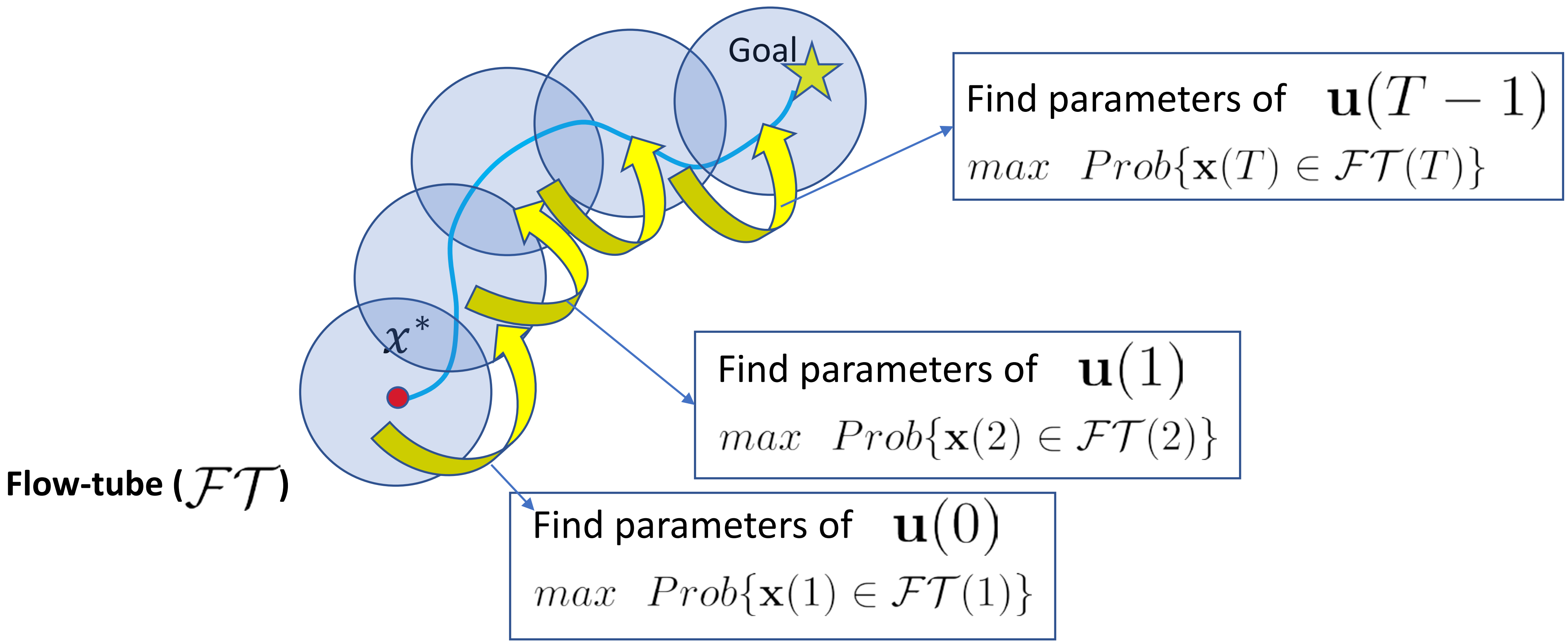


Trajectories and Control inputs of the vehicle for different realization of uncertainties





- Instead of maximizing the chance of reaming inside the tube, we can look for **chance constrained** controllers.



Example: Chance Constrained Formulation

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

Source of uncertainties: Initial states $\big(x_1(0), x_2(0), x_3(0)\big) \sim pr_{x_0}(x_1, x_2, x_3)$

Uncertain Parameter $\omega(k) \sim pr_{\omega_k}(\omega)$

- Suppose at time k: $\big(x_1(k), x_2(k), x_3(k)\big) \sim U([-0.1\,, 0.1]^3)$ $\qquad \omega_k \sim Beta(2{,}5)$
- We want to find a set of control inputs at time k that steer states $(x_1(k+1), x_2(k+1), x_3(k+1))$ to the neighborhood of the given way-point $(0{,}0{,}0.9)$, i.e. a ball around the way-point $1^2 - \left(\frac{x_1-0}{0.03}\right)^2 - \left(\frac{x_2-0}{0.02}\right)^2 - \left(\frac{x_3-0.9}{0.4}\right)^3 \geq 0$, with a probability greater or equal to $1 - \Delta$.

Example: Chance Constrained Formulation

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
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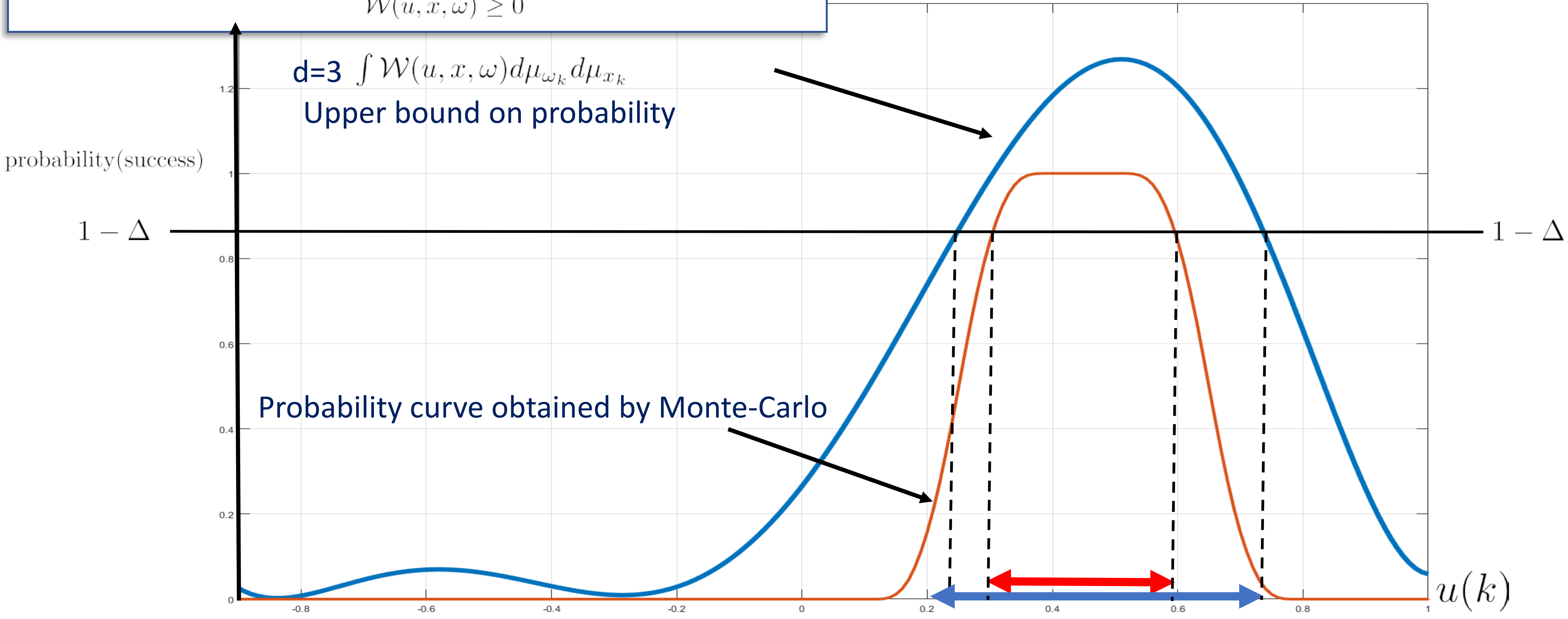
$$U_{cc} = \{u(k) : \text{Probability}(Success) \geq 1-\Delta\}$$
$$= \left\{ u(k) :\ \text{Probability}\left(1 - \left(\frac{x_1(k+1)}{0.03}\right)^2 - \left(\frac{x_2(k+1)}{0.02}\right)^2 - \left(\frac{x_3(k+1)}{0.4}\right)^2 \geq 0\right) \geq 1-\Delta \right\}$$

Example: Control of Uncertain Nonlinear System

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R},\mathcal{W}(u,x,\omega)\in\mathbb{R}_d[u,\omega]}{\text{minimize}} \quad \int \mathcal{W}(u,x,\omega)d\mu_{\omega_k}d\mu_{x_k}du$$
$$\text{subject to} \quad \mathcal{W}(u,x,\omega)-1\geq 0 \quad \forall (u,x,\omega)\in\mathcal{K}$$
$$\mathcal{W}(u,x,\omega)\geq 0$$

$x=[x_1(k),x_2(k)]$

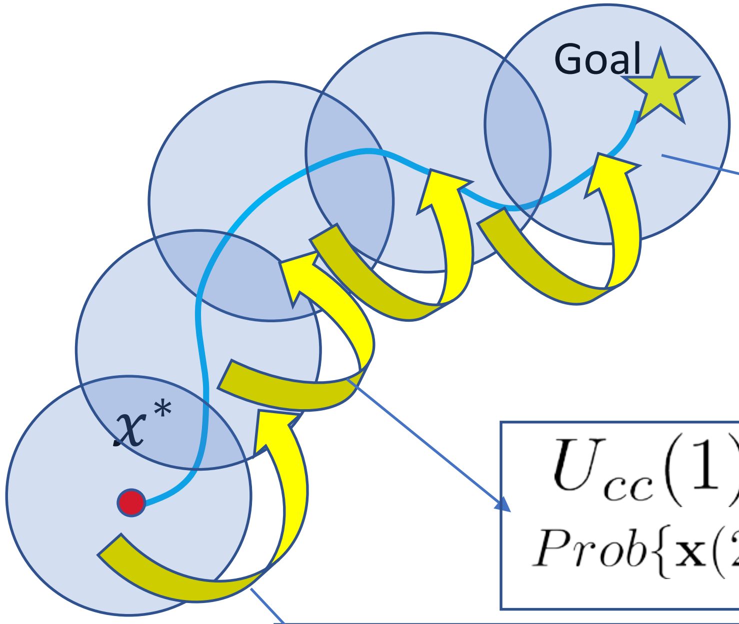


Outer approximation of $U_{cc}=\{u(k): Prob(Success)\geq 1-\Delta\}$**:** $\{u(k): \int \mathcal{W}(u,x,\omega)d\mu_{\omega_k}d\mu_{x_k}\geq 1-\Delta\}$

- We can also fin the inner approximation (Lecture 7)

https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_3_SOS_ChanceConstrained

chance constrained flow-tube based control:



Flow-tube ($\mathcal{FT}$)

$U_{cc}(T-1)$ Set of all control input for which:

$Prob\{\mathbf{x}(T) \in \mathcal{FT}(T) \geq 1 - \Delta\}$

$U_{cc}(1)$ Set of all control input for which:

$Prob\{\mathbf{x}(2) \in \mathcal{FT}(2) \geq 1 - \Delta\}$

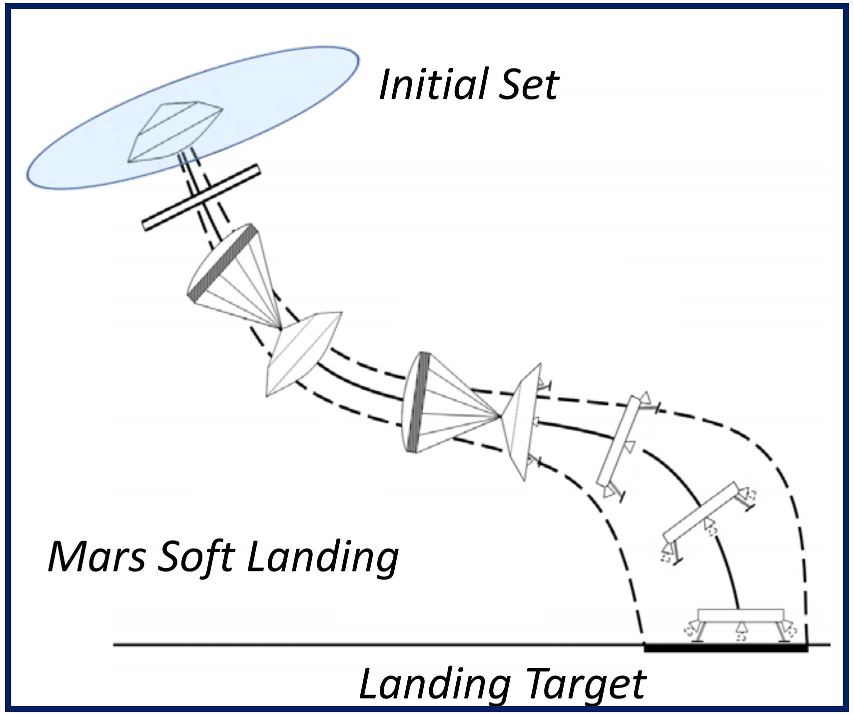
$U_{cc}(0)$ Set of all control input for which:

$Prob\{\mathbf{x}(1) \in \mathcal{FT}(1) \geq 1 - \Delta\}$

Topics:

- Introduction
- Polynomial Representation of Obstacles and Dynamical Systems
- Risk Bounded Trajectory Planning in Uncertain Environments
- Control of Probabilistic Nonlinear Systems
  - Nonlinear State Feedback Control
  - Receding Horizon Control
- Flow-Tube based Control of Probabilistic Nonlinear Systems
- Chance Constrained Backward Reachability Set
- Continuous-Time Path Planning in Uncertain Environments

Backward Reachability Set Analysis of Probabilistic Nonlinear Systems



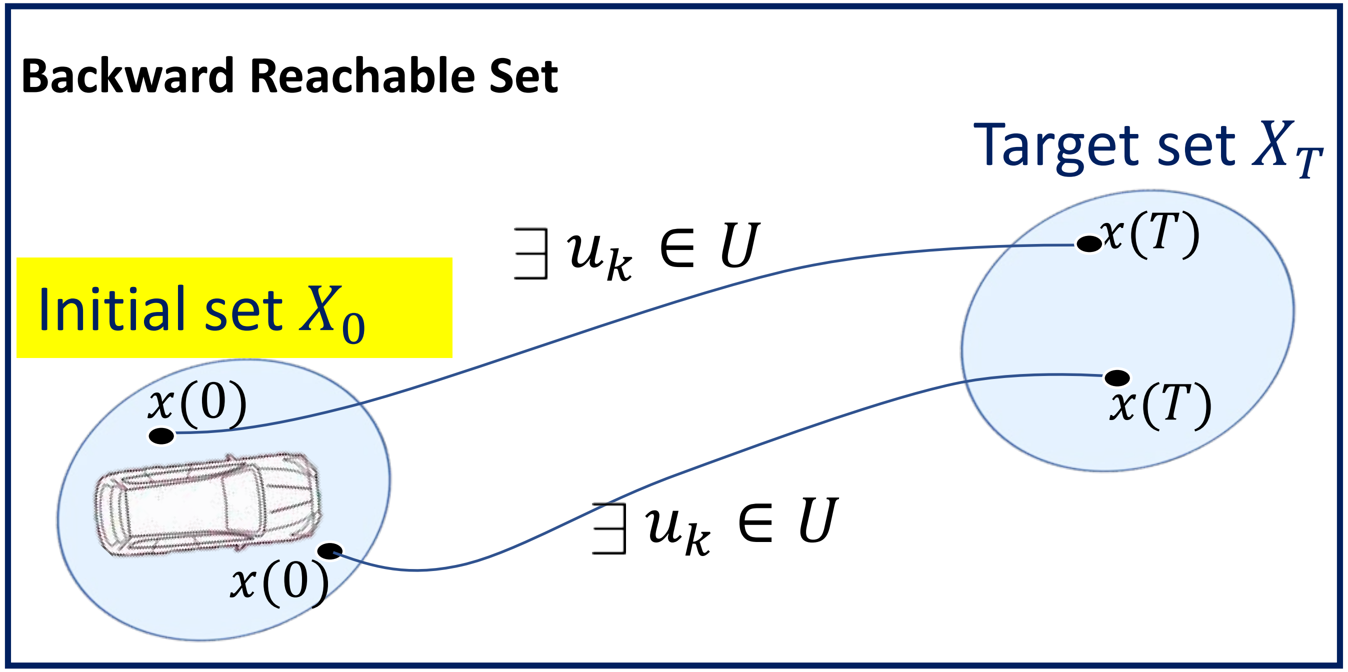
Reachability Set Analysis of Dynamical System

Continuous State space model $x_{k+1} = f(x_k, u_k)$ states inputs

$u_k \in U$ Input Constraints

Target Set X_T

Goal: Find a set of initial states X_0 for which set X_T is reachable in T time steps under input constraints $u_k \in U$



Reachability Set Analysis of Dynamical System

Continuous State space model

$$x_{k+1} = f(x_k, u_k)$$

states inputs

$u_k \in U$

Input Constraints

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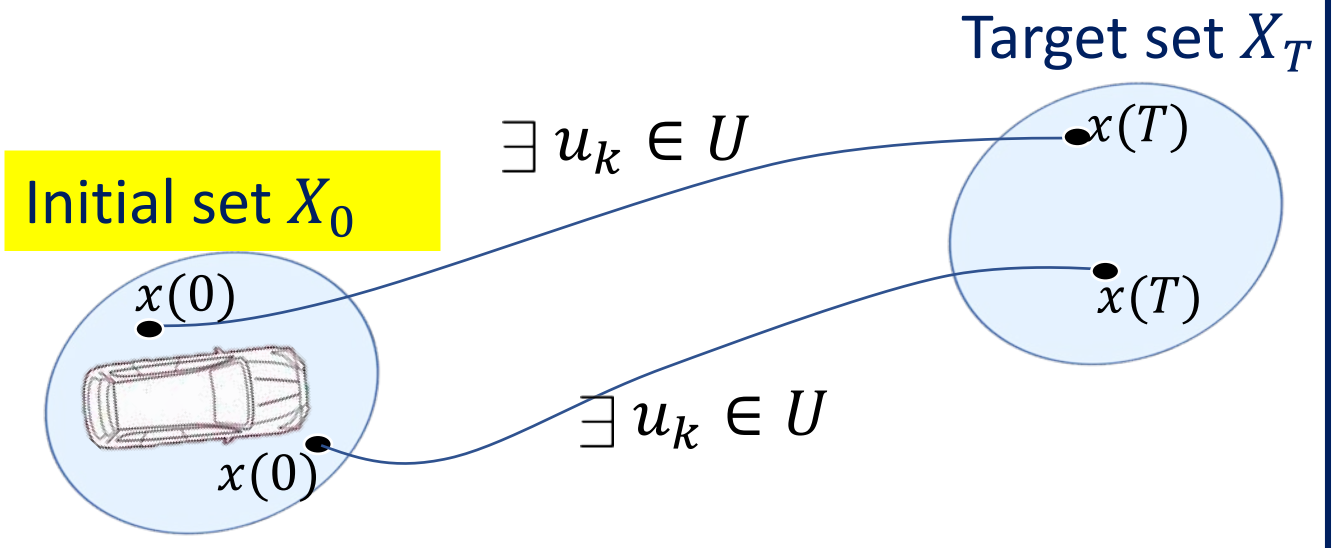
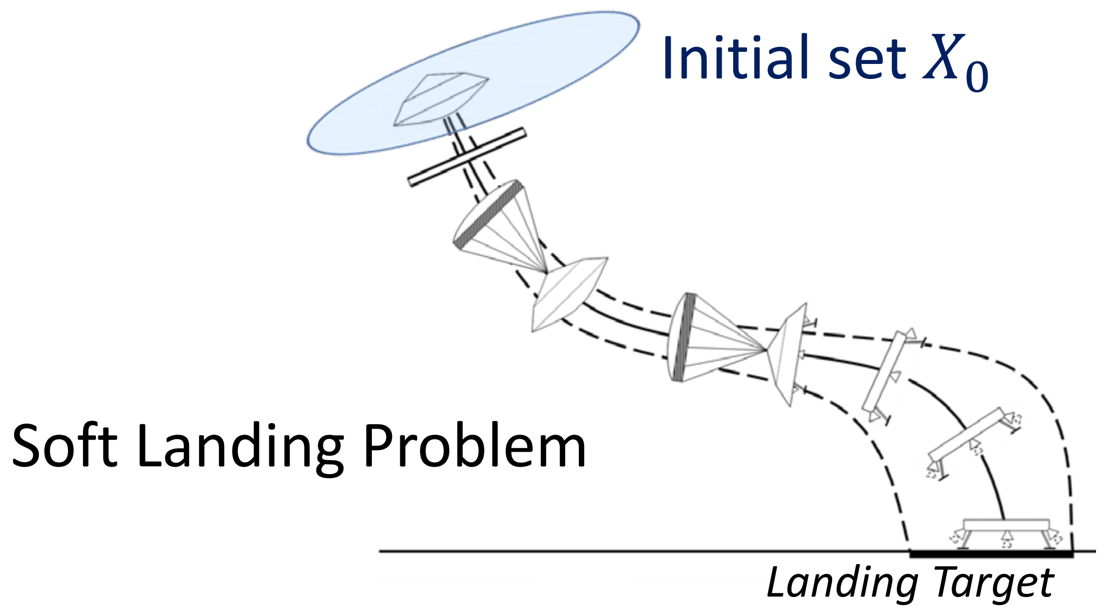
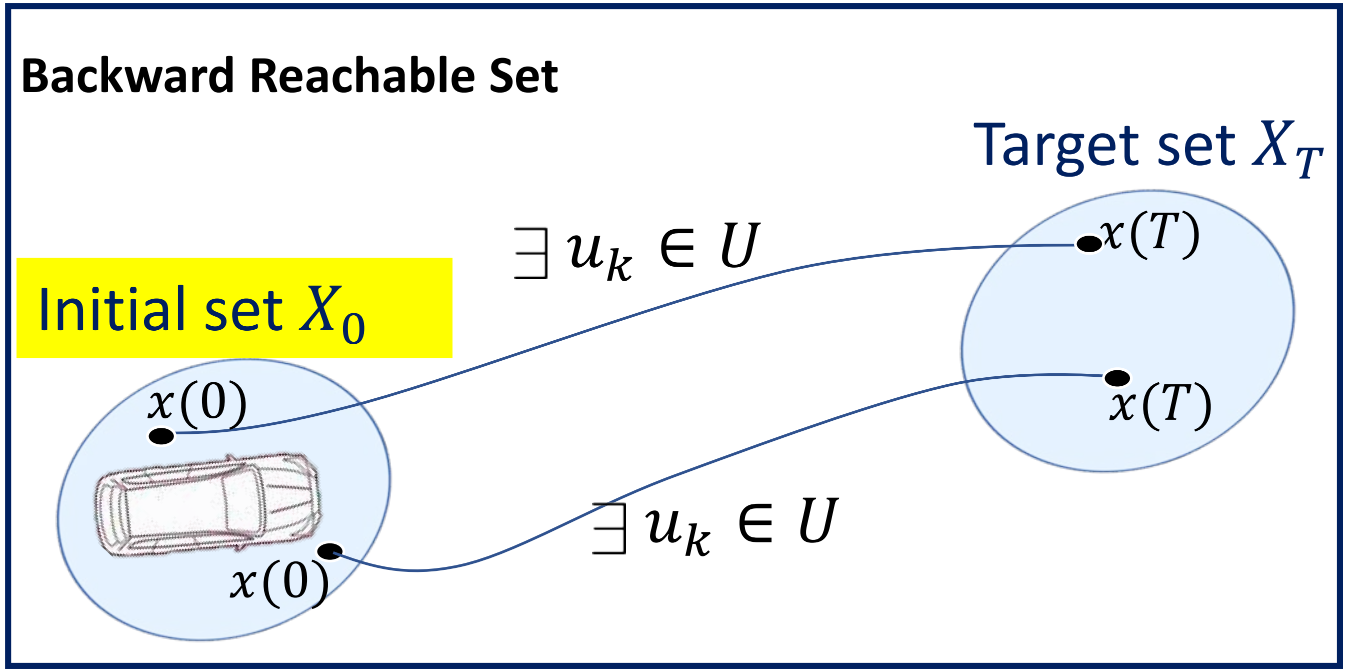
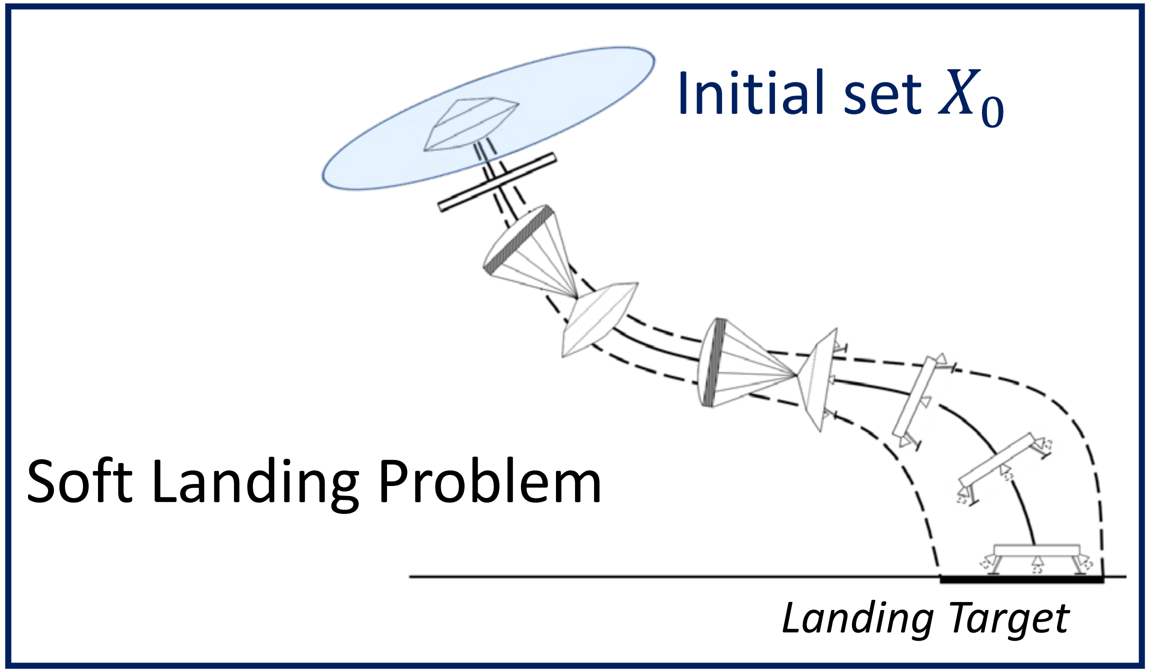
Backward Reachable Set









- To avoid the target set (pedestrian), vehicle at time 0 should be outside of the initial set X_0.
- To reach to the landing site, mars lander should start the landing process from of the initial set X_0.

Chance Constrained Backward Reachability

Continuous State space model $x_{k+1} = f(x_k, u_k, \omega_k)$

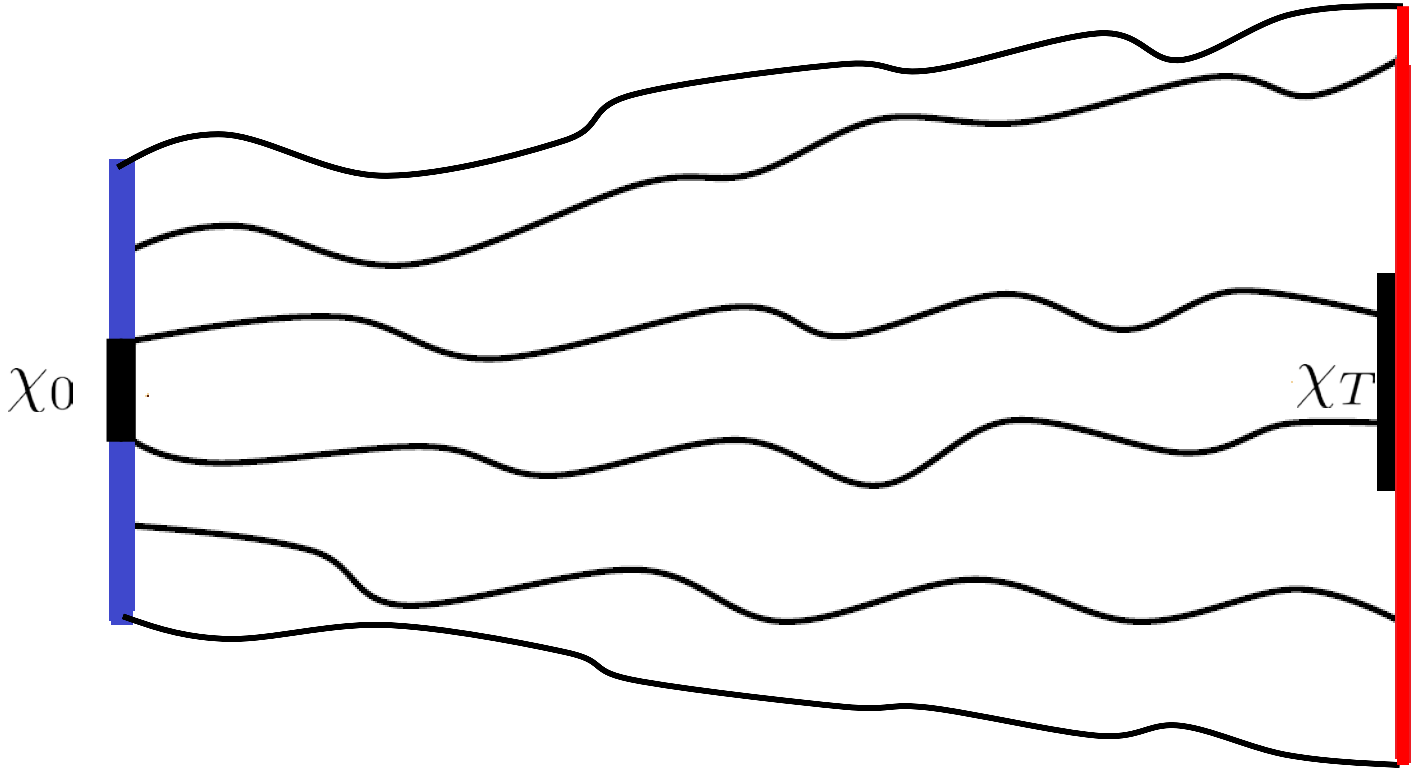
states inputs Uncertainty ~ p(ω_k):probability distribution

$u_k \in U$

Input Constraints

Target Set $X_T(\omega_T)$ Uncertainty ~ p(ω_T):probability distribution

- Uncertain dynamical system: $x_{k+1} = f(x_k, u_k, \omega_k)$



$\text{Probability}(x_T \in \chi_T(\omega)) \geq 1 - \Delta$

$$\chi_0^{\Delta} = \{x_0 : \text{Probability}(x_T \in \chi_T(\omega)) \geq 1 - \Delta\}$$

Chance Constrained Backward Reachability

Continuous State space model $x_{k+1} = f(x_k, u_k, \omega_k)$

states inputs Uncertainty ~ $p(\omega_k)$:probability distribution

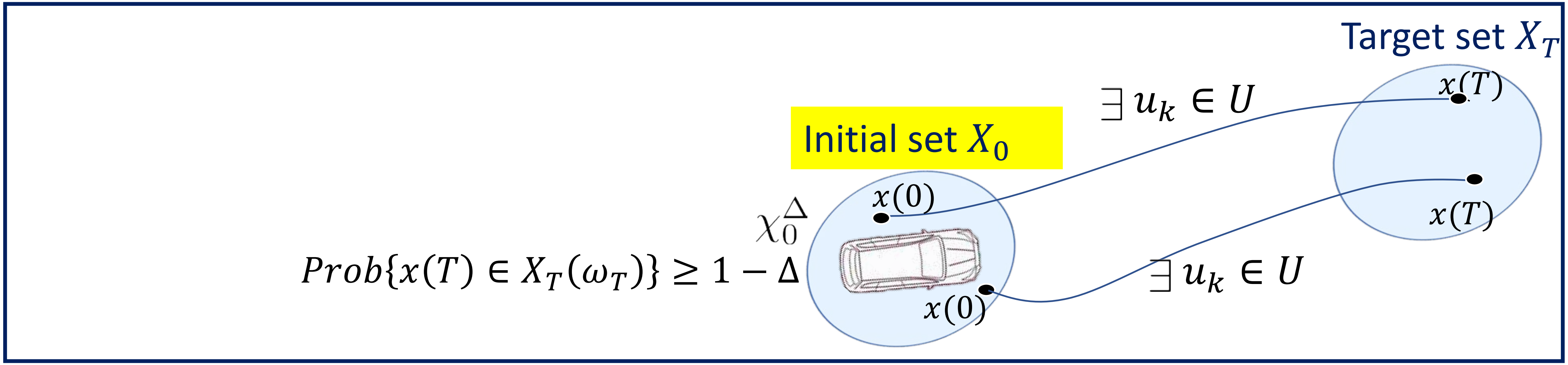
$u_k \in U$

Input Constraints

Target Set $X_T(\omega_T)$ Uncertainty ~ $p(\omega_T)$:probability distribution

Goal: Find a set of initial states X_0 for which Probability that set X_T is reachable in T time steps under input constraints is greater than $1 - \Delta$.

$$\chi_0^{\Delta} = \{x_0 : \text{Probability}(x_T \in \chi_T(\omega)) \geq 1 - \Delta\}$$



Chance Constrained Backward Reachability

Goal: Find a set of initial states X_0 for which Probability that set X_T is reachable in T time steps under input constraints is greater than $1 - \Delta$.

$$\chi_0^{\Delta} = \{x_0 : \text{Probability}(x_T \in \chi_T(\omega)) \geq 1 - \Delta\}$$

- Target Set: $\chi_T(v) = \{x : g(x, v) \geq 0\}$
- States at time step T: $x_T = p_T(x_0, u_k|_{k=0}^{T-1}, \omega_k|_{k=0}^{T-1})$

Chance Constrained Backward Reachability

Goal: Find a set of initial states X_0 for which Probability that set X_T is reachable in T time steps under input constraints is greater than $1 - \Delta$.

$$\chi_0^{\Delta} = \{x_0 : \text{Probability}(x_T \in \chi_T(\omega)) \geq 1 - \Delta\}$$

- Target Set: $\chi_T(v) = \{x : g(x, v) \geq 0\}$
- States at time step T: $x_T = p_T(x_0, u_k|_{k=0}^{T-1}, \omega_k|_{k=0}^{T-1})$
- $x_T \in \chi_T(v) \xrightarrow[g(x_T, v) \geq 0]{} \underbrace{g\left(p_T(x_0, u_k|_{k=0}^{T-1}, \omega_k|_{k=0}^{T-1}), v\right)}_{\mathcal{P}(x_0, u_k|_{k=0}^{T-1}, \omega_k|_{k=0}^{T-1}, v)} \geq 0$

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Design parameter

?

Uncertain parameters with known probability distribution

Chance Constrained Backward Reachability

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Design parameter

Uncertain parameters with known probability distribution

$u_k \sim Uniform$ over the constraint set U

Chance Constrained Backward Reachability

Goal: Find a set of initial states X_0 for which Probability that set X_T is reachable in T time steps under input constraints is greater than $1-\Delta$.

$$\chi_0^{\Delta} = \{x_0 : \text{Probability}\left(\mathcal{P}(\underbrace{x_0}_{\text{Design parameter}}, \underbrace{u_k|_{k=0}^{T-1}}_{u_k \sim Uniform \text{ over the constraint set } U}, \underbrace{\omega_k|_{k=0}^{T-1}, v}_{\text{Uncertain parameters with known probability distribution}}) \geq 0\right) \geq 1-\Delta\}$$

χ_0^{Δ} : Chance constrained set with respect to uncertainties u_k, ω_k, v and design variable x_0.

$$\chi_0^{\Delta} = \{x_0 : \text{Probability}\left(\mathcal{P}(x_0, u_k|_{k=0}^{T-1}, \omega_k|_{k=0}^{T-1}, v) \geq 0\right) \geq 1-\Delta\}$$

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Using the results of Lecture 7:Nonlinear Chance Constrained and Chance Optimization: $\Downarrow$ $\chi_0^{\Delta} = \{x_0 \in \mathbb{R}^n : \mathcal{P}(x) \geq 1-\Delta\}$

- A. Jasour, B. Williams, “Chance Constrained Backward Reachable Set For Probabilistic Nonlinear Systems”, (To appear)

Topics:

- ➢ Introduction
- ➢ Polynomial Representation of Obstacles and Dynamical Systems
- ➢ Risk Bounded Trajectory Planning in Uncertain Environments
- ➢ Control of Probabilistic Nonlinear Systems
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- ➢ Flow-Tube based Control of Probabilistic Nonlinear Systems
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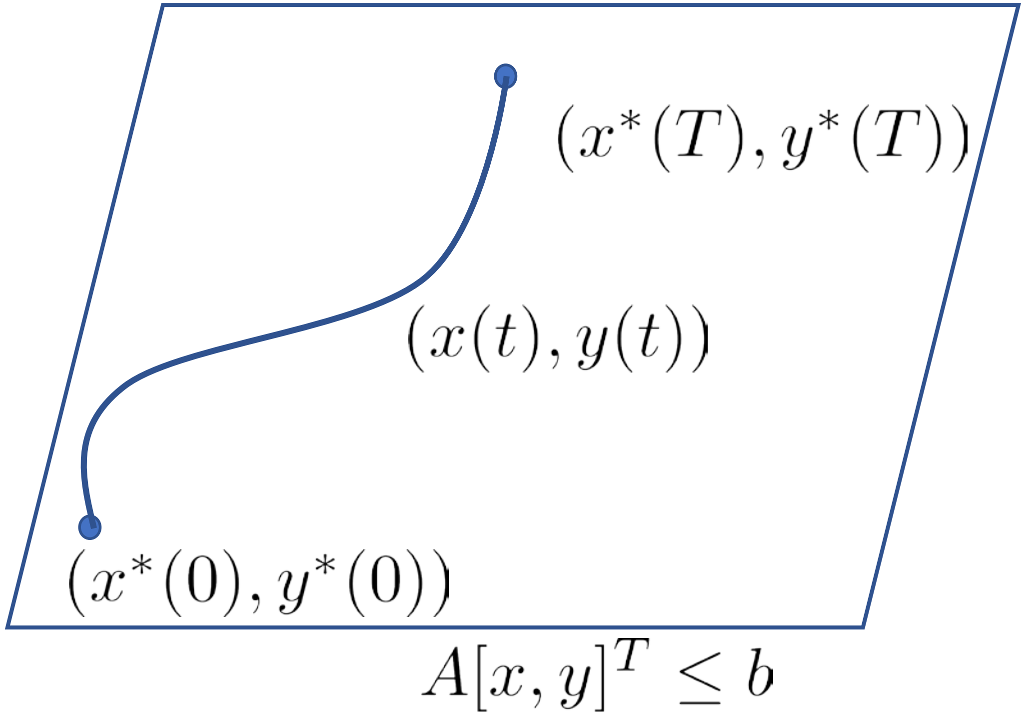
- ➢ Continuous-Time Path Planning

Sum of Squares Based Continuous Time Path Planning

- To design trajectories for dynamical systems, we can rely on motion planning algorithms (e.g., trajectory optimization, rrt^*, PRM,….)

- In this section, we look at one possible Sum-of-Squares based technique.

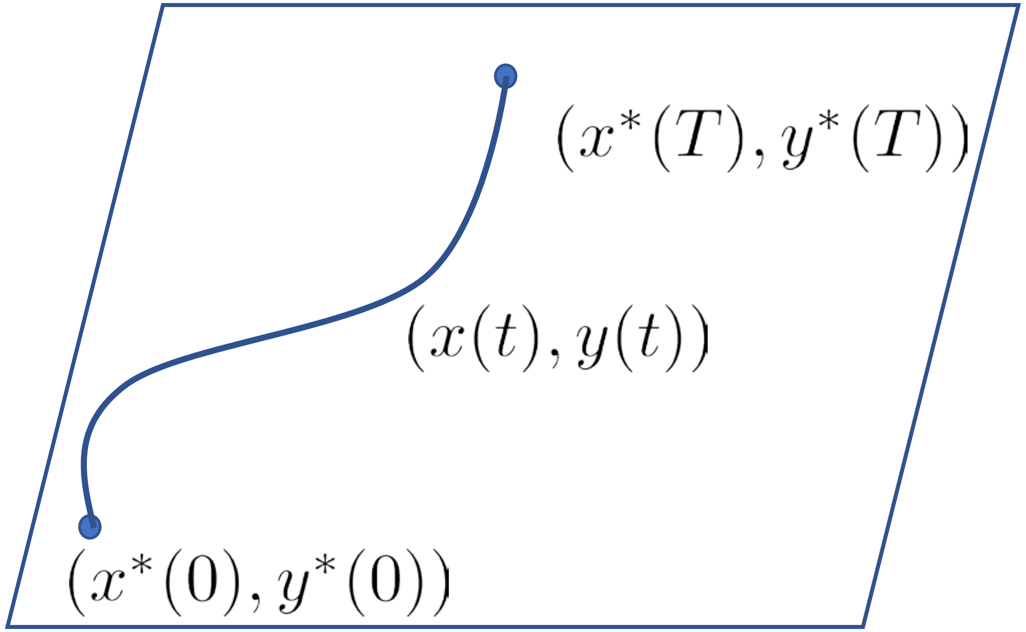
➢ Given, initial and goal points inside a convex polytope (safe region), Find a polynomial trajectory that i) Connects the given points and ii) is safe over its entire length (remains inside the safe region).



➢ Given, initial and goal points inside a convex polytope (safe region), Find a polynomial trajectory that i) Connects the given points and ii) is safe over its entire length (remains inside the safe region).

polynomial trajectory :

$$\begin{cases} x(t) = p_x(t) = \sum_{i=1}^{d} c_{x_i} t^i & \text{Polynomial in time with unknown coefficients } c_{x_i} \\ y(t) = p_y(t) = \sum_{i=1}^{d} c_{y_i} t^i & \text{Polynomial in time with unknown coefficients } c_{y_i} \end{cases}$$



$A[x, y]^T \leq b$

- Given, initial and goal points inside a convex polytope (safe region), Find a polynomial trajectory that i) Connects the given points and ii) is safe over its entire length (remains inside the safe region).

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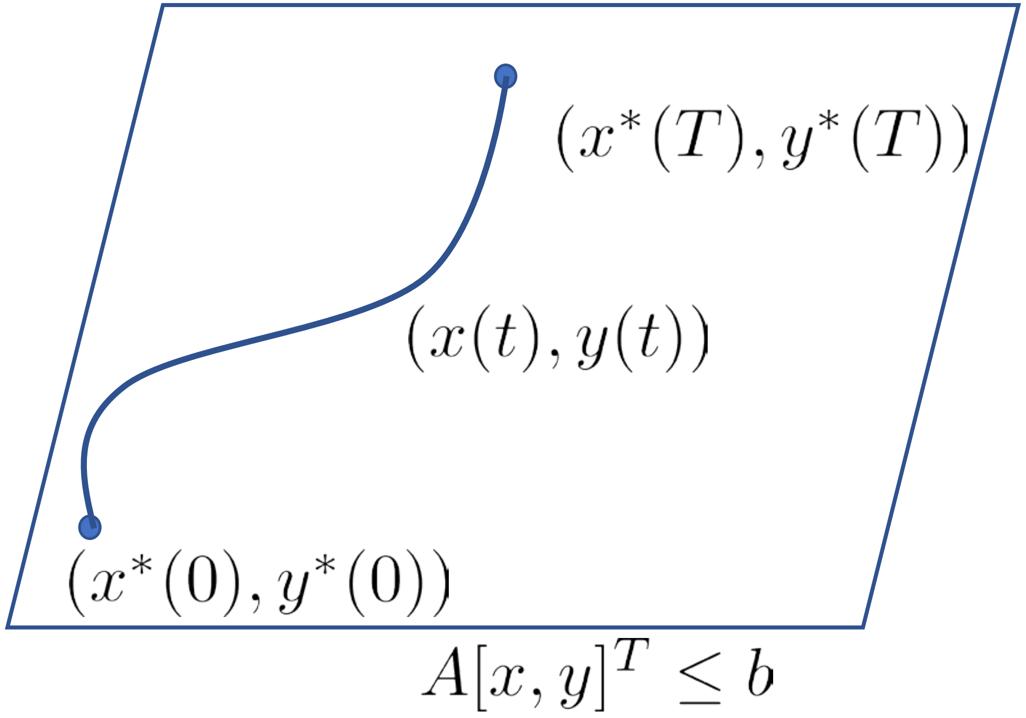
Boundary conditions:

$$\left.\begin{array}{l} (x^*(0), y^*(0)) = (p_x(0), p_y(0)) \\ (x^*(T), y^*(T)) = (p_x(T), p_y(T)) \end{array}\right\} \text{Linear constraints}$$



- Given, initial and goal points inside a convex polytope (safe region), Find a polynomial trajectory that i) Connects the given points and ii) is safe over its entire length (remains inside the safe region).

polynomial trajectory :

$$\begin{cases} x(t) = p_x(t) = \sum_{i=1}^{d} c_{x_i} t^i \\ y(t) = p_y(t) = \sum_{i=1}^{d} c_{y_i} t^i \end{cases}$$

Polynomial in time with unknown coefficients c_{x_i}

Polynomial in time with unknown coefficients c_{y_i}

Boundary conditions:

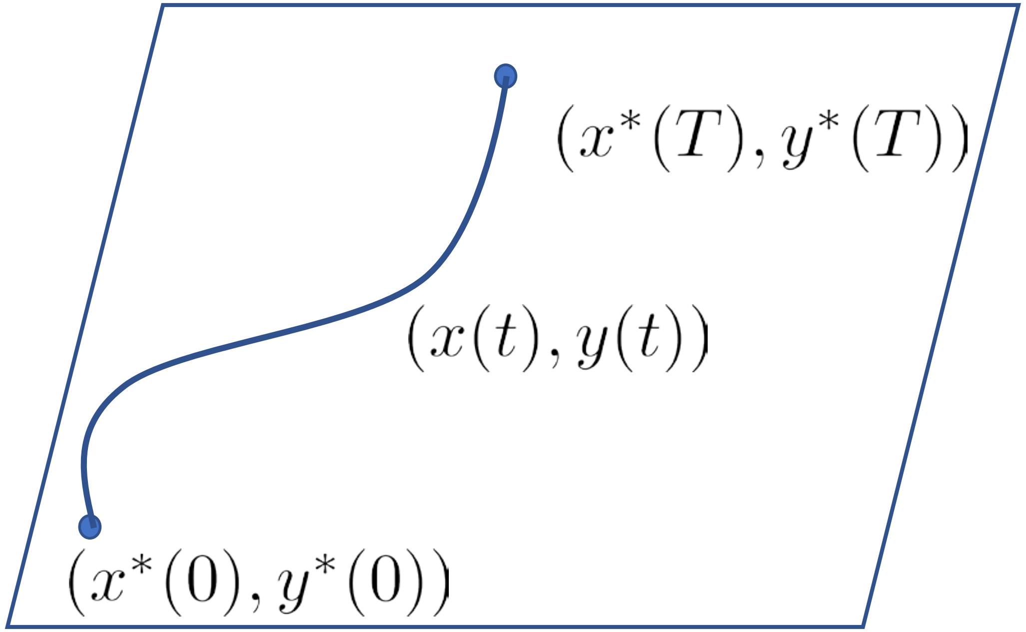
$$(x^*(0), y^*(0)) = (p_x(0), p_y(0))$$
$$(x^*(T), y^*(T)) = (p_x(T), p_y(T))$$

Linear constraints

Safety Constraints:

$$b - A[x(t), y(t)]^T \geq 0 \quad \forall t = [0, T]$$

SOS constraints



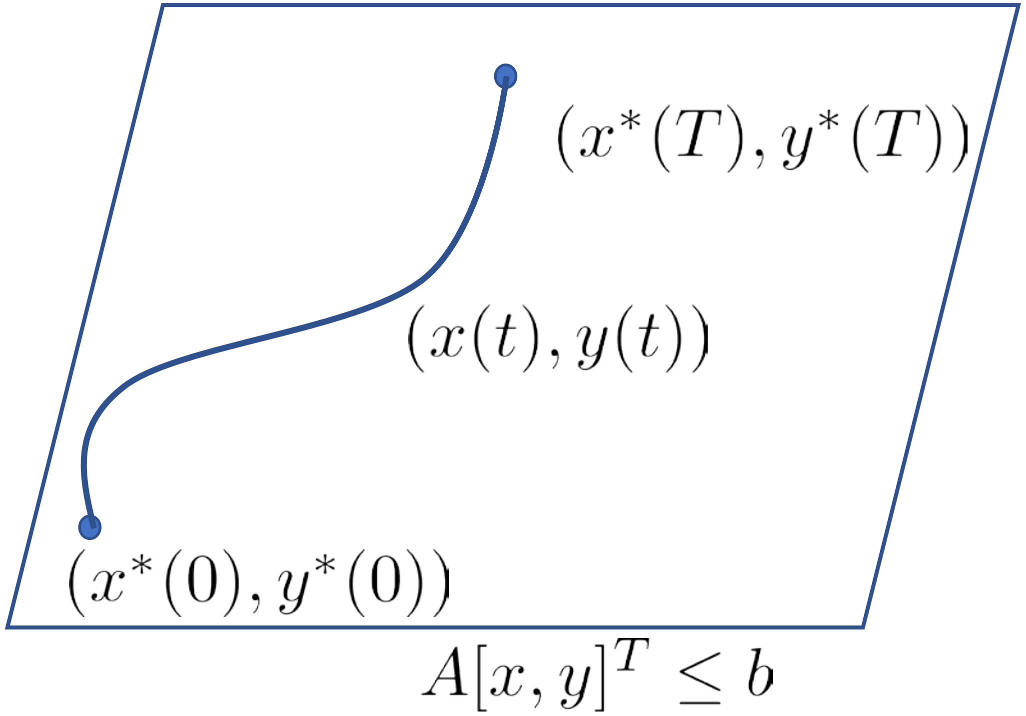
$$A[x, y]^T \leq b$$

- Similarly, we can add additional constraints on velocity, acceleration

- Given, initial and goal points inside a convex polytope (safe region), Find a polynomial trajectory that i) Connects the given points and ii) is safe over its entire length (remains inside the safe region).

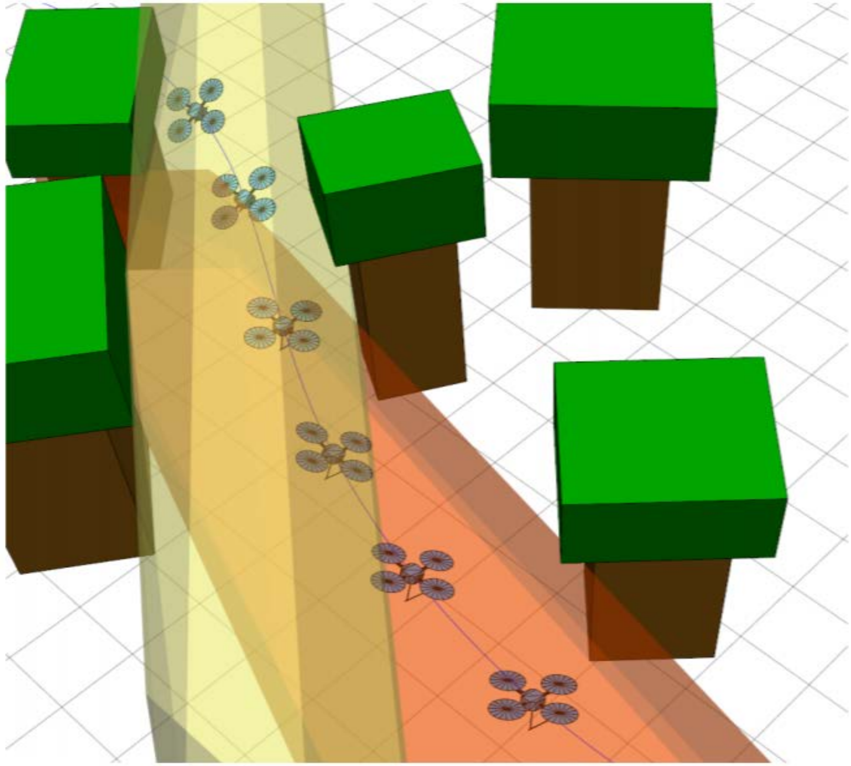
polynomial trajectory : $x(t) = p_x(t) = \sum_{i=1}^{d} c_{x_i} t^i \qquad y(t) = p_y(t) = \sum_{i=1}^{d} c_{y_i} t^i$

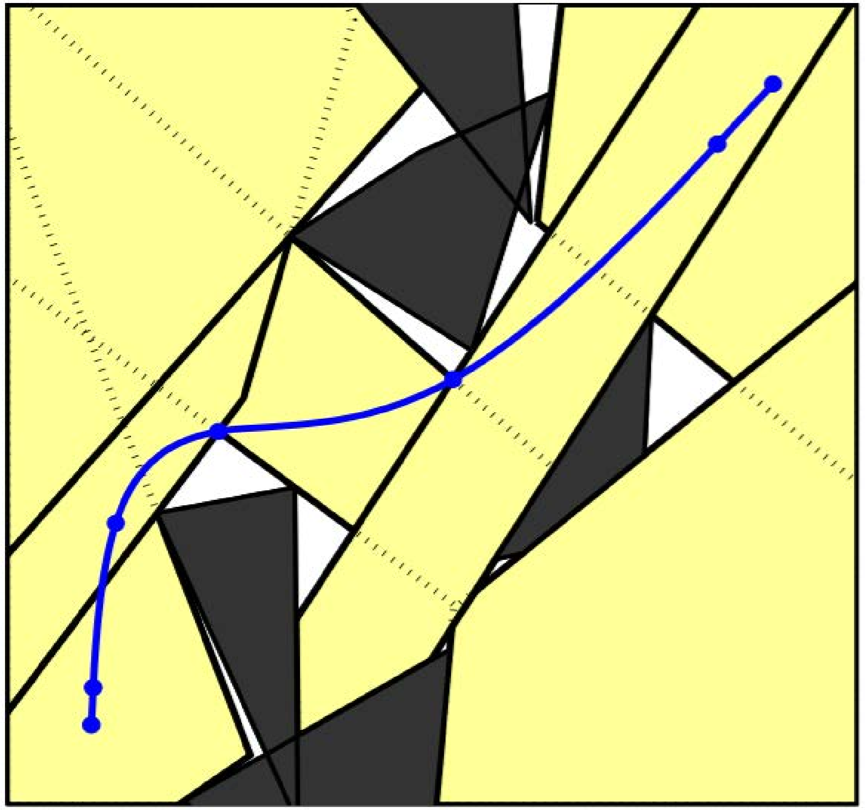
- SOS based SDP in the coefficients of the trajectories $p_x(t)$ and $p_y(t)$



- ➢ Given, i) a set of convex regions that covers the obstacle-free space, and ii) initial and goal point
- ➢ We can formulate the trajectory planning problem as mixed-integer SDP.

    - integer variable for each convex region to choose the sequence of regions to construct the trajectory.
    - SDP for each convex region.





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- A. Jasour, B. Williams, "Chance Constrained Backward Reachable Set For Probabilistic Nonlinear Systems", (To appear)