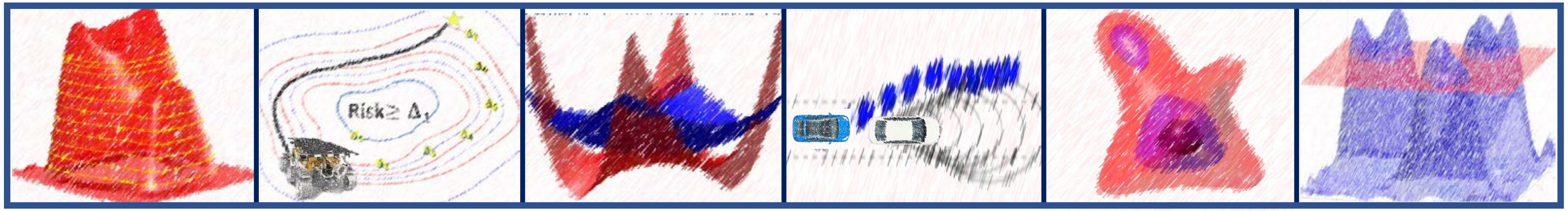
Lecture 10

Probabilistic Nonlinear Safety Verification

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

Ashkan Jasour



Topics

- Design Problems Under Uncertainty
- Probabilistic Safety Verification Problems

Design Problems Under Uncertainty

Nonlinear Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \ i = 1, ..., n_g \end{aligned}$$

Chance Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{pr(\omega)}(\ p_i(x,\omega) \geq 0, \ i = 1, ..., n_p\ ) \\ \text{subject to} \quad & g_i(x) \geq 0, \ i = 1, ..., n_g \end{aligned}$$

Chance Constrained Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \text{Probability}_{pr(\omega)}(\ g_i(x,\omega) \geq 0, \ i = 1, ..., n_g\ ) \geq 1 - \Delta \end{aligned}$$

Robust Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x,\omega) \geq 0, \ i = 1, ..., n_g, \quad \forall \omega \in \Omega \end{aligned}$$

Distributionally Robust Chance Constrained Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \omega \sim \text{pr}(\omega, a), \ a \in \mathcal{A} \\ & \text{Probability}_{\text{pr}(\omega,a)}(\ g_i(x,\omega) \geq 0, \ i = 1, ..., n_g\ ) \geq 1 - \Delta, \quad \forall a \in \mathcal{A} \end{aligned}$$

Design Problems Under Uncertainty

Nonlinear Optimization

$$\begin{array}{ll}\underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x)\\ \text{subject to} & g_i(x)\geq 0,\ i=1,...,n_g\end{array}$$

⇨ SOS / Moment Based SDP Relaxation

Chance Optimization

$$\begin{array}{ll}\underset{x\in\mathbb{R}^n}{\text{maximize}} & \text{Probability}_{pr(\omega)}(\ p_i(x,\omega)\geq 0,\ i=1,...,n_p\ )\\ \text{subject to} & g_i(x)\geq 0,\ i=1,...,n_g\end{array}$$

⇨ SOS / Moment Based SDP Relaxation

Chance Constrained Optimization

$$\begin{array}{ll}\underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x)\\ \text{subject to} & \text{Probability}_{pr(\omega)}(\ g_i(x,\omega)\geq 0,\ i=1,...,n_g\ )\geq 1-\Delta\end{array}$$

⇨ SOS / Moment Based SDP Relaxation

Robust Optimization

$$\begin{array}{ll}\underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x)\\ \text{subject to} & g_i(x,\omega)\geq 0,\ i=1,...,n_g,\quad \forall\omega\in\Omega\end{array}$$

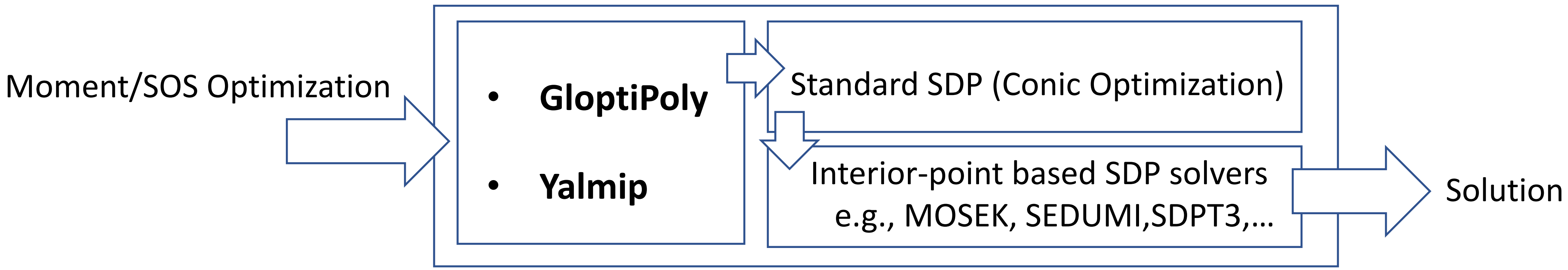
⇨ SOS / Moment Based SDP Relaxation

Distributionally Robust Chance Constrained Optimization

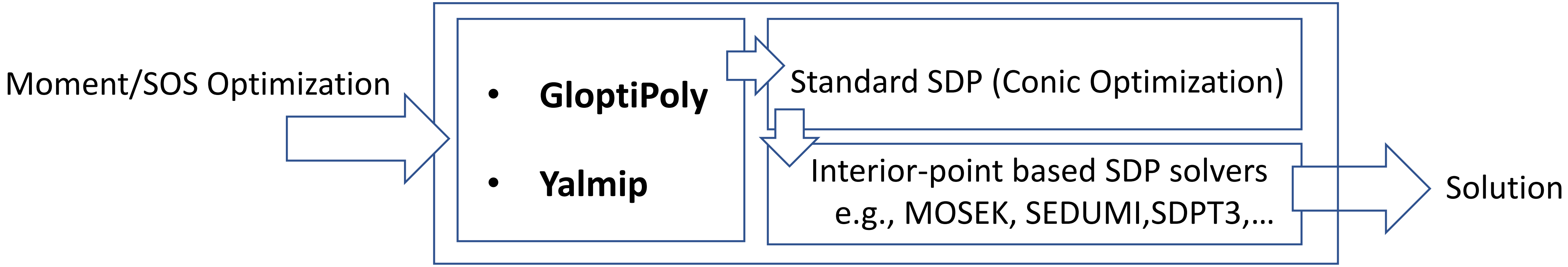
$$\begin{array}{ll}\underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x)\\ \text{subject to} & \omega\sim\text{pr}(\omega,a),\ a\in\mathcal{A}\\ & \text{Probability}_{\text{pr}(\omega,a)}(\ g_i(x,\omega)\geq 0,\ i=1,...,n_g\ )\geq 1-\Delta,\quad \forall a\in\mathcal{A}\end{array}$$

⇨ SOS / Moment Based SDP Relaxation

- SOS and Moment based SDP for nonlinear optimization under uncertainty.
- **GloptiPoly** package for Moment SDPs and **Yalmip** package for SOS SDPs.

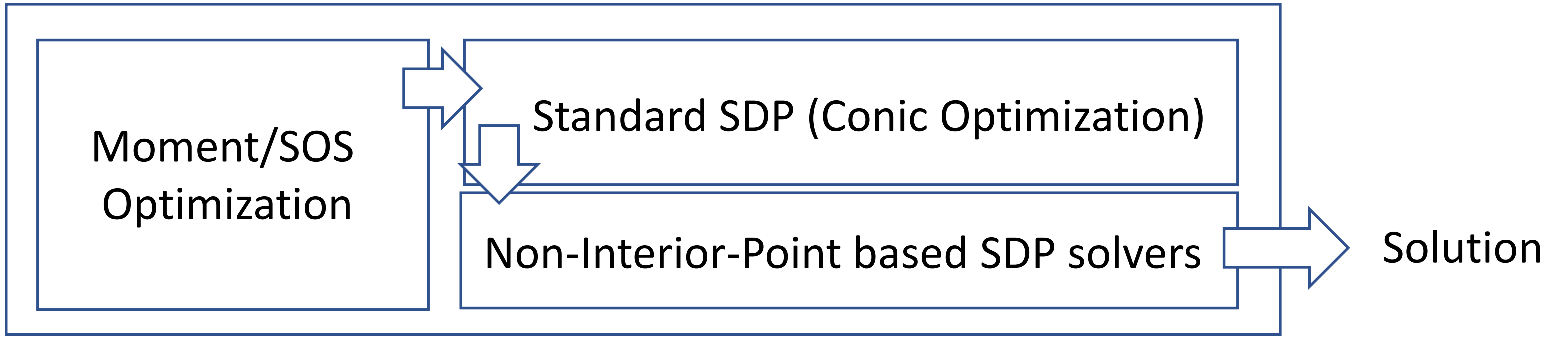


- SOS and Moment based SDP for nonlinear optimization under uncertainty.
- **GloptiPoly** package for Moment SDPs and **Yalmip** package for SOS SDPs.



- In order to use "**Non-Interior-Point**" based SDP solvers to solve **large scale SDPs**, we need to either:

  1) Extract the constructed SDP from Ylamip/GloptiPoly or **2)** Construct the Moment/SOS SDP directly.



➢ To construct the SOS SDP:

Example: $\min_{x} p(x)$

SOS program:

$$\begin{array}{ll} \underset{\gamma}{\text{maximize}} & \gamma \\ \text{subject to} & p(x) - \gamma \in SOS \end{array}$$

Yalmip SOS:

```
sdpvar x y gamma

p = (1+x*y)^2-x*y+(1-y)^2

F = sos(p- gamma)

solvesos(F,- gamma,[], gamma)
```

➢ To construct the SOS SDP:

Example: $\min_{x} p(x)$

SOS program:

$$\begin{aligned} \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in SOS \end{aligned}$$

Yalmip SOS:

```
sdpvar x y gamma

p = (1+x*y)^2-x*y+(1-y)^2

F = sos(p- gamma)

solvesos(F,- gamma,[], gamma)
```

SOS SDP

```
sdpvar x y gamma

p = (1+x)^4 + (1-y)^2;

v = monolist([x y],degree(p)/2);
X = sdpvar(length(v));
p_sos = v'*X*v;

F = [coefficients((p- gamma)-p_sos,[x y]) == 0, X >= 0]
```

monomials

SOS polynomial

Linear constraints

PSD

- **To construct the SOS SDP:**

Example: $\min_{x} p(x)$

SOS program:

$$\begin{aligned} \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in SOS \end{aligned}$$

Yalmip SOS:

```
sdpvar x y gamma

p = (1+x*y)^2-x*y+(1-y)^2

F = sos(p- gamma)

solvesos(F,- gamma,[], gamma)
```

SOS SDP

```
sdpvar x y gamma

p = (1+x)^4 + (1-y)^2;

v = monolist([x y],degree(p)/2);
X = sdpvar(length(v));
p_sos = v'*X*v;

F = [coefficients((p- gamma)-p_sos,[x y]) == 0, X >= 0]
```

monomials

SOS polynomial

Linear constraints

PSD

Lecture 5: Duality of SOS and Moment Based SDPs, Page 82

Standard SDP

$$\begin{aligned} \underset{X \in \mathbb{R}^{n\times n}}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A \bullet X = b \quad \text{Linear constraints} \\ & X \succcurlyeq 0 \quad \text{PSD} \end{aligned}$$

To construct the Moment SDP:

Example $\mathrm{P}^* = \underset{x \in \mathbb{R}^2}{\text{minimize}} \quad 3 + 2x_1 + 2x_2 + 3x_1^2 + 2x_1x_2 + 3x_2^2 + x_1^4 + x_2^4$

Moment SDP

$$\mathrm{P}^{*2}_{mom} = \underset{y}{\text{minimize}} \quad 3 + 2y_{10} + 2y_{01} + 3y_{20} + 2y_{11} + 3y_{02} + y_{40} + y_{04}$$

subject to

$$y_{00} = 1 \qquad \mathbf{M}_2(y) = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{01} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succcurlyeq 0$$

GloptiPoly

```
mpol x1 x2
p =3+2*x1+2*x2+3*x1^2+2*x1*x2+3*x2^2+x1^4+x2^4;
P = msdp(min(p))
[status,obj] = msol(P)
```

➢ To construct the Moment SDP:

Example $\mathrm{P}^* = \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad 3+2x_1+2x_2+3x_1^2+2x_1x_2+3x_2^2+x_1^4+x_2^4$

Moment SDP

$$\mathrm{P}^{*2}_{mom} = \underset{y}{\text{minimize}} \quad 3+2y_{10}+2y_{01}+3y_{20}+2y_{11}+3y_{02}+y_{40}+y_{04}$$

subject to $\quad y_{00}=1 \qquad \mathbf{M}_2(y)=\begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{01} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succcurlyeq 0$

GloptiPoly

```
mpol x1 x2
p =3+2*x1+2*x2+3*x1^2+2*x1*x2+3*x2^2+x1^4+x2^4;
P = msdp(min(p))
[status,obj] = msol(P)
```

Moment SDP

Vector y ➡ Moment and Localizing Matrices ➡ **Standard Dual SDP**

$$\underset{y}{\text{maximize}} \quad b^T y$$

$$\text{subject to} \quad C - \textstyle\sum_{i=1}^m y_i A_i \succcurlyeq 0$$

Lecture 5: Duality of SOS and Moment Based SDPs, Page 82

➢ To construct the Moment SDP:

Example $$\mathrm{P}^* = \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad 3+2x_1+2x_2+3x_1^2+2x_1x_2+3x_2^2+x_1^4+x_2^4$$

Moment SDP

$$\mathrm{P}^{*2}_{mom} = \underset{y}{\text{minimize}} \quad 3+2y_{10}+2y_{01}+3y_{20}+2y_{11}+3y_{02}+y_{40}+y_{04}$$

$$\text{subject to} \quad y_{00} = 1 \qquad \mathbf{M}_2(y) = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{01} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succcurlyeq 0$$

GloptiPoly

```
mpol x1 x2
p =3+2*x1+2*x2+3*x1^2+2*x1*x2+3*x2^2+x1^4+x2^4;
P = msdp(min(p))
[status,obj] = msol(P)
```

Moment SDP

Vector y ➡ Moment and Localizing Matrices ➡ **Standard Dual SDP**

$$\underset{y}{\text{maximize}} \quad b^T y$$
$$\text{subject to} \quad C - \sum_{i=1}^m y_i A_i \succcurlyeq 0$$

Lecture 5: Duality of SOS and Moment Based SDPs, Page 82

Example: First-order Augmented Lagrangian Algorithm for Moment SDP based Chance Optimization

https://github.com/jasour/rarnop19/tree/master/Lecture9_SDP_Algorithms/First-Order-Algorithm_Moment_ChancOpt

Topics

- ➢ Design Problems Under Uncertainty
- ➢ Probabilistic Safety Verification Problems

Probabilistic Safety Verification

Nonlinear Probabilistic Systems

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

- Source of uncertainties: $x_0 \sim pr(x_0), \quad \omega_k \sim pr(\omega_k)$

$\mathbf{x}(k) = [x_1(k), ..., x_n(k)]^T \in \chi \subset \mathbb{R}^n$ → States

$\mathbf{u}(k) = [u_1(k), ..., u_m(k)]^T \in \mathcal{U} \subset \mathbb{R}^m$ → Given Control Inputs

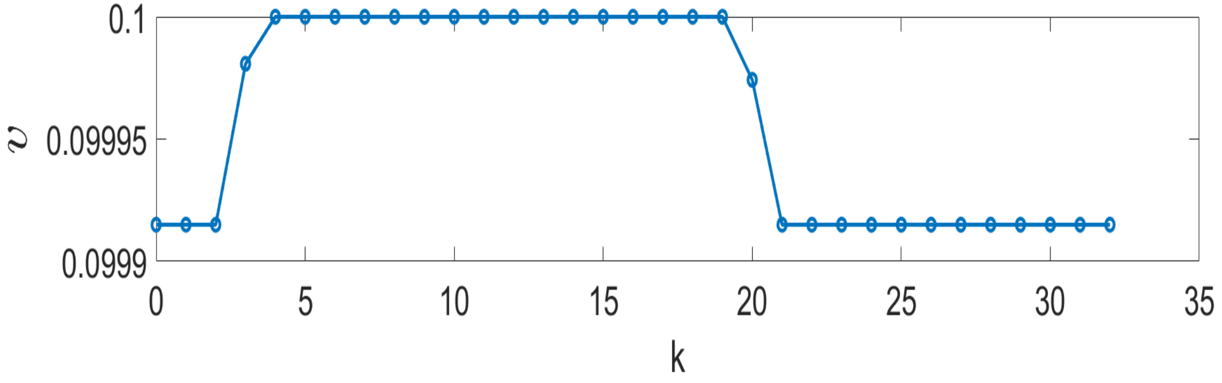
$\omega(k) = [\omega_1(k), ..., \omega_l(k)]^T \in \Omega \subset \mathbb{R}^l$ → Probabilistic uncertainty $\sim pr(\omega_k)$

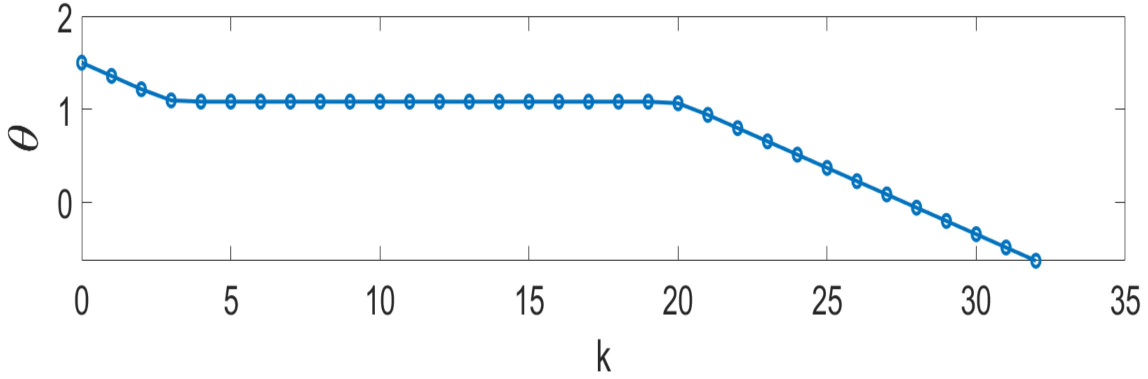
Example:

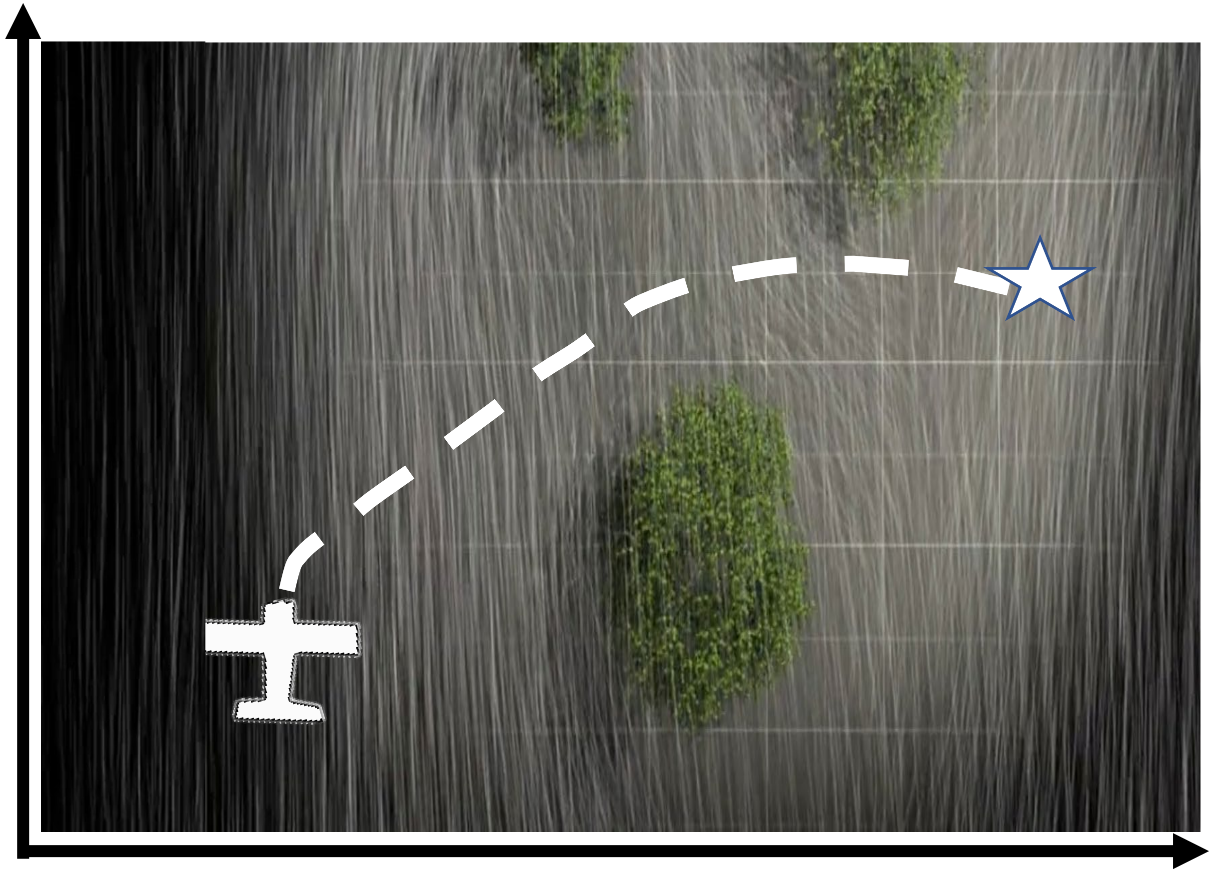
$$x_{k+1} = x_k + v_k \cos(\theta_k)$$
$$y_{k+1} = y_k + v_k \sin(\theta_k)$$

states: (x, y) position
control inputs: (θ, v) yaw angle and velocity

Planned Control Inputs:



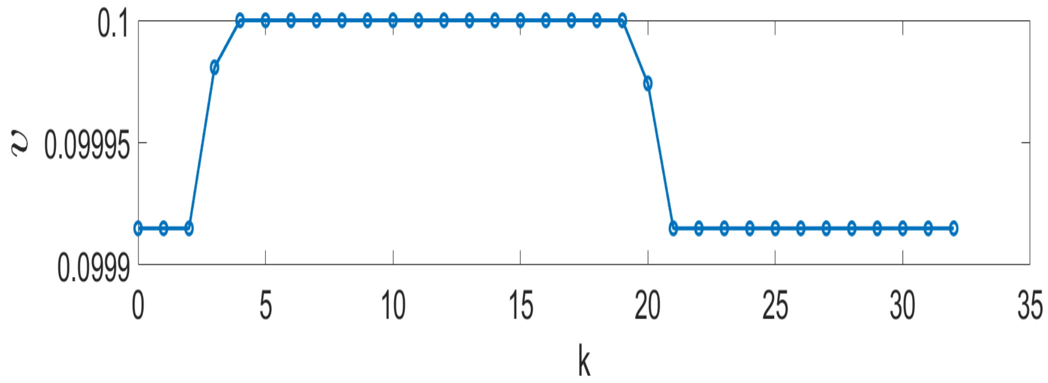


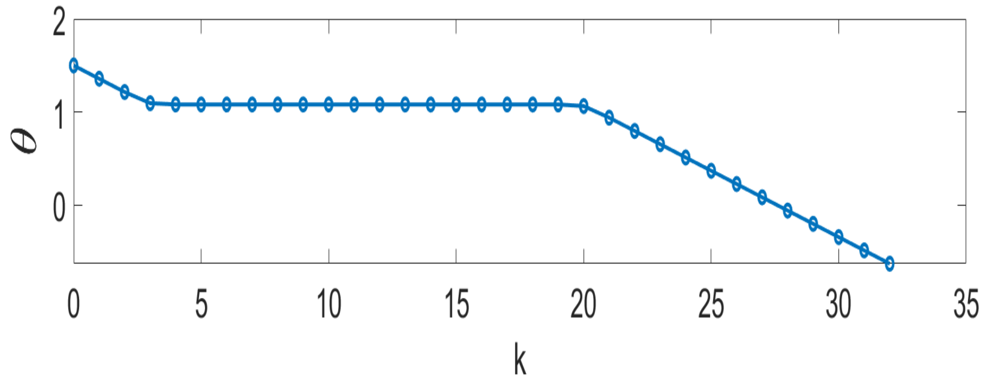


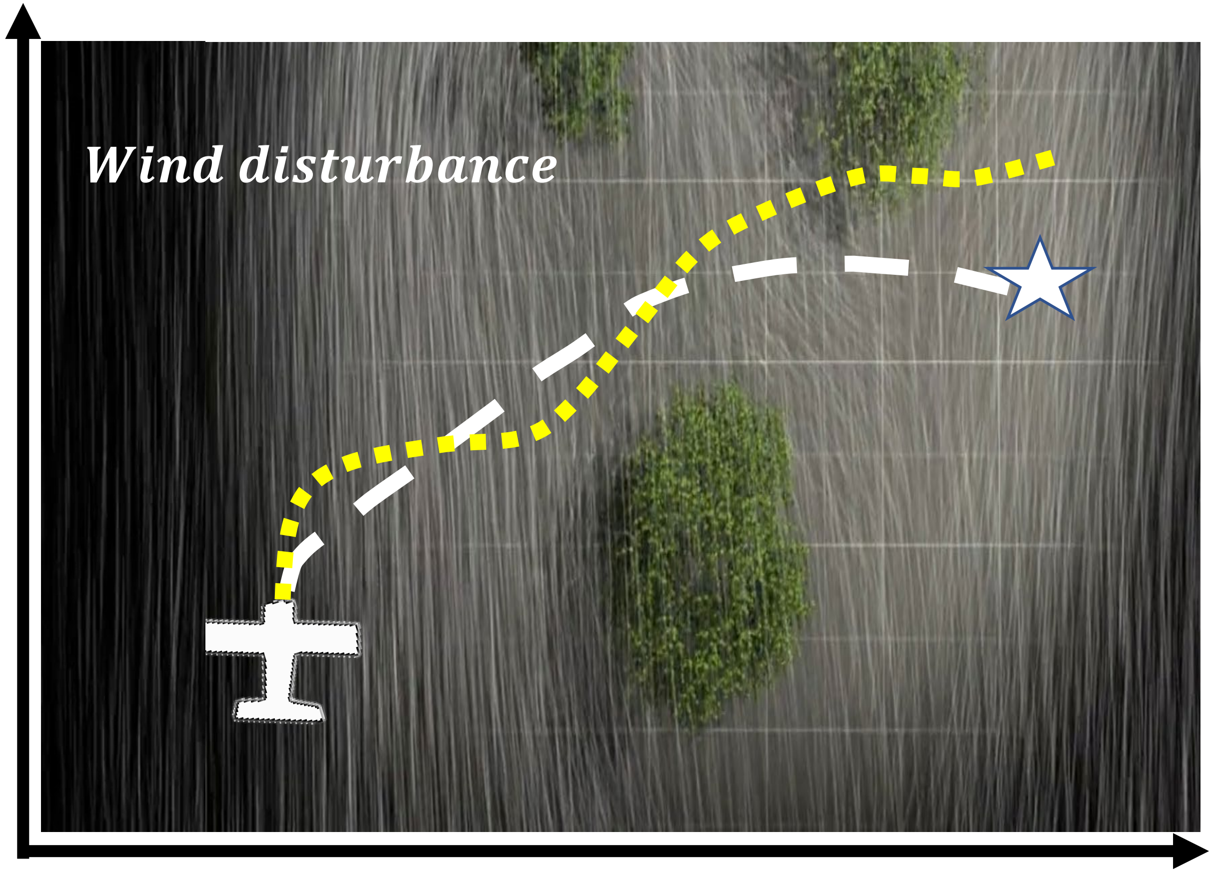
$$x_{k+1} = x_k + (v_k + \omega_{1_k}) \cos(\theta_k + \omega_{2_k}) + \omega_{3_k}$$
$$y_{k+1} = y_k + (v_k + \omega_{1_k}) \sin(\theta_k + \omega_{2_k}) + \omega_{4_k}$$

states: (x, y) position
control inputs: (θ, v) yaw angle and velocity
uncertainty: $(\underbrace{\omega_1, \omega_2}_{\text{Control Noise}}, \underbrace{\omega_3, \omega_4}_{\text{Wind Disturbance}})$

Planned Control Inputs:

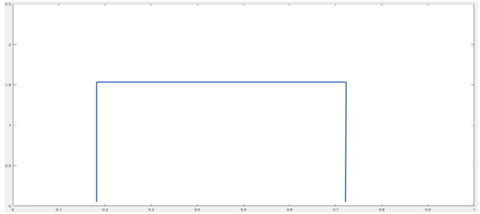




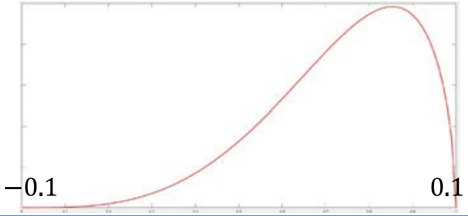


$$x_{k+1} = x_k + (v_k + \omega_{1_k})\cos(\theta_k + \omega_{2_k}) + \omega_{3_k}$$
$$y_{k+1} = y_k + (v_k + \omega_{1_k})\sin(\theta_k + \omega_{2_k}) + \omega_{4_k}$$

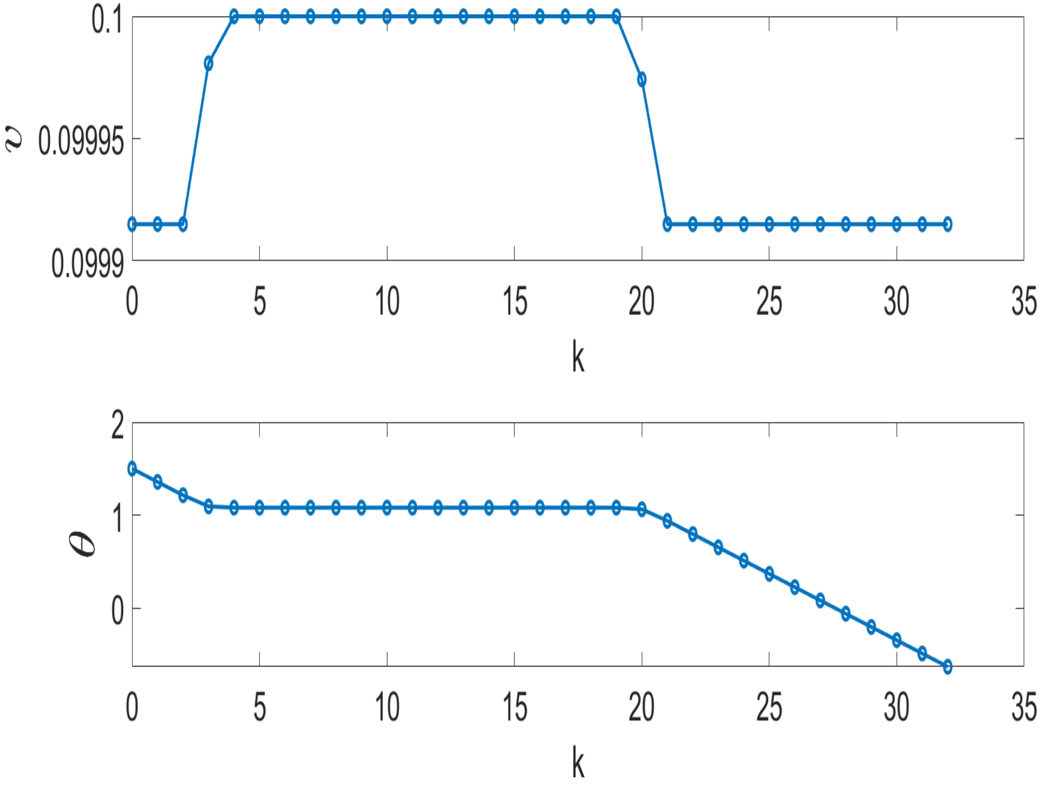
Probability distributions: $\begin{cases} \omega_1 \sim Uniform[-0.1, 0.1] \\ \omega_2 \sim Uniform[-1, 1] \end{cases}$

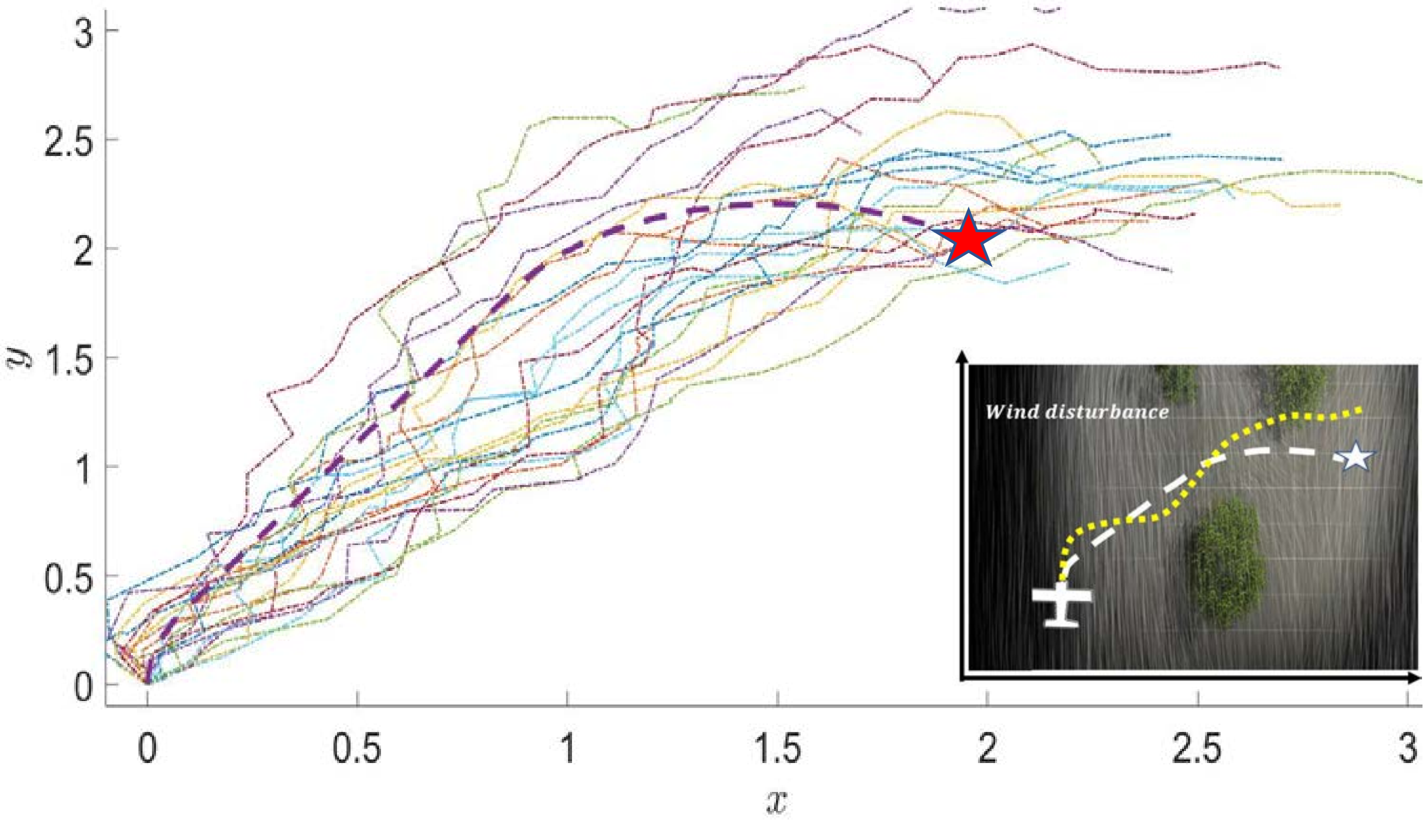


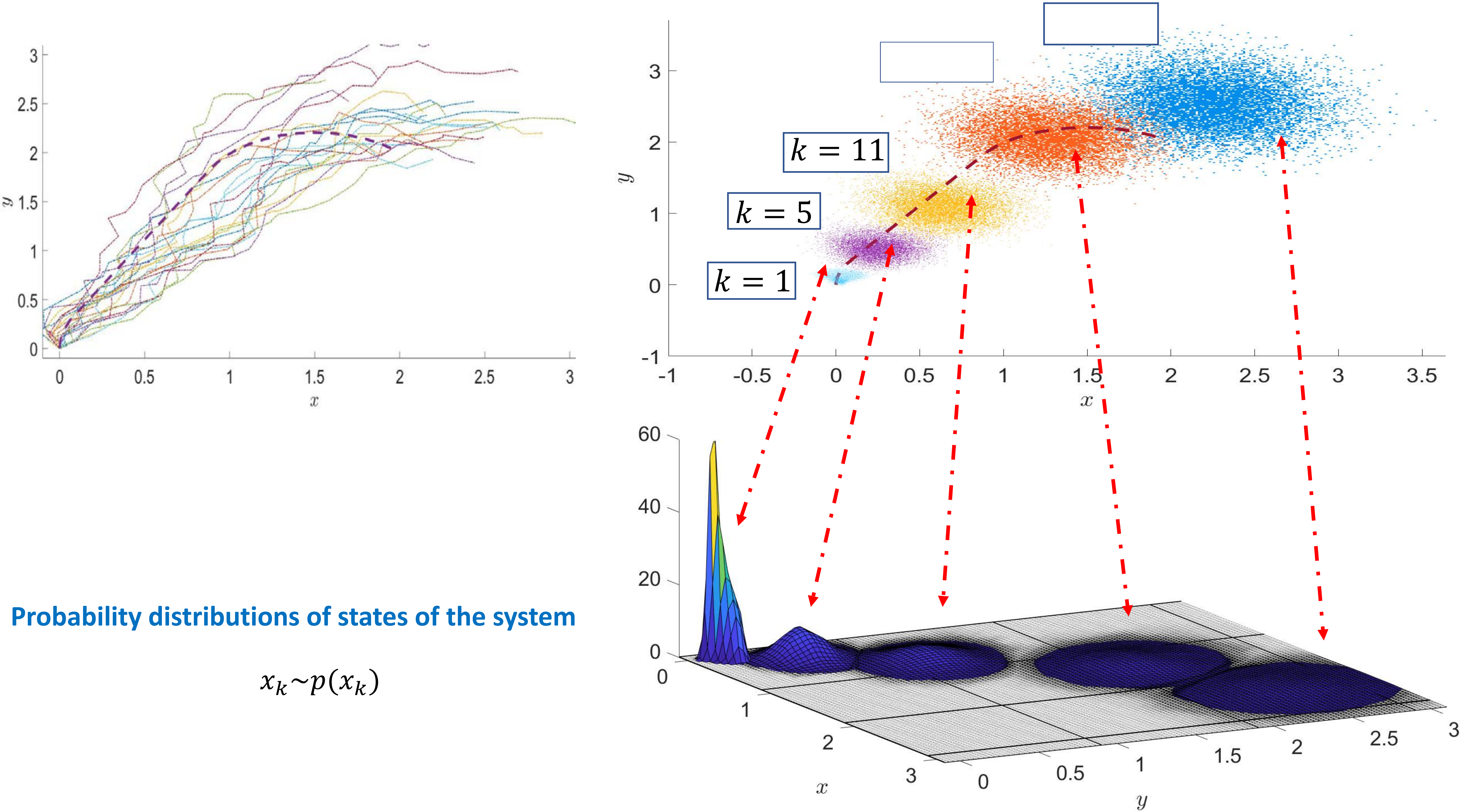
$\omega_3, \omega_4 \sim Beta[-0.1, 0.1]$



Planned Control Inputs:





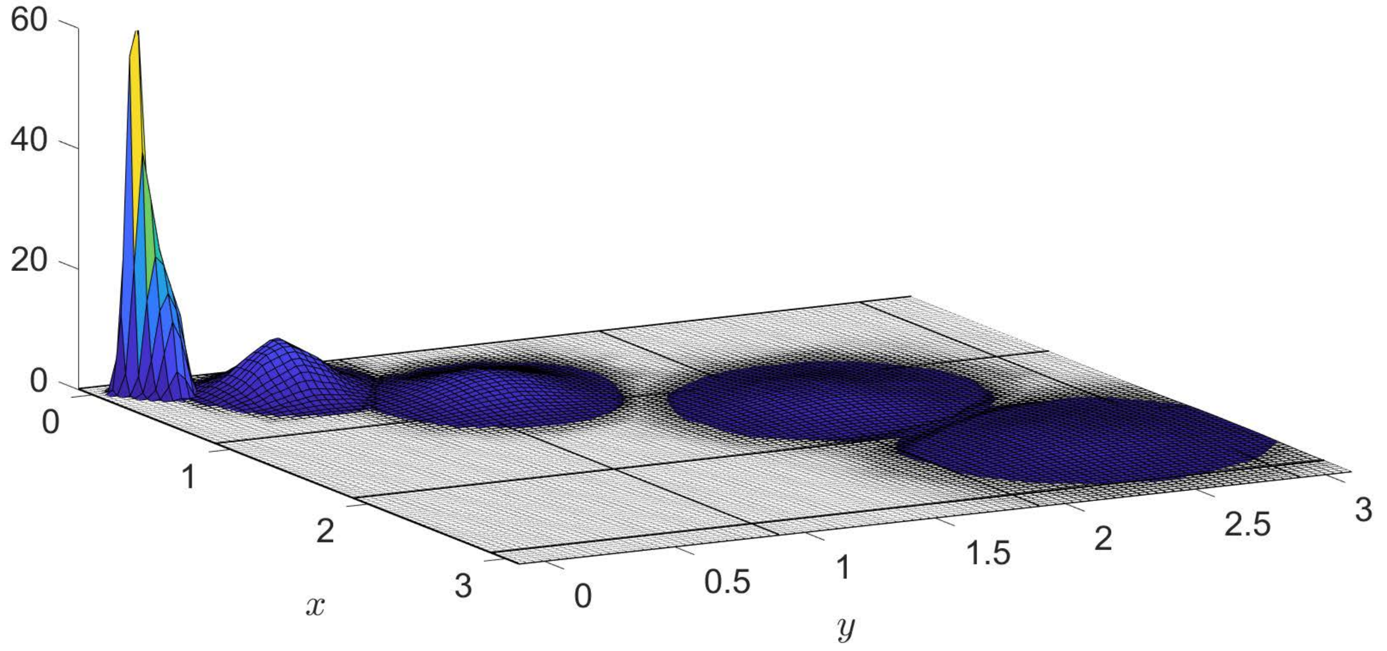


Probability distributions of states of the system

$$x_k \sim p(x_k)$$

- For **safety verification**, we need to obtain probability distributions of the states of the system.

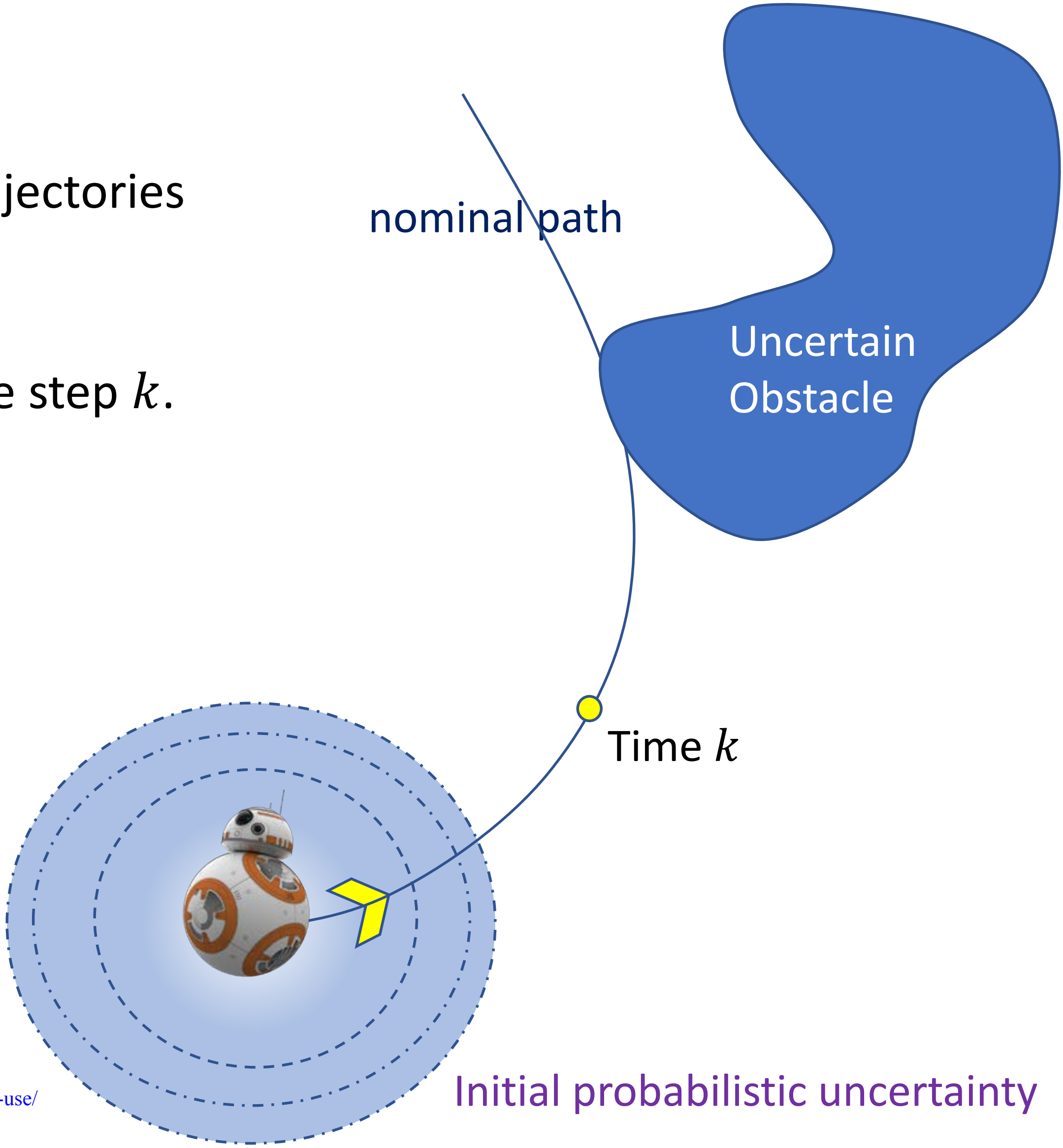
- For this, we need to propagate **initial probability distribution** of the states through **nonlinear dynamics** of the system.



Probabilistic Safety Verification

- **Given:**
  - Nonlinear uncertain dynamical system
  - Probity distribution of uncertainties
  - Candidate Plan, i.e., nominal control inputs and trajectories
  - Nonlinear safety constraints

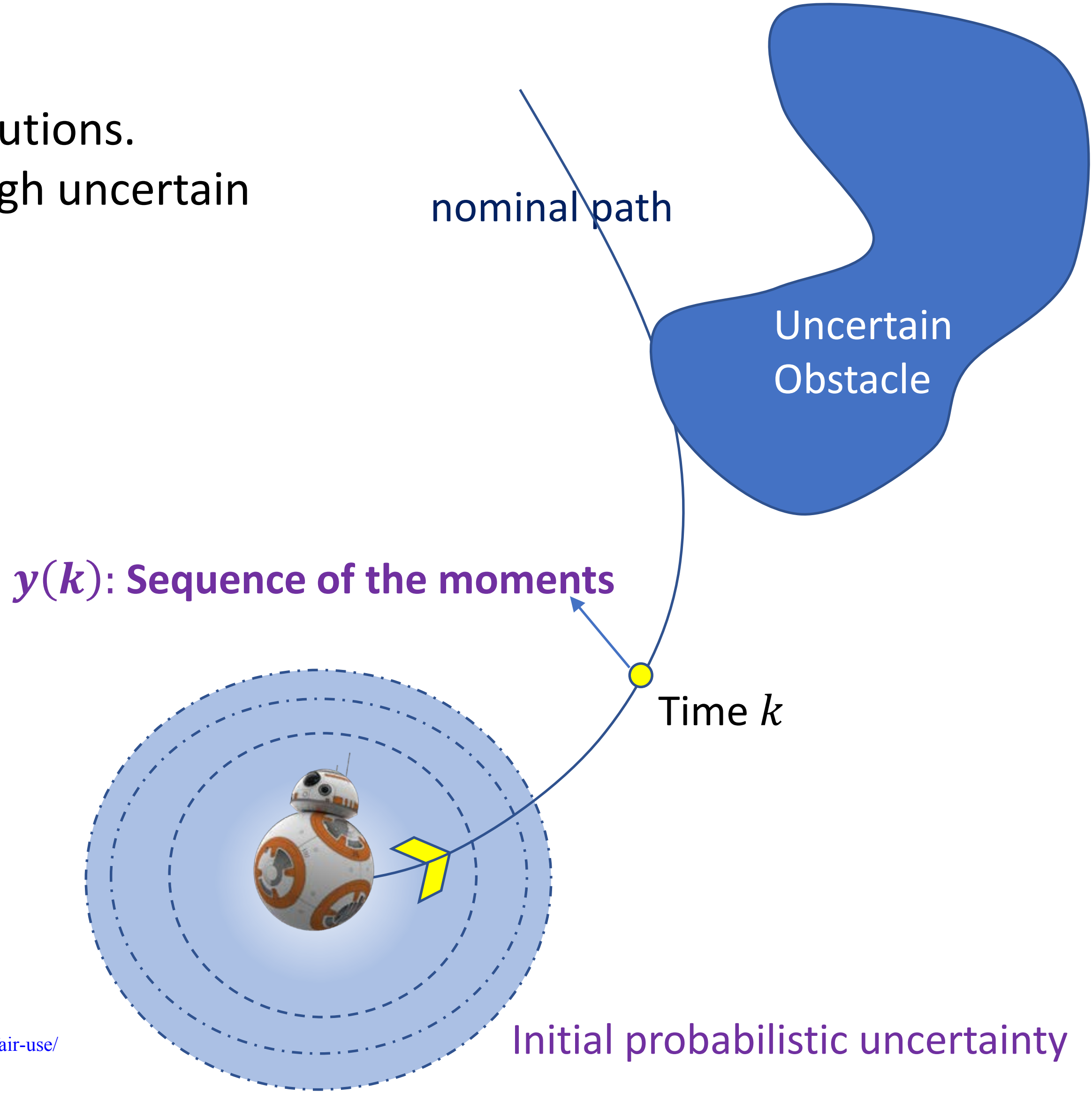
- **Goal:** We want to validate the safety of the system at each time step k.



Probabilistic Safety Verification

➢ For this purpose:

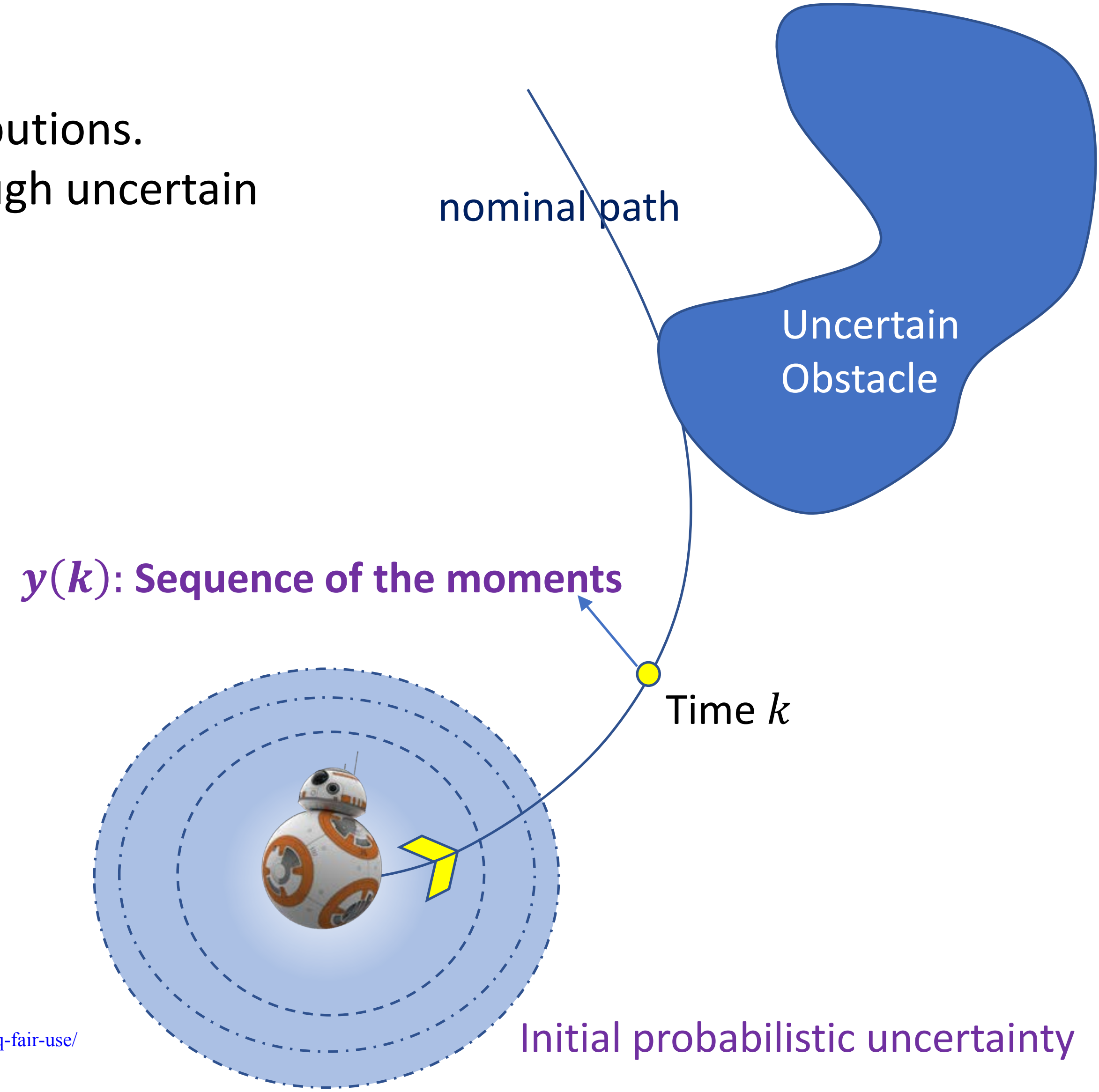
- We use moment representation of the probability distributions.
- We propagate the moments of initial uncertainties through uncertain dynamics of the system.



Probabilistic Safety Verification

- For this purpose:
  - We use moment representation of the probability distributions.
  - We propagate the moments of initial uncertainties through uncertain dynamics of the system.

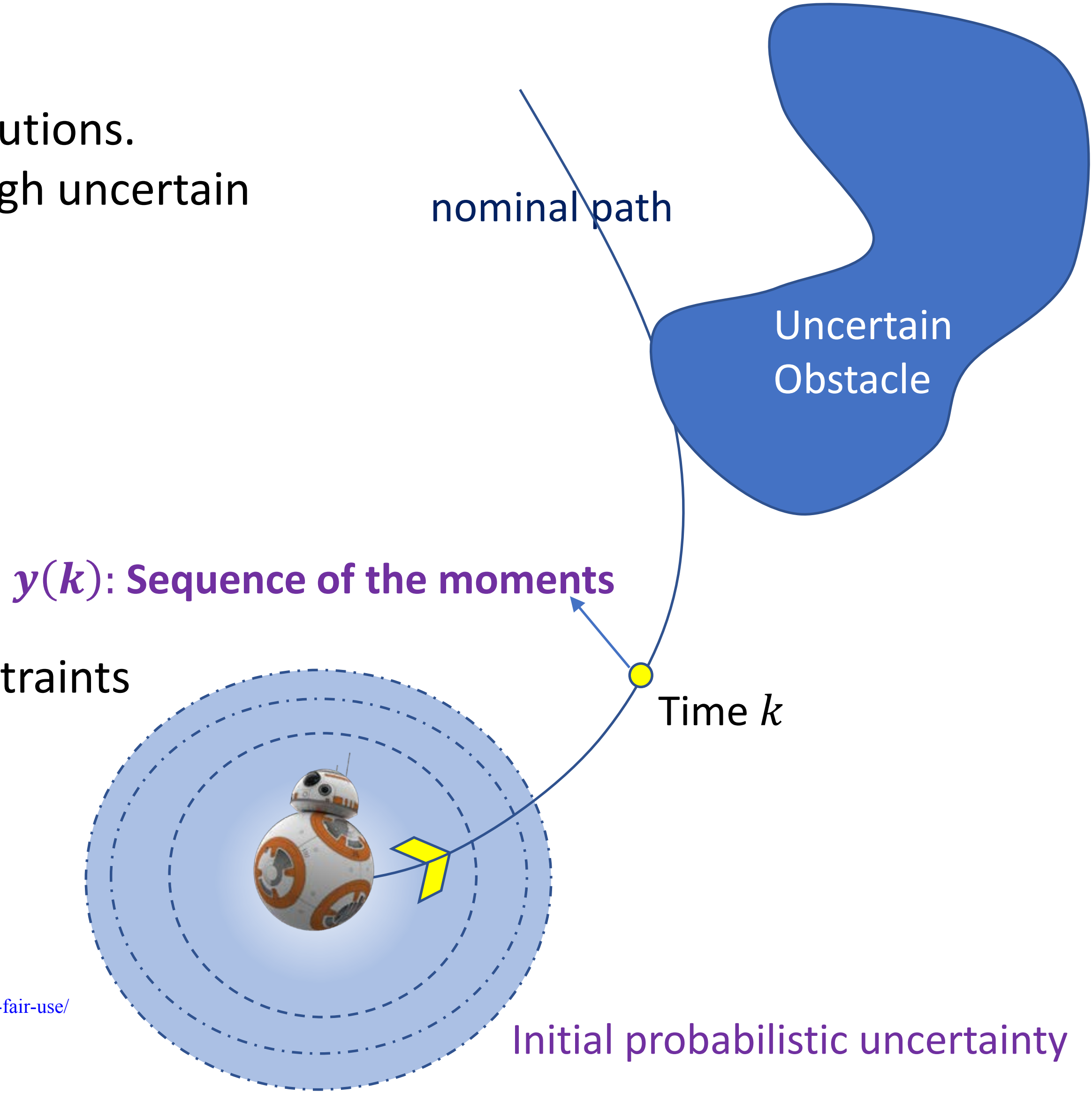
- Using the information of the moments at time step k:



Probabilistic Safety Verification

- For this purpose:
  - We use moment representation of the probability distributions.
  - We propagate the moments of initial uncertainties through uncertain dynamics of the system.

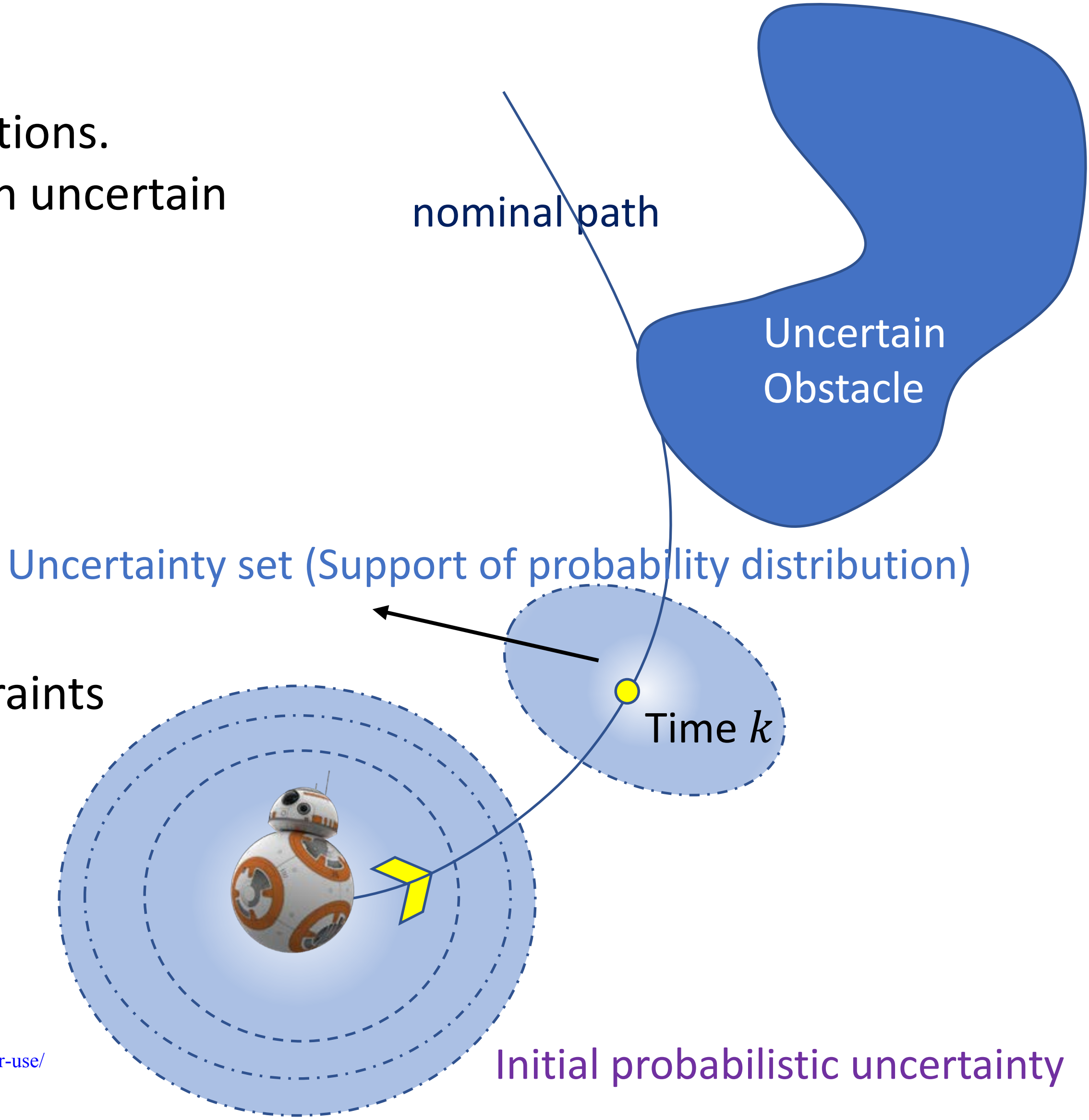
- Using the information of the moments at time step k:
  - We calculate Risk, i.e., probability of satisfying safety constraints



Probabilistic Safety Verification

- For this purpose:
  - We use moment representation of the probability distributions.
  - We propagate the moments of initial uncertainties through uncertain dynamics of the system.

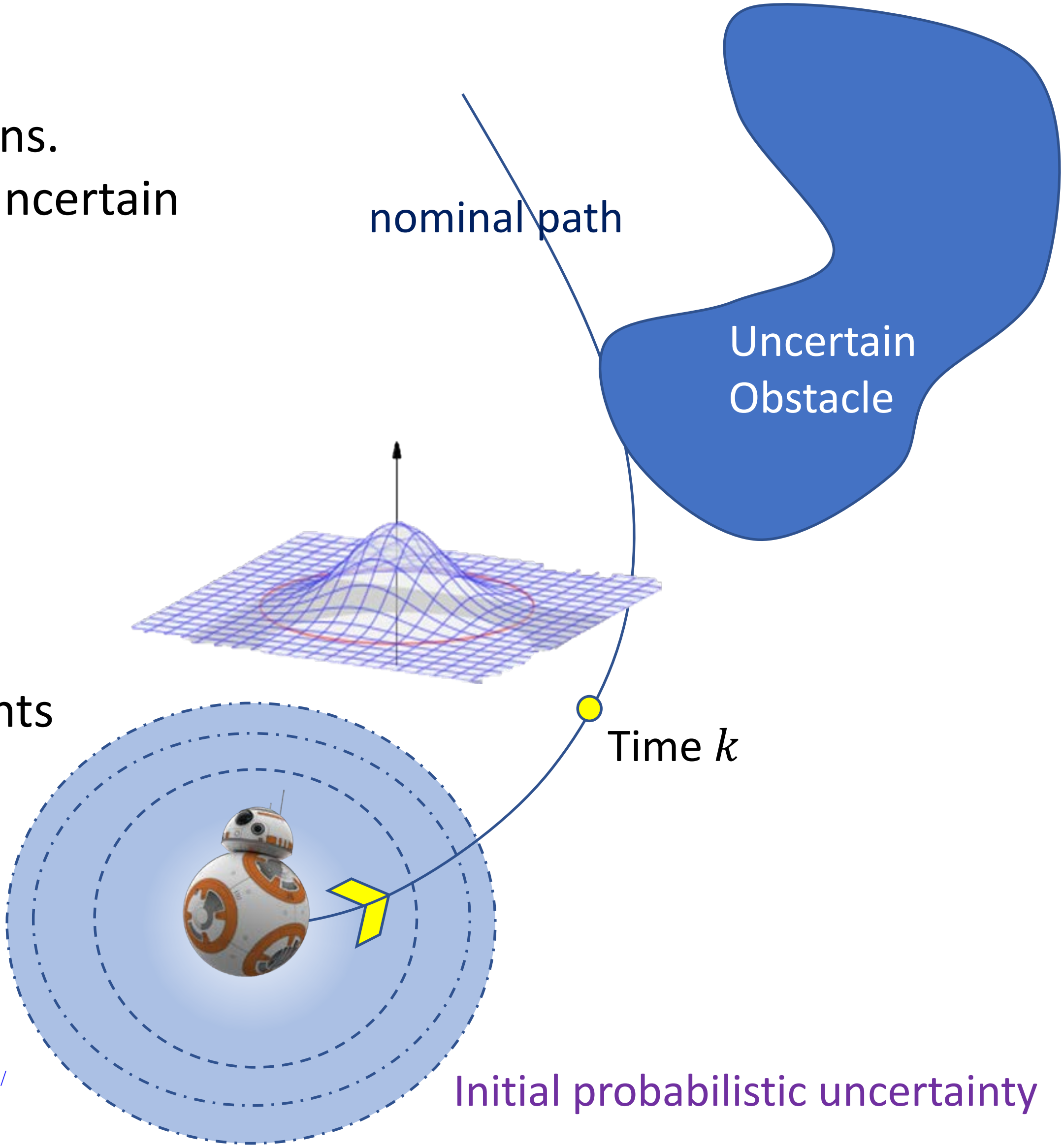
- Using the information of the moments at time step k:
  - We calculate Risk, i.e., probability of satisfying safety constraints
  - Uncertainty Set, i.e., reachable set



Probabilistic Safety Verification

- For this purpose:
  - We use moment representation of the probability distributions.
  - We propagate the moments of initial uncertainties through uncertain dynamics of the system.

- Using the information of the moments at time step k:
  - We calculate Risk, i.e., probability of satisfying safety constraints
  - Uncertainty Set, i.e., reachable set
  - Probability distribution of states of the system



Probabilistic Safety Verification

- Main Idea is to find a **polynomial approximation** of

  i) Indictor functions and ii) Probability density function

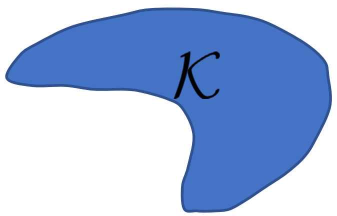
Probabilistic Safety Verification

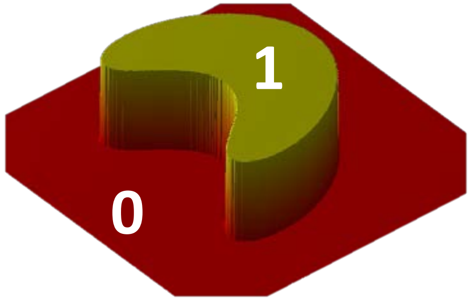
➢ Main Idea is to find a **polynomial approximation** of

i) Indictor functions and ii) Probability density function

1) To calculate the **risk**, we find polynomial approximation of the indictor function of the safety constraint set

$$\mathbf{P}^*_{\mathbf{risk}} = \mathrm{E}[\mathbf{I}_{\mathcal{K}}]$$





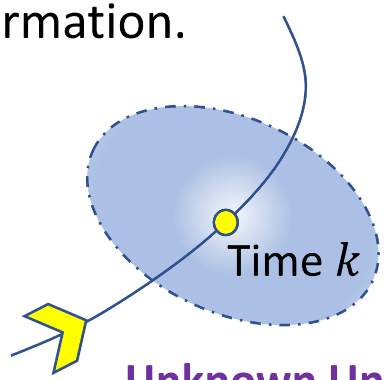
- D. Henrion, J. B. Lasserre, and C. Savorgnan "Approximate Volume and Integration for Basic Semialgebraic Sets", SIAM Rev., 51(4), 722–743, 2009.
- A. Jasour, A. Hofmann, B. C. Williams, "Moment-Sum-Of-Squares Approach for Fast Risk Estimation in Uncertain Environments", IEEE Conference on Decision and Control, 2018.

Probabilistic Safety Verification

- Main Idea is to find a **polynomial approximation** of

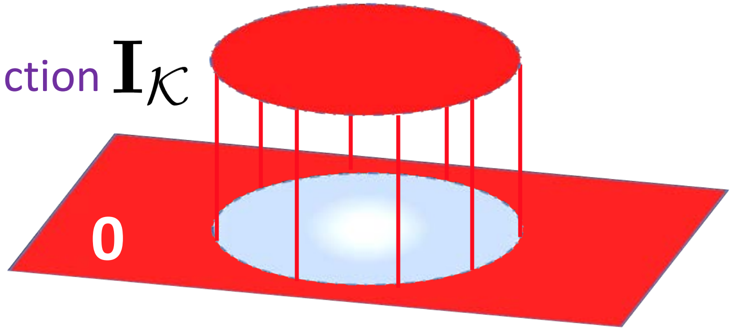
  i) Indictor functions and ii) Probability density function

2) To obtain **Uncertainty Set**, we find polynomial approximation of the indictor function of the uncertainty set using the moment information.



Unknown Uncertainty Set

Indicator function $\mathbf{I}_{\mathcal{K}}$



$$\text{Uncertainty Set: } \mathcal{K} = \{x \in \mathbb{R}^n : \mathbf{I}_{\mathcal{K}}(x) = 1\}$$

Semialgebraic approximation of the Uncertainty Set:

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

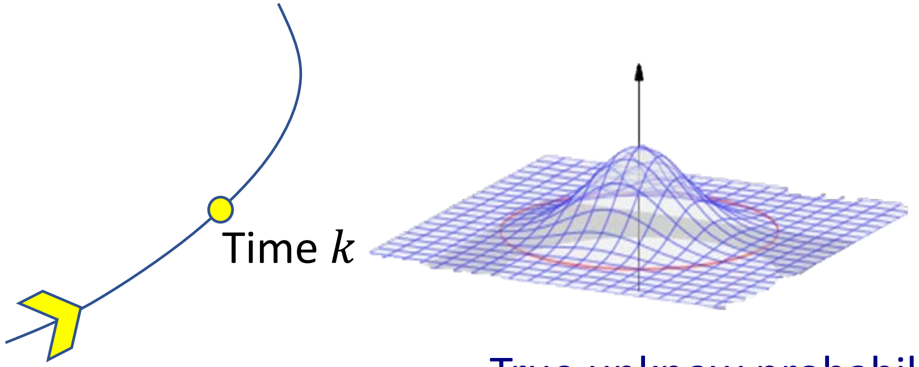
- A. Jasour, C. Lagoa, "Reconstruction of Support of a Measure From Its Moments", 53 rd IEEE Conference on Decision and Control, Los Angeles, California, 2014

Probabilistic Safety Verification

- Main Idea is to find a **polynomial approximation** of

  i) Indictor functions and ii) Probability density function

3) We find polynomial approximation of the probability density function using the moment information.



$$\underset{p_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2$$

True unknow probability density function (pdf)

Polynomial approximation of pdf

- D. Henrion, J. B. Lasserre, M. Mevissen "Mean Squared Error Minimization for Inverse Moment Problems", Journal Applied Mathematics and Optimization archive Volume 70 Issue 1, Pages 83-110, 2014.

Probabilistic Safety Verification:

1) Uncertainty (Moment) propagation through nonlinear uncertain dynamics

2) Risk estimation in presence of nonlinear safety constraints

3) Uncertainty set construction from the moment information

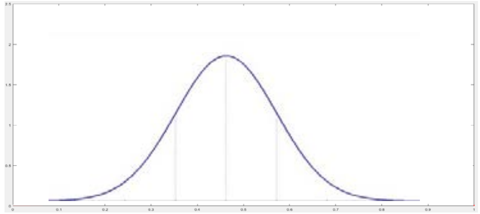
4) Probability density function construction from the moment information

1) Uncertainty (Moment) propagation through nonlinear uncertain dynamics

Linear systems and Gaussian Uncertainty Propagation:

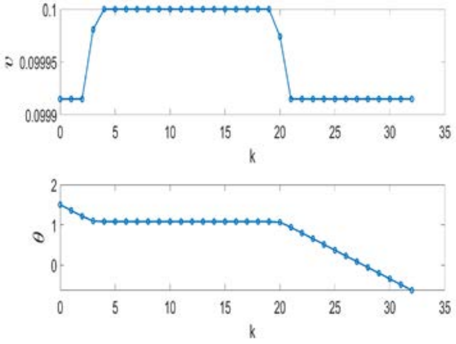
$$x_{k+1} = x_k + v_{x_k} + \omega_{1_k}$$
$$y_{k+1} = y_k + v_{y_k} + \omega_{2_k}$$

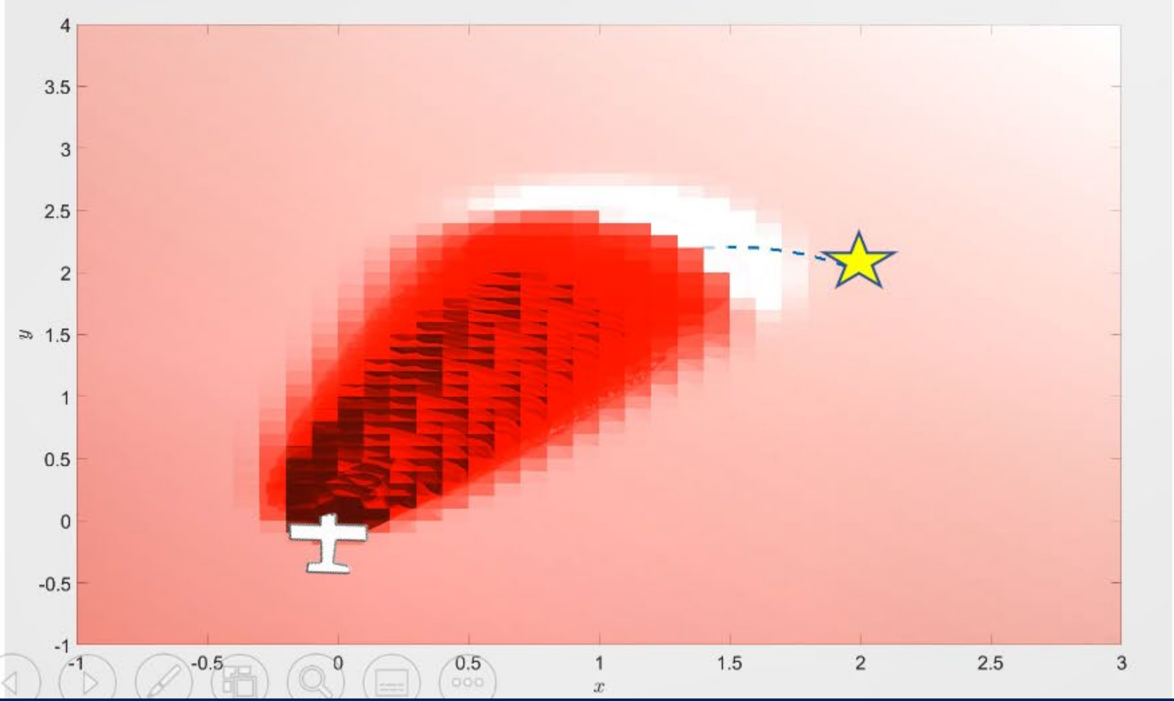
$x_0 \sim Normal(0, 0.0001)$
$y_0 \sim Normal(0, 0.0001)$
$\omega_1 \sim Normal(0, 0.05)$
$\omega_2 \sim Normal(0, 0.05)$

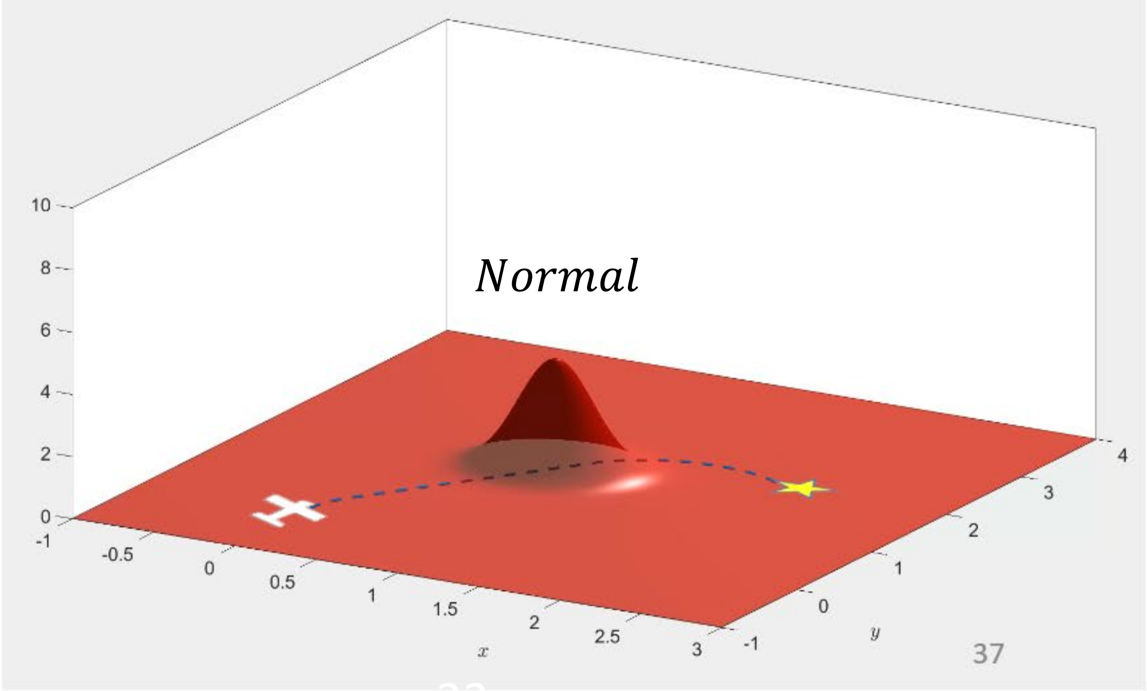


control inputs: velocity v_{x_k} and v_{y_k}

$$v_{x_k} = v_k \cos(\theta_k)$$
$$v_{y_k} = v_k \sin(\theta_k)$$





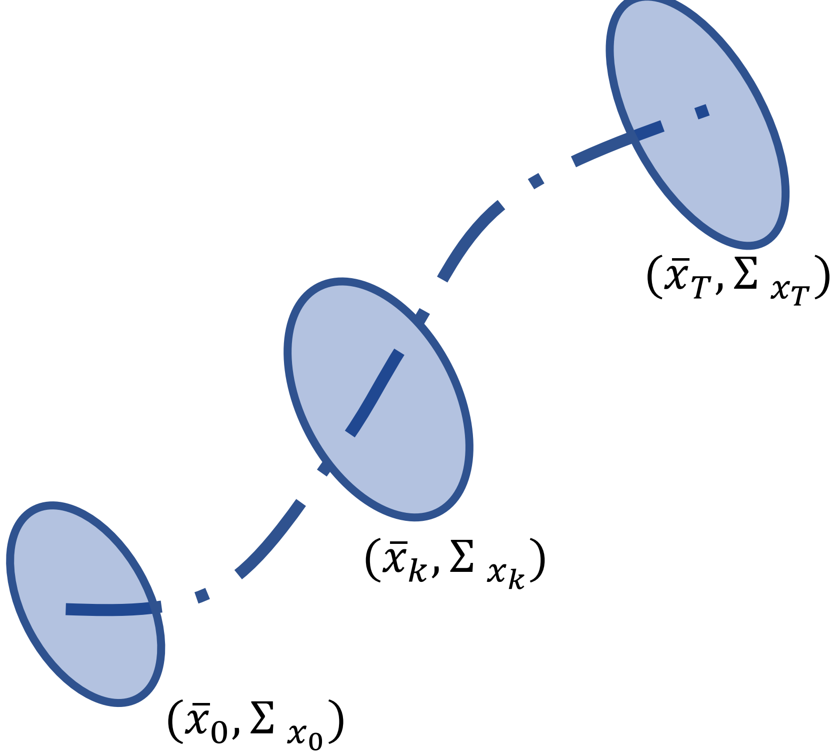


Linear systems and Gaussian Uncertainty Propagation:

$$x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$x_0 = N(\bar{x}_0, \Sigma_{x_0})$$

$$\omega_k = N(\bar{\omega}, \Sigma_{\omega_k})$$



- We only need **first** and **second** moments to represent probability distributions

Distribution of states at time k: $N(\bar{x}_k, \Sigma_{x_k})$

- **Mean:** $\mathrm{E}[x_{k+1}] = \bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \bar{\omega}_k$

- **Covariance:**

$$\mathrm{E}[(x_{k+1} - \mathrm{E}^2[x_{k+1}]] = \Sigma_{x_{k+1}} = A_k \Sigma_{x_k} A_k^T + B_{\omega_k} \Sigma_{\omega_k} B_{\omega_k}^T$$

Nonlinear Probabilistic Systems

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$

$\mathbf{x}(k) = [x_1(k), ..., x_n(k)]^T \in \chi \subset \mathbb{R}^n$ → States

$\mathbf{u}(k) = [u_1(k), ..., u_m(k)]^T \in \mathcal{U} \subset \mathbb{R}^m$ → Given Control Inputs

$\omega(k) = [\omega_1(k), ..., \omega_l(k)]^T \in \Omega \subset \mathbb{R}^l$ → Probabilistic uncertainty $\sim pr(\omega_k)$

- We need **higher order moments** to represent probability distributions.

Nonlinear Probabilistic Systems

Example:

$$x(k+1) = \delta x^2(k) + \omega(k)$$

$\mathrm{x}(0) \sim pr(x)$ Unmodeled parameter$\sim pr(\delta)$ Disturbance at time k $\sim pr(\omega_k)$

- We want to find the moments of probability distribution of the state at time step $k = 2$.

Nonlinear Probabilistic Systems

Example:

$$x(k+1) = \delta x^2(k) + \omega(k)$$

$x(0) \sim pr(x)$ Unmodeled parameter $\sim pr(\delta)$ Disturbance at time $k \sim pr(\omega_k)$

- We want to find the moments of probability distribution of the state at time step $k = 2$.

- By recursion, states at time $k = 2$: $$x(2) = \underbrace{\delta^3 x(0)^4 + \delta\omega(0)^2 + 2\delta^2 x(0)^2\omega(0) + \omega(1)}_{\text{Initial states, unmodeled parameter, and disturbances } k=0,1}$$

Nonlinear Probabilistic Systems

Example:

$$x(k+1) = \delta x^2(k) + \omega(k)$$

$x(0) \sim pr(x)$ Unmodeled parameter $\sim pr(\delta)$ Disturbance at time $k \sim pr(\omega_k)$

➢ We want to find the moments of probability distribution of the state at time step $k = 2$.

- By recursion, states at time $k = 2$: $$x(2) = \underbrace{\delta^3 x(0)^4 + \delta\omega(0)^2 + 2\delta^2 x(0)^2\omega(0) + \omega(1)}_{\text{Initial states, unmodeled parameter, and disturbances } k=0,1}$$

- Moment of order α at time $k = 2$: $$y_{x_\alpha}(2) = \mathrm{E}[x^\alpha(2)] = \mathrm{E}[\left(\delta^3 x(0)^4 + \delta\omega(0)^2 + 2\delta^2 x(0)^2\omega(0) + \omega(1)\right)^\alpha]$$

Nonlinear Probabilistic Systems

Example:

$$x(k+1) = \delta x^2(k) + \omega(k)$$

$\mathrm{x}(0) \sim pr(x)$ Unmodeled parameter $\sim pr(\delta)$ Disturbance at time $k \sim pr(\omega_k)$

- We want to find the moments of probability distribution of the state at time step $k = 2$.

- By recursion, states at time $k = 2$: $x(2) = \underbrace{\delta^3 x(0)^4 + \delta\omega(0)^2 + 2\delta^2 x(0)^2\omega(0) + \omega(1)}_{\text{Initial states, unmodeled parameter, and disturbances } k=0,1}$

- Moment of order α at time $k = 2$: $y_{x_\alpha}(2) = \mathrm{E}[x^\alpha(2)] = \mathrm{E}[\big(\delta^3 x(0)^4 + \delta\omega(0)^2 + 2\delta^2 x(0)^2\omega(0) + \omega(1)\big)^\alpha]$

  - Moment of order $\alpha = 1$ $y_{x_{\alpha=1}}(2) = \underbrace{y_{\delta_3} y_{x_4}(0) + y_{\delta_1} y_{\omega_2}(0) + 2y_{\delta_2} y_{x_2}(0) y_{\omega_1}(0) + y_{\omega_1}(1)}$

Given moments of uncertainties: $\mathrm{E}[\delta^\alpha] = y_{\delta_\alpha}$ $\mathrm{E}[\omega^\alpha(k)] = y_{\omega_\alpha}(k)$ $\mathrm{E}[x^\alpha(0)] = y_{x_\alpha}(0)$

Nonlinear Probabilistic Systems

Example:

$$x(k+1) = \delta x^2(k) + \omega(k)$$

x(0) $\sim pr(x)$ Unmodeled parameter$\sim pr(\delta)$ Disturbance at time k $\sim pr(\omega_k)$

- We want to find the moments of probability distribution of the state at time step $k = 2$.

- By recursion, states at time $k = 2$: $x(2) = \underbrace{\delta^3 x(0)^4 + \delta\omega(0)^2 + 2\delta^2 x(0)^2\omega(0) + \omega(1)}_{\text{Initial states, unmodeled parameter, and disturbances } k=0,1}$

- Moment of order α at time $k = 2$: $y_{x_\alpha}(2) = \mathrm{E}[x^\alpha(2)] = \mathrm{E}[\left(\delta^3 x(0)^4 + \delta\omega(0)^2 + 2\delta^2 x(0)^2\omega(0) + \omega(1)\right)^\alpha]$

  - Moment of order $\alpha = 1$ $y_{x_{\alpha=1}}(2) = \underbrace{y_{\delta_3} y_{x_4}(0) + y_{\delta_1} y_{\omega_2}(0) + 2y_{\delta_2} y_{x_2}(0) y_{\omega_1}(0) + y_{\omega_1}(1)}$

Given moments of uncertainties: $\mathrm{E}[\delta^\alpha] = y_{\delta_\alpha}$ $\mathrm{E}[\omega^\alpha(k)] = y_{\omega_\alpha}(k)$ $\mathrm{E}[x^\alpha(0)] = y_{x_\alpha}(0)$

Assumption: uncertainties are independent

$$\mathrm{E}[x^{\alpha_1}(0)\delta^{\alpha_2}\omega^{\alpha_3}(k)\omega^{\alpha_4}(k')] = \mathrm{E}[x^{\alpha_1}(0)]\mathrm{E}[\delta^{\alpha_2}]\mathrm{E}[\omega^{\alpha_3}(k)]\mathrm{E}[\omega^{\alpha_4}(k')] = y_{x_{\alpha_1}}(0)y_{\delta_{\alpha_2}}y_{\omega_{\alpha_3}}(k)y_{\omega_{\alpha_4}}(k')$$

Otherwise, we need to use joint moments of uncertainties.

Nonlinear Probabilistic Systems

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$

$\mathbf{x}(k) = [x_1(k), ..., x_n(k)]^T \in \chi \subset \mathbb{R}^n$ → States

$\mathbf{u}(k) = [u_1(k), ..., u_m(k)]^T \in \mathcal{U} \subset \mathbb{R}^m$ → Given Control Inputs

$\omega(k) = [\omega_1(k), ..., \omega_l(k)]^T \in \Omega \subset \mathbb{R}^l$ → Probabilistic uncertainty $\sim pr(\omega_k)$

Nonlinear Probabilistic Systems

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$

$\mathbf{x}(k) = [x_1(k), \ldots, x_n(k)]^T \in \chi \subset \mathbb{R}^n$ → States

$\mathbf{u}(k) = [u_1(k), \ldots, u_m(k)]^T \in \mathcal{U} \subset \mathbb{R}^m$ → Given Control Inputs

$\omega(k) = [\omega_1(k), \ldots, \omega_l(k)]^T \in \Omega \subset \mathbb{R}^l$ → Probabilistic uncertainty $\sim pr(\omega_k)$

Uncertainty Propagation: Moment propagation

- By recursion of the dynamical model, we can write the states in terms of the uncertain parameters and control input as follows:

$$x_i(k) = P_{f_k\,i}(\mathbf{x}(0), \mathbf{u}(j)|_{j=0}^{k-1}, \omega(j)|_{j=0}^{k-1}), \;\; i = 1, \ldots, n$$

Then, moment of order α reads as

$$y_{x\,\alpha}(k) = E[x_1^{\alpha_1}(k) \ldots x_n^{\alpha_n}(k)] = E[P_{f_k\,1}^{\;\alpha_1}(.) \ldots P_{f_k\,n}^{\;\alpha_n}(.)]$$

Nonlinear Probabilistic Systems

Uncertainty Propagation: Moment propagation

$$y_{x_\alpha}(k) = \mathrm{E}[x_1^{\alpha_1}(k)...x_n^{\alpha_n}(k)] = E[P_{f_k}{}_1^{\alpha_1}(.)...P_{f_k}{}_n^{\alpha_n}(.)]$$

- Given the control inputs up to time step $k-1$ and moments of uncertainties

Moments of initial states:

$$y_{x_\beta}(0) = \mathrm{E}[\underbrace{x_1^{\beta_1}(0)...x_n^{\beta_n}(0)}_{\text{Initial states}}]$$

Moments of uncertainty ω_k at time step j:

$$y_{\omega_\gamma}(j) = \mathrm{E}[\underbrace{\omega_1^{\gamma_1}(j)...\omega_l^{\gamma_l}(j)}_{\text{uncertainties}}], j = 0, ..., k-1$$

Nonlinear Probabilistic Systems

Uncertainty Propagation: Moment propagation

$$y_{x\alpha}(k) = \mathrm{E}[x_1^{\alpha_1}(k)...x_n^{\alpha_n}(k)] = E[P_{f_k}{}_1^{\alpha_1}(.)...P_{f_k}{}_n^{\alpha_n}(.)]$$

- Given the control inputs up to time step $k-1$ and moments of uncertainties

Moments of initial states:

$$y_{x\beta}(0) = \mathrm{E}[\underbrace{x_1^{\beta_1}(0)...x_n^{\beta_n}(0)}_{\text{Initial states}}]$$

Moments of uncertainty ω_k at time step j:

$$y_{\omega\gamma}(j) = \mathrm{E}[\underbrace{\omega_1^{\gamma_1}(j)...\omega_l^{\gamma_l}(j)}_{\text{uncertainties}}], j = 0, ..., k-1$$

and also considering that uncertainties at each time step k are independent, i.e.,

$$\mathrm{E}[\underbrace{x_1^{\beta_1}(0)...x_n^{\beta_n}(0)}_{\text{Initial states}}\underbrace{\omega_1^{\gamma 1}(j)...\omega_l^{\gamma_l}(j)}_{\text{Uncertainties at time } j}\underbrace{\omega_1^{\zeta_1}(j')...\omega_l^{\zeta_l}(j')}_{\text{Uncertainties at time } j'}] = \underbrace{y_{x\beta}(0)}_{\text{Moments of initial states}}\underbrace{y_{\omega\gamma}(j)}_{\text{Moments of uncertainty } \omega \text{ at time } j}\underbrace{y_{\omega\zeta}(j')}_{\text{Moments of uncertainty } \omega \text{ at time } j'}$$

Nonlinear Probabilistic Systems

Uncertainty Propagation: Moment propagation

$$y_{x_\alpha}(k) = \mathrm{E}[x_1^{\alpha_1}(k) \ldots x_n^{\alpha_n}(k)] = E[P_{f_k}{}_1^{\alpha_1}(.) \ldots P_{f_k}{}_n^{\alpha_n}(.)]$$

- Given the control inputs up to time step $k-1$ and moments of uncertainties

Moments of initial states:

$$y_{x_\beta}(0) = \mathrm{E}[\underbrace{x_1^{\beta_1}(0) \ldots x_n^{\beta_n}(0)}_{\text{Initial states}}]$$

Moments of uncertainty ω_k at time step j:

$$y_{\omega_\gamma}(j) = \mathrm{E}[\underbrace{\omega_1^{\gamma_1}(j) \ldots \omega_l^{\gamma_l}(j)}_{\text{uncertainties}}], j = 0, \ldots, k-1$$

and also considering that uncertainties at each time step k are independent, i.e.,

$$\mathrm{E}[\underbrace{x_1^{\beta_1}(0) \ldots x_n^{\beta_n}(0)}_{\text{Initial states}} \underbrace{\omega_1^{\gamma 1}(j) \ldots \omega_l^{\gamma_l}(j)}_{\text{Uncertainties at time } j} \underbrace{\omega_1^{\zeta_1}(j') \ldots \omega_l^{\zeta_l}(j')}_{\text{Uncertainties at time } j'}] = \underbrace{y_{x_\beta}(0)}_{\text{Moments of initial states}} \underbrace{y_{\omega_\gamma}(j)}_{\text{Moments of uncertainty } \omega \text{ at time } j} \underbrace{y_{\omega_\zeta}(j')}_{\text{Moments of uncertainty } \omega \text{ at time } j'}$$

We can rewrite the moments of states at time

$$y_{x_\alpha}(k) = \sum_j \overset{\text{Known coefficients}}{c_j} \left\{ y_{x_{\beta_j}}(0) y_{\omega_{\gamma_j}}(0) \ldots y_{\omega_{\eta_j}}(k-1) \right\}$$

Nonlinear Probabilistic Systems

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states inputs Probabilistic uncertainty $\sim pr(\omega_k)$

- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$

Moments of states at time k

Known coefficients

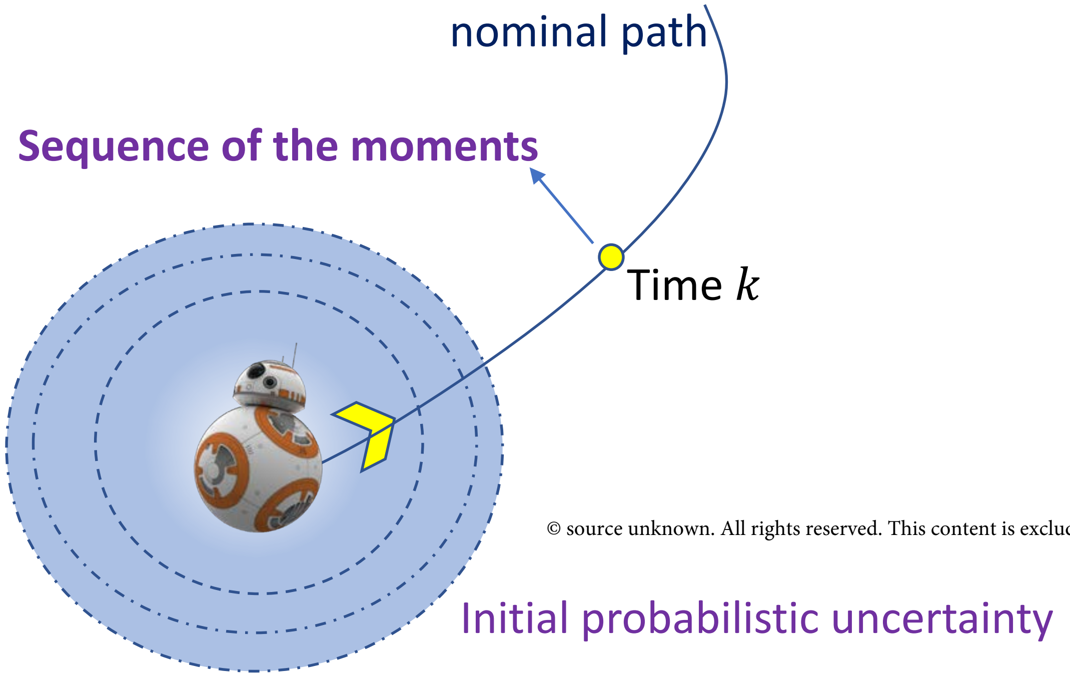
$$y_{x^\alpha}(k) = \sum_j c_j \left\{ y_{x^{\beta_j}}(0) y_{\omega^{\gamma_j}}(0) \ldots y_{\omega^{\eta_j}}(k-1) \right\}$$

Moments of initial states

Moments of uncertainty ω at time 0

⋮

Moments of uncertainty ω at time $k-1$



Nonlinear Probabilistic Systems

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states　　inputs　　Probabilistic uncertainty $\sim pr(\omega_k)$

- Source of uncertainties: $x_0 \sim pr(x_0)$, $\omega_k \sim pr(\omega_k)$

Moments of states at time k

Known coefficients

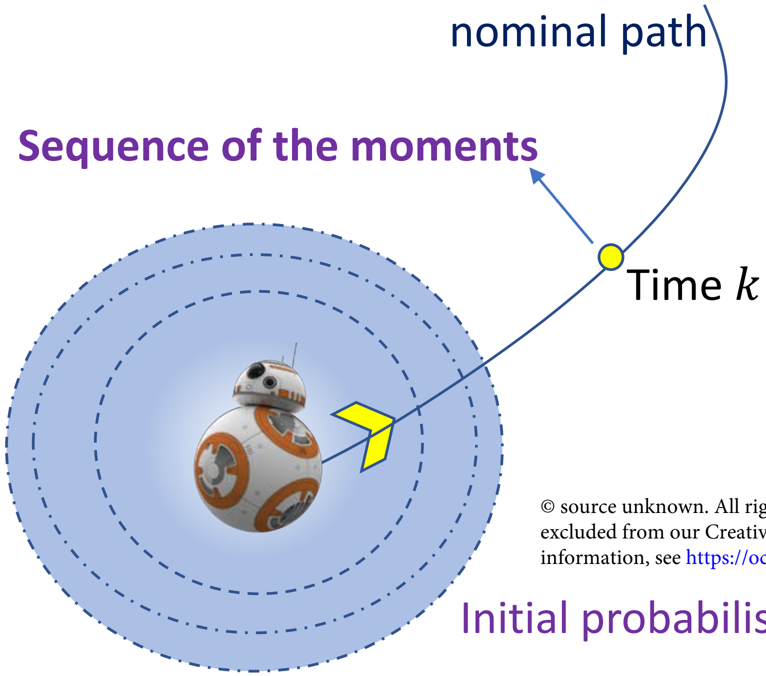
$$y_{x^\alpha}(k) = \sum_j c_j \left\{ y_{x^{\beta_j}}(0) y_{\omega^{\gamma_j}}(0) \ldots y_{\omega^{\eta_j}}(k-1) \right\}$$

Moments of initial states

Moments of uncertainty ω at time 0

.
.
.

Moments of uncertainty ω at time $k-1$



Initial probabilistic uncertainty

Propagated moments

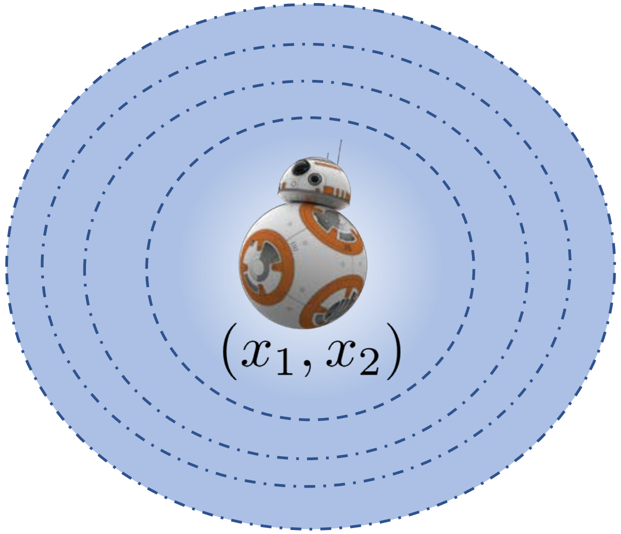
- Risk at time k
- Uncertainty Set at time k
- Probability distribution at time k

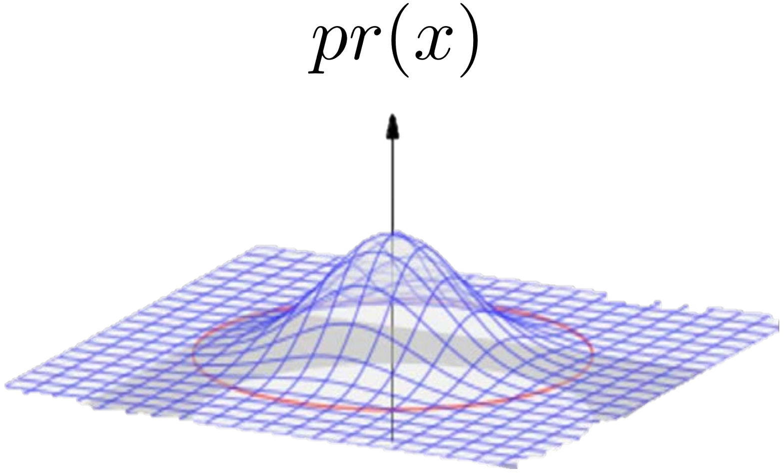
2) Risk estimation in presence of nonlinear safety constraints

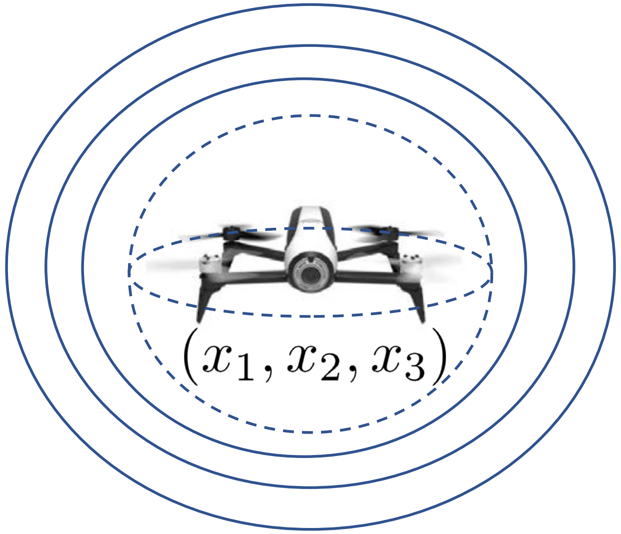
Let

- $x \in \mathbb{R}^n$:
  - Multivariate random variable with known probability distribution $pr(x)$
  - Represent an uncertain position of a robot

Example:





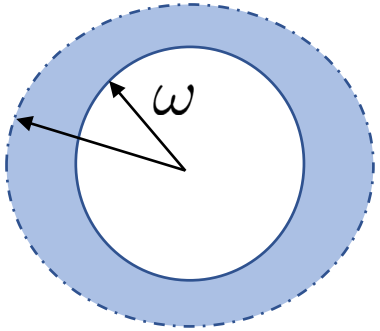


- $\chi(\omega)$: Uncertain unsafe region represented by a semi-algebraic set (represent safety constraints)

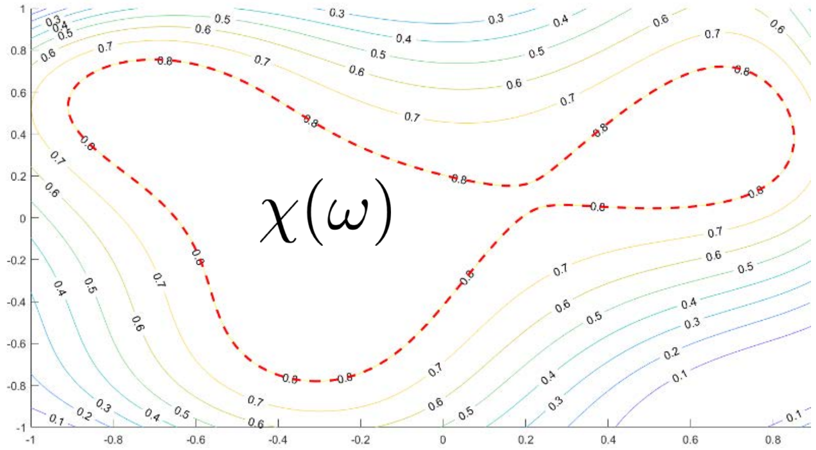
$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x,\omega) \leq u_j, j = 1, \ldots, \ell \}$$

- Uncertain parameter: $\omega \in \mathbb{R}^m \sim pr(\omega)$
- Polynomial: $\mathcal{P}_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$
- $l_j, u_j \in \mathbb{R}$

Example: Obstacle with uncertain location/size/geometry



$$\chi(\omega) := \{(x_1,x_2) : \omega^2 - x_1^2 - x_2^2 \geq 0 \}$$



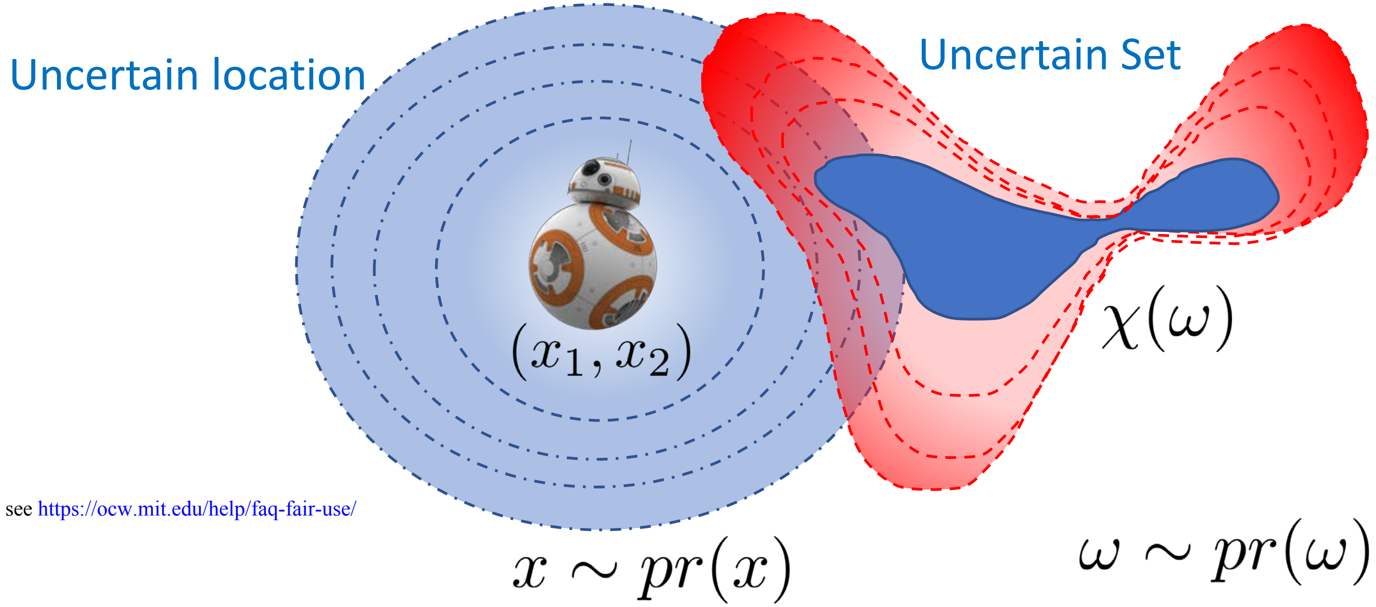
Nonconvex Uncertain Set

Given:

- $x \in \mathbb{R}^n \sim pr(x)$
- $\omega \in \mathbb{R}^m \sim pr(\omega)$
- Uncertain Set $\chi(\omega)$

Risk: probability of collision with obstacle (probability of violating safety constraints)

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$



Given:

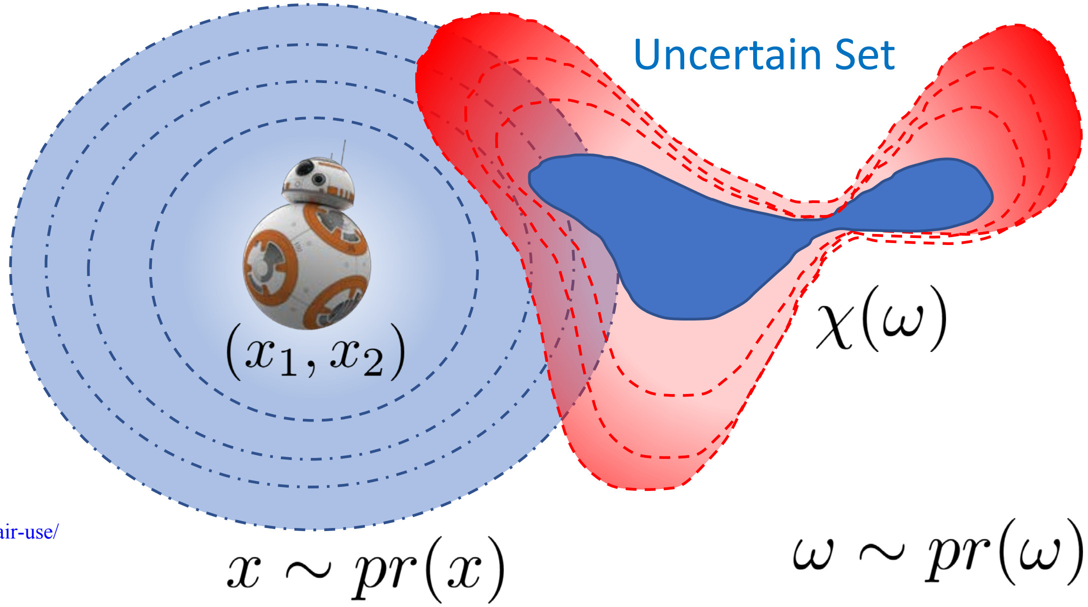
- $x \in \mathbb{R}^n \sim pr(x)$
- $\omega \in \mathbb{R}^m \sim pr(\omega)$
- Uncertain Set $\chi(\omega)$

Risk: probability of collision with obstacle (probability of violating safety constraints)

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

Find: Lower/Upper bounds of the risk using the moments information

$$\mathbf{P}^{\mathbf{L}}_{\mathbf{risk}} \leq \mathbf{P}^*_{\mathbf{risk}} \leq \mathbf{P}^{\mathbf{U}}_{\mathbf{risk}}$$



Given:

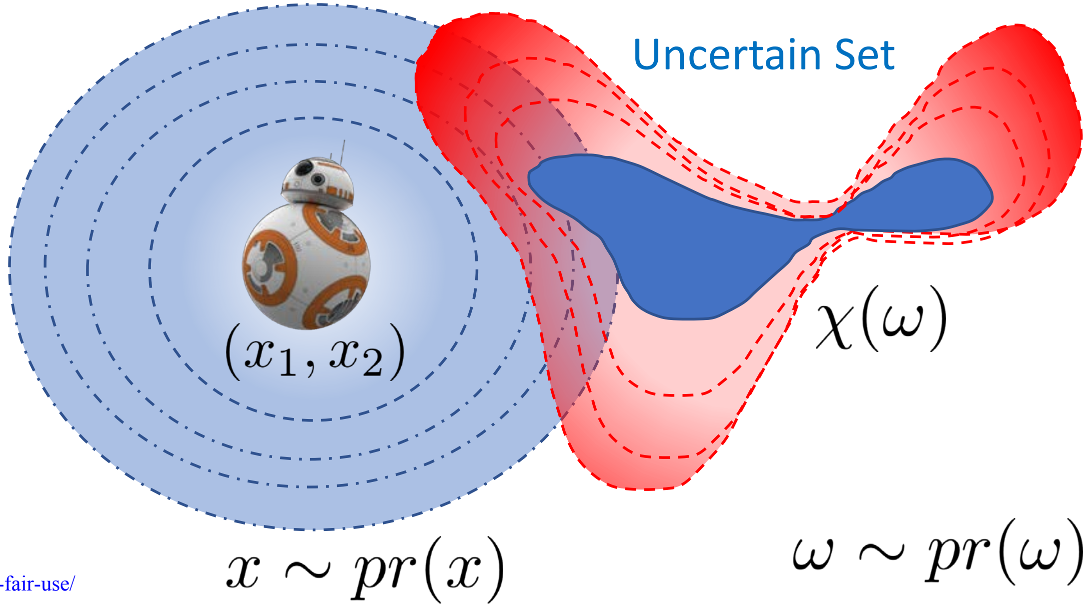
- $x \in \mathbb{R}^n \sim pr(x)$
- $\omega \in \mathbb{R}^m \sim pr(\omega)$
- Uncertain Set $\chi(\omega)$

Risk: probability of collision with obstacle (probability of violating safety constraints)

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

Find: Lower/Upper bounds of the risk using the moments information

$$\mathbf{P}^{\mathbf{L}}_{\mathbf{risk}} \leq \mathbf{P}^*_{\mathbf{risk}} \leq \mathbf{P}^{\mathbf{U}}_{\mathbf{risk}}$$



Why Upper and Lower bound of Risk ?

Challenges:

- Risk calculation problem is computationally challenging
- It involves a multivariate integral over a nonconvex set

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

$$= \int_{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\}} pr(x)pr(\omega)dx_1 \ldots dx_n d\omega_1 \ldots d\omega_m$$

- Multivariate integral
- In general, It does not have any analytical solution

Approaches:

- Sampling Based Methods
- Indicator Function Based Methods

- Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

➢ Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\}} pr(x)pr(\omega)dx_1 \ldots dx_n d\omega_1 \ldots d\omega_m$$

- N samples from the distributions of uncertainties $(x,\omega)^j|_{j=1}^{N}$

- Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\mathbf{P}^{*}_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1 ... dx_n d\omega_1 ... d\omega_m$$

- N samples from the distributions of uncertainties $(x,\omega)^j|_{j=1}^{N}$

- $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} = \frac{1}{N}\sum_{j=1}^{N} \mathbf{I}_{\mathcal{K}}$

$$\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & (x,\omega) \in \mathcal{K} = \{(x,\omega) : \{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\} \\ 0 & \text{Otherwise} \end{cases}$$

- Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \le \mathcal{P}_j(x,\omega) \le u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1 \ldots dx_n d\omega_1 \ldots d\omega_m$$

- N samples from the distributions of uncertainties $(x,\omega)^j|_{j=1}^N$

- $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} = \frac{1}{N}\sum_{j=1}^{N} \mathbf{I}_{\mathcal{K}}$ $\qquad$ $\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & (x,\omega) \in \mathcal{K} = \left\{(x,\omega) : \{l_j \le \mathcal{P}_j(x,\omega) \le u_j\}_{j=1}^{\ell}\right\} \\ 0 & \text{Otherwise} \end{cases}$

- Law of large numbers $\qquad \lim_{N\to\infty} \mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \to \mathbf{P}_{\mathbf{risk}}$

- Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1 \dots dx_n d\omega_1 \dots d\omega_m$$

- N samples from the distributions of uncertainties $(x,\omega)^j|_{j=1}^N$

- $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} = \frac{1}{N}\sum_{j=1}^{N} \mathbf{I}_{\mathcal{K}}$ $\qquad$ $\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & (x,\omega) \in \mathcal{K} = \{(x,\omega) : \{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\} \\ 0 & \text{Otherwise} \end{cases}$

- Law of large numbers $\qquad \lim_{N\to\infty} \mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \to \mathbf{P}_{\mathbf{risk}}$

- For safety $\qquad \text{Probability}(\text{Failure}) \leq \Delta \qquad \text{Probability}(\text{Success}) \geq 1 - \Delta$

Acceptable Risk Level

- Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m$$

- N samples from the distributions of uncertainties $(x,\omega)^j|_{j=1}^N$

- $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} = \frac{1}{N}\sum_{j=1}^{N} \mathbf{I}_{\mathcal{K}}$ $\qquad$ $\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & (x,\omega) \in \mathcal{K} = \{(x,\omega) : \{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\} \\ 0 & \text{Otherwise} \end{cases}$

- Law of large numbers $\quad \lim_{N\to\infty} \mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \to \mathbf{P}_{\mathbf{risk}}$

- For safety $\quad \text{Probability(Failure)} \leq \Delta \qquad \text{Probability(Success)} \geq 1-\Delta$

Acceptable Risk Level

$\Uparrow \qquad \Uparrow$

Estimation of Probability: $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \leq \Delta \qquad \mathbf{P}^{\mathbf{N}}_{\mathbf{success}} \geq 1-\Delta$

➢ Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\ \}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m$$

- N samples from the distributions of uncertainties $(x,\omega)^j|_{j=1}^N$
- $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} = \frac{1}{N}\sum_{j=1}^N \mathbf{I}_{\mathcal{K}}$ $\qquad \mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & (x,\omega) \in \mathcal{K} = \{(x,\omega) : \{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\ \} \\ 0 & \text{Otherwise} \end{cases}$
- Law of large numbers $\qquad \lim_{N\to\infty}\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \to \mathbf{P}_{\mathbf{risk}}$

➢ For safety $\qquad \text{Probability}(\text{Failure}) \leq \Delta \qquad \text{Probability}(\text{Success}) \geq 1-\Delta$

Acceptable Risk Level

Estimation of Probability: $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \leq \Delta \qquad \mathbf{P}^{\mathbf{N}}_{\mathbf{success}} \geq 1-\Delta$

- Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\text{Probability}(\text{Failure}) \leq \Delta \qquad \text{Probability}(\text{Success}) \geq 1 - \Delta$$

Estimation of Probability: $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \leq \Delta$ $\qquad$ $\mathbf{P}^{\mathbf{N}}_{\mathbf{success}} \geq 1 - \Delta$

$$\text{Probability}(\text{Failure}) \leq \Delta \qquad \text{Probability}(\text{Success}) \geq 1 - \Delta$$

Upper bound Estimation: $\mathbf{P}^{\mathbf{U}}_{\mathbf{risk}} \leq \Delta$ $\qquad$ **Lower bound Estimation:** $\mathbf{P}^{\mathbf{L}}_{\mathbf{risk}} \leq \Delta$

- Sampling Based Methods like Monte Carlo relies on **finite** number of **uncertainty samples**

$$\text{Probability}(\text{Failure}) \leq \Delta \qquad \text{Probability}(\text{Success}) \geq 1 - \Delta$$

Estimation of Probability: $\mathbf{P}^{\mathbf{N}}_{\mathbf{risk}} \leq \Delta$ $\qquad$ $\mathbf{P}^{\mathbf{N}}_{\mathbf{success}} \geq 1 - \Delta$

$$\text{Probability}(\text{Failure}) \leq \Delta \qquad \text{Probability}(\text{Success}) \geq 1 - \Delta$$

Upper bound Estimation: $\mathbf{P}^{\mathbf{U}}_{\mathbf{risk}} \leq \Delta$ $\qquad$ **Lower bound Estimation:** $\mathbf{P}^{\mathbf{L}}_{\mathbf{risk}} \leq \Delta$

- **Sampling** based methods like Monte Carlo **Do Not** provide such upper/lower bound estimations of the risk. (They provide probabilistic upper/lower bound)

- Hence, **chance constraints are not guaranteed** to be satisfied.

Approaches:

- Sampling Based Methods

- Indicator Function Based Methods

➢ Indicator Function Based Methods

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \le \mathcal{P}_j(x,\omega) \le u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m$$

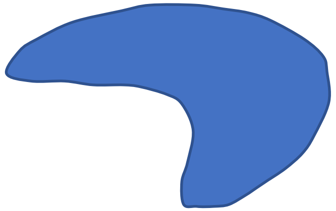
$$= \int_{\mathcal{K}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m$$

$$= \int \mathbf{I}_{\mathcal{K}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m = \mathrm{E}[\mathbf{I}_{\mathcal{K}}]$$

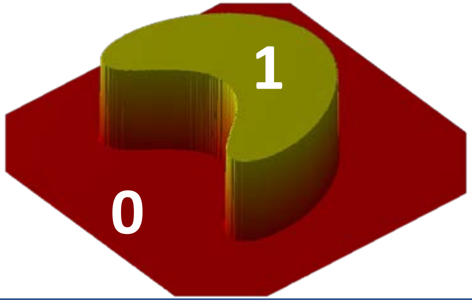
Indicator function

$$\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & \forall (x,\omega) \in \mathcal{K}, \\ 0 & \forall (x,\omega) \notin \mathcal{K} \end{cases}$$

Set $\mathcal{K}$



Indicator function $\mathbf{I}_{\mathcal{K}}$



Indicator Function Based Methods

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \le \mathcal{P}_j(x,\omega) \le u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m$$

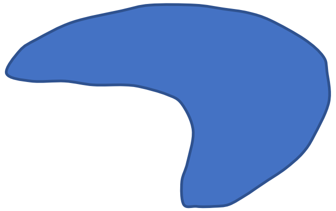
$$= \int_{\mathcal{K}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m$$

$$= \int \mathbf{I}_{\mathcal{K}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m = \mathrm{E}[\mathbf{I}_{\mathcal{K}}]$$

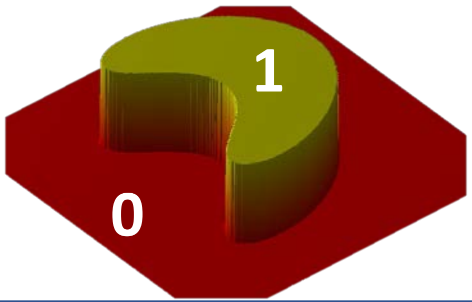
Indicator function

$$\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & \forall (x,\omega) \in \mathcal{K}, \\ 0 & \forall (x,\omega) \notin \mathcal{K} \end{cases}$$

Set $\mathcal{K}$



Indicator function $\mathbf{I}_{\mathcal{K}}$



- If we describe (approximate) the indicator function, we can calculate the integral easily.

➢ Indicator Function Based Methods

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \mathrm{E}[\mathbf{I}_{\mathcal{K}}]$$

- **Probability Bounds and Indictor function approximations**

  1) Markov Bound

  2) Chebyshev Bound

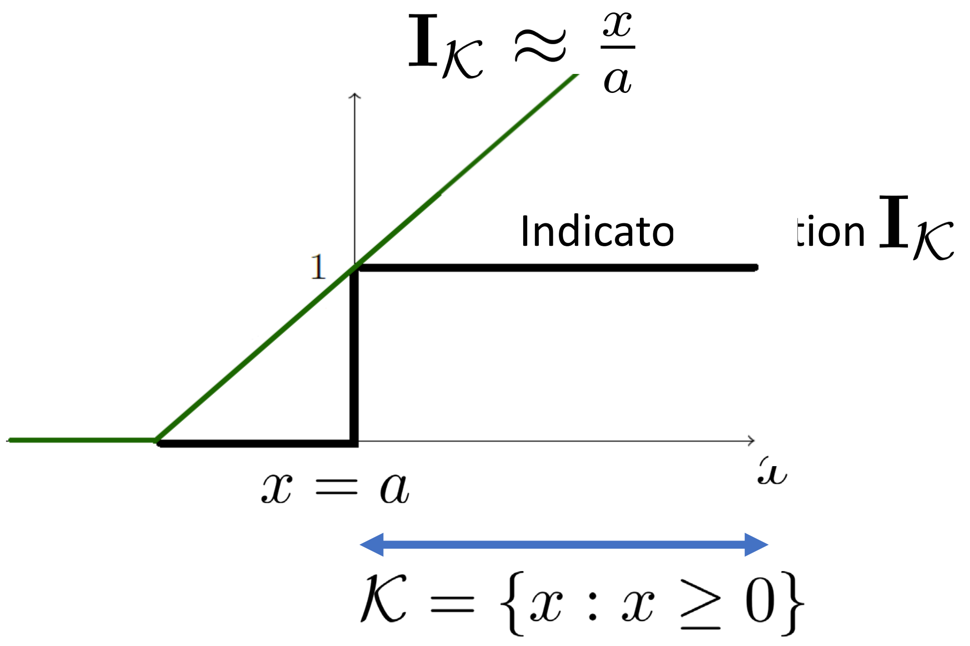
  3) Chernoff Bound

1) Markov Bound

- Let $x \geq 0$ be a random variable with probability distribution $pr(x)$

$$\text{Probability}_{pr(x)}\{x \geq a\} \leq \mathrm{E}[\mathbf{I}_{\mathcal{K}}] = \frac{\mathrm{E}[x]}{a}$$

- $\frac{x}{a}$ is an upper bound approximation of the indicator function



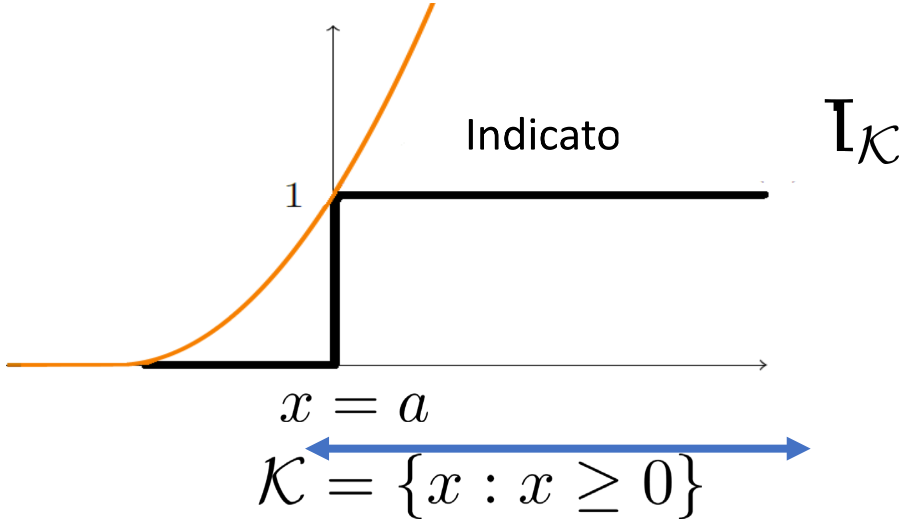
1) Markov Bound- Higher order extension

- Let $x \geq 0$ be a random variable with probability distribution $pr(x)$

$$\text{Probability}_{pr(x)}\{x \geq a\} \leq \mathrm{E}[\mathbf{I}_{\mathcal{K}}] = \frac{\mathrm{E}[|x|^n]}{a^n}$$

- $\frac{|x|^n}{a^n}$ is an upper bound approximation of the indicator function

$$\mathbf{I}_{\mathcal{K}} \approx \frac{|x|^n}{a^n}$$



2) Chebyshev Bound

- Let x be a random variable with probability distribution $pr(x)$

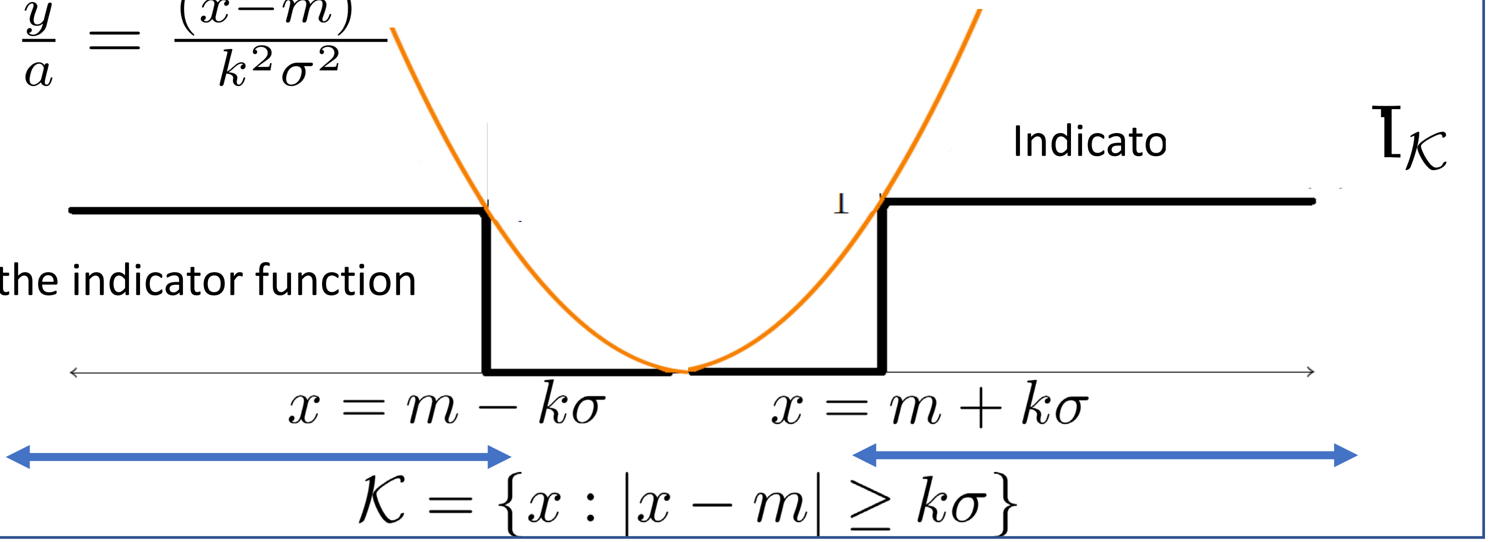
$$\text{Probability}_{pr(x)}\{|x-m| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$m: \ \text{mean}$$
$$\sigma^2: \mathrm{E}[(x-m)^2]$$

$$\text{Probability}_{pr(x)}\{|x-m| \geq k\sigma\} = \text{Probability}_{pr(x)}\{\overbrace{(x-m)^2}^{y} \geq \overbrace{k^2\sigma^2}^{a}\} \xRightarrow[\text{Markov Bound}]{} \leq \mathrm{E}[\mathbf{I}_{\mathcal{K}}] = \frac{\mathrm{E}[y]}{a} = \frac{1}{k^2}\frac{\mathrm{E}[(x-m)^2]}{\sigma^2} = \frac{1}{k^2}$$

$$\mathbf{I}_{\mathcal{K}} \approx \frac{y}{a} = \frac{(x-m)^2}{k^2\sigma^2}$$

- $\frac{(x-m)^2}{k^2\sigma^2}$ is an upper bound approximation of the indicator function



3) Chernoff bound

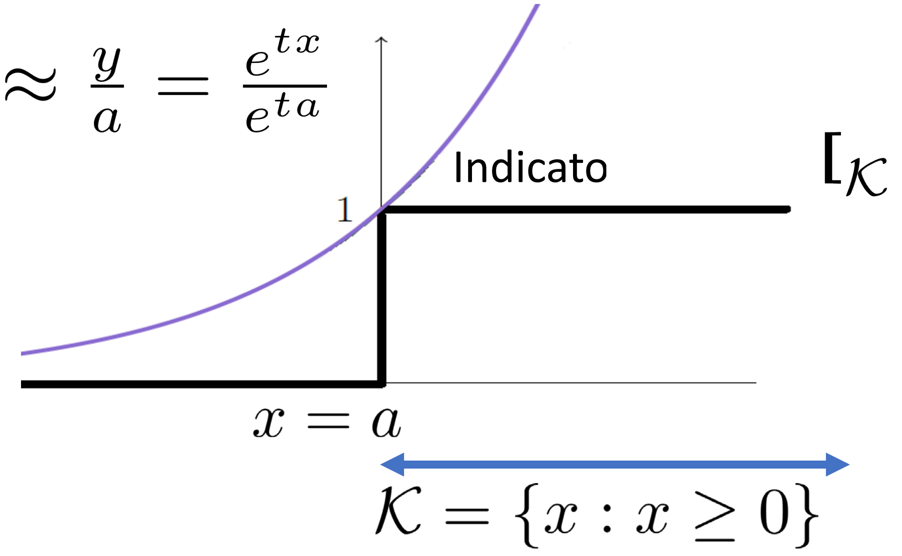
- Let x be a random variable with probability distribution $pr(x)$

$$\text{Probability}_{pr(x)}\{x \geq a\} \leq \frac{\mathrm{E}[e^{tx}]}{e^{ta}} \qquad \text{for every } t > 0$$

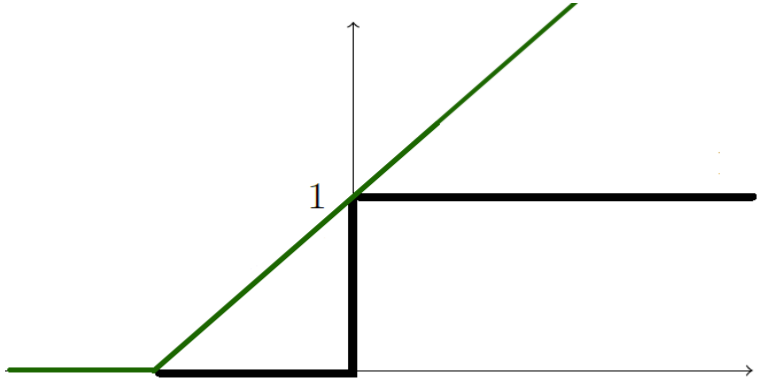
$$\text{Probability}_{pr(x)}\{x \geq a\} = \text{Probability}_{pr(x)}\{\overbrace{e^{tx}}^{y} \geq \overbrace{e^{ta}}^{a}\} \xrightarrow[\text{Markov Bound}]{} \leq \mathrm{E}[\mathbf{I}_{\mathcal{K}}] = \frac{\mathrm{E}[y]}{a} = \frac{\mathrm{E}[e^{tx}]}{e^{ta}}$$

- $\frac{e^{tx}}{e^{ta}}$ is an upper bound approximation of the indicator function

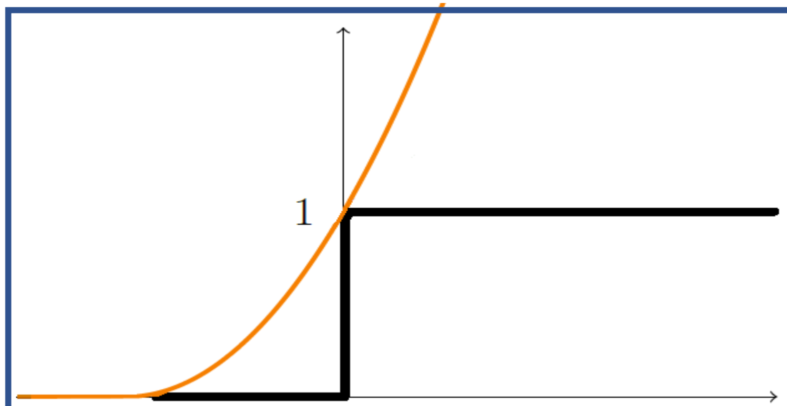
$$\mathbf{I}_{\mathcal{K}} \approx \frac{y}{a} = \frac{e^{tx}}{e^{ta}}$$



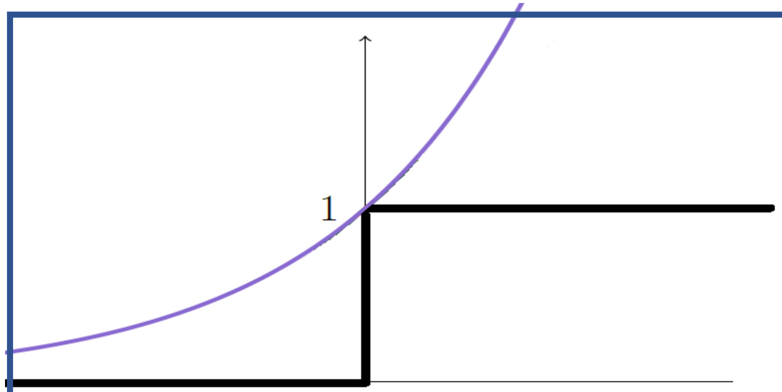
Probability Bounds and Indictor function approximations



Markov Bound



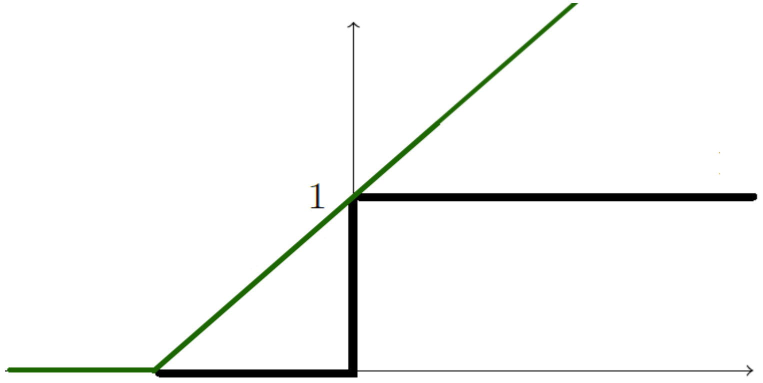
Chebyshev Bound



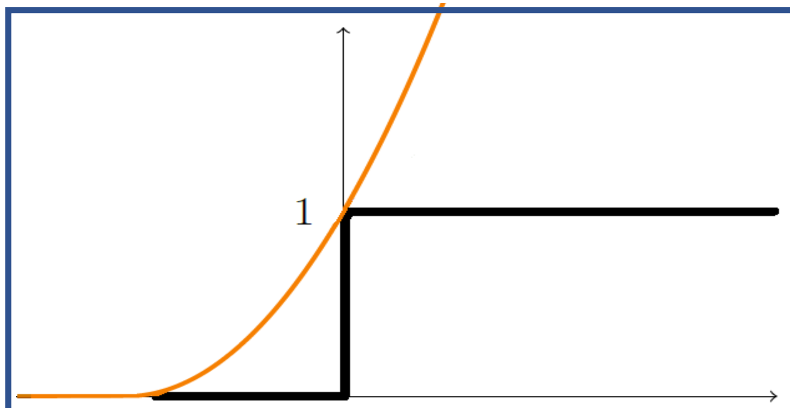
Chernoff Bound

- Poor polynomial indictor functions.

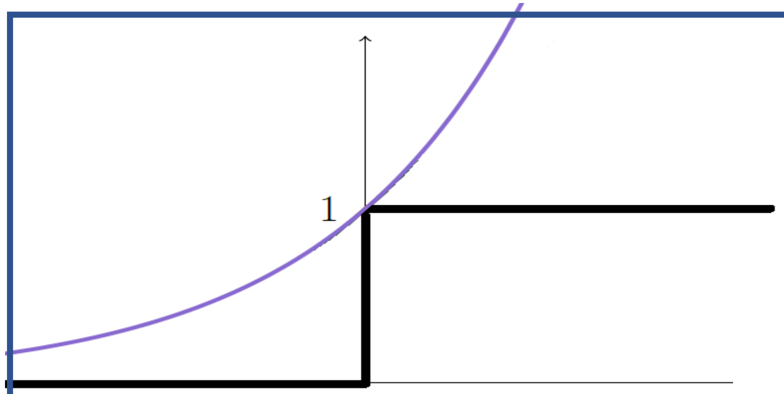
Probability Bounds and Indictor function approximations



Markov Bound



Chebyshev Bound

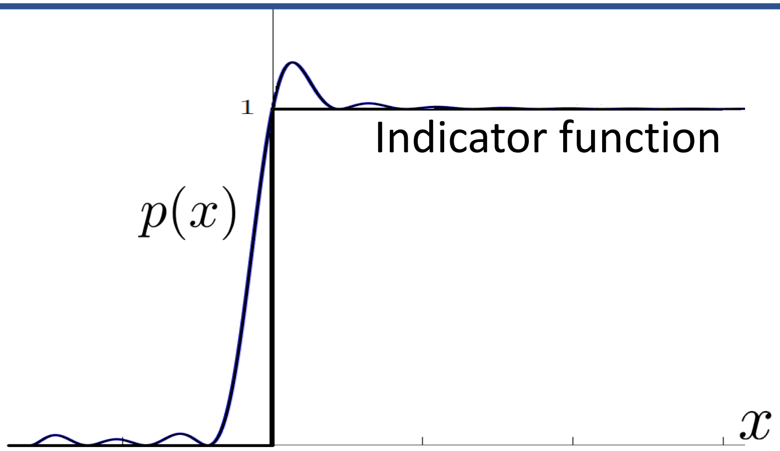


Chernoff Bound

- To improve the probability bounds, we need to look for better approximation of Indicator function

Polynomial Approximation of Indictor function

$$\text{Probability}_{pr(x)}\{x \geq a\} \leq \mathrm{E}[p(x)]$$



➢ Indicator Function Based Methods

$$\mathbf{P}^*_{\mathbf{risk}} := \mathrm{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \mathrm{E}[\mathbf{I}_{\mathcal{K}}]$$

- **Probability Bounds and Indictor function approximations**

  1) Markov Bound

  2) Chebyshev Bound

  3) Chernoff Bound

  4) Polynomial based Bound

Polynomial based Probability Bound

Polynomial based Probability Bound

- Sum-of-Squares Program for Upper/Lower Bound Probability Estimation
- Moment Program for Upper/Lower Bound Probability Estimation (Dual Optimization)
- Modified SOS/Moment Program for Probability Estimation in High Dimensions

Sum-of-Squares Program for Upper/Lower Bound Probability Estimation

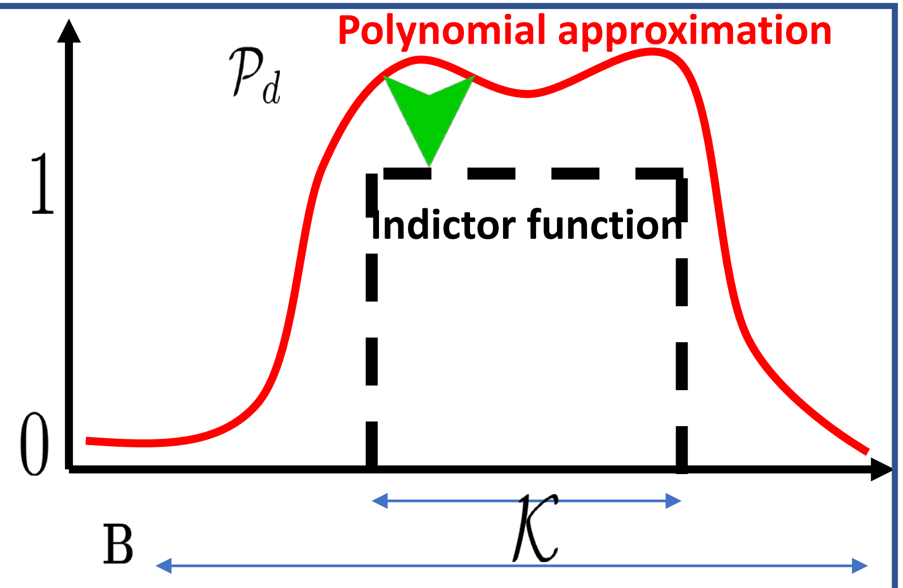
Upper Bound Probability Estimation

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\} = \int_{\underbrace{\{(x,\omega):\{l_j \le \mathcal{P}_j(x,\omega)\le u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m$$

$$\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & \forall (x,\omega) \in \mathcal{K}, \\ 0 & \forall (x,\omega) \notin \mathcal{K} \end{cases}$$

$$= \int \mathbf{I}_{\mathcal{K}} pr(x)pr(\omega)dx_1...dx_n d\omega_1...d\omega_m = \mathrm{E}[\mathbf{I}_{\mathcal{K}}]$$

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega)dxd\omega \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \ge 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \ge 0 \end{aligned}$$



B : simple box that contains safety set

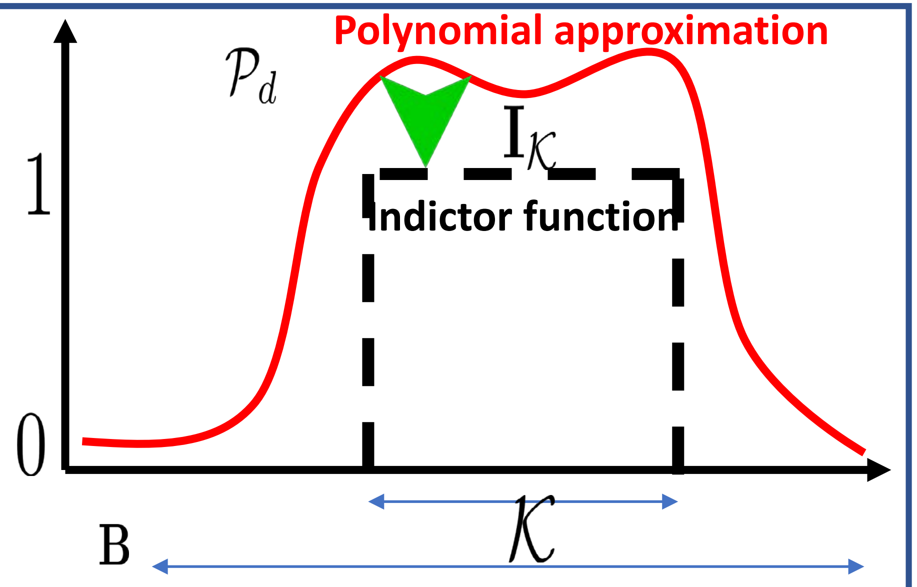
Assumption: After rescaling of polynomials $\mathrm{B} = [-1,1]^{n+m}$

Upper Bound Probability Estimation

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x,\omega)dxd\omega$$

$$\text{subject to} \quad \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega)\in\mathcal{K}$$

$$\mathcal{P}_d(x,\omega) \geq 0$$



Polynomial $\mathcal{P}_d(x,\omega)$ is an upper bound approximation of indicator function $\mathrm{I}_{\mathcal{K}}$ and monotonically converges as its degree d increases.

$$\mathbf{P}_{\mathbf{risk}} \leq \mathrm{E}[\mathcal{P}_d(x,\omega)] \qquad \lim_{d\to\infty} \mathrm{E}[\mathcal{P}_d(x,\omega)] = \mathbf{P}_{\mathbf{risk}}$$

F. Dabbene, D. Henrion, ”Set approximation via minimum-volume polynomial sublevel sets”, European Control Conference (ECC), Switzerland, 2013.
D. Henrion, J. B. Lasserre, C. Savorgnan, ”Approximate volume and integration for basic semialgebraic sets”, SIAM Review, 51(4), pp.722–743, 2009.

Upper Bound Probability Estimation

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x,\omega)\,dx\,d\omega$$
$$\text{subject to} \quad \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K}$$
$$\mathcal{P}_d(x,\omega) \geq 0$$

$$\mathbf{P}_{\mathbf{risk}} \leq \mathrm{E}[\mathcal{P}_d(x,\omega)]$$

$$\lim_{d\to\infty} \mathrm{E}[\mathcal{P}_d(x,\omega)] = \mathbf{P}_{\mathbf{risk}}$$

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x,\omega)pr(x)pr(\omega)\,dx\,d\omega \longrightarrow \mathrm{E}[\mathcal{P}_d(x,\omega)]$$
$$\text{subject to} \quad \left.\begin{array}{l}\mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ \mathcal{P}_d(x,\omega) \geq 0\end{array}\right\} \text{SOS Conditions}$$

Upper Bound Probability Estimation

- Probability distributions of uncertainties

$$x \in \mathbb{R}^n \sim pr(x) \qquad \omega \in \mathbb{R}^m \sim pr(\omega)$$

- Uncertain safety set

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x,\omega) \leq u_j, j = 1, \dots, \ell\ \}$$

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

Upper Bound

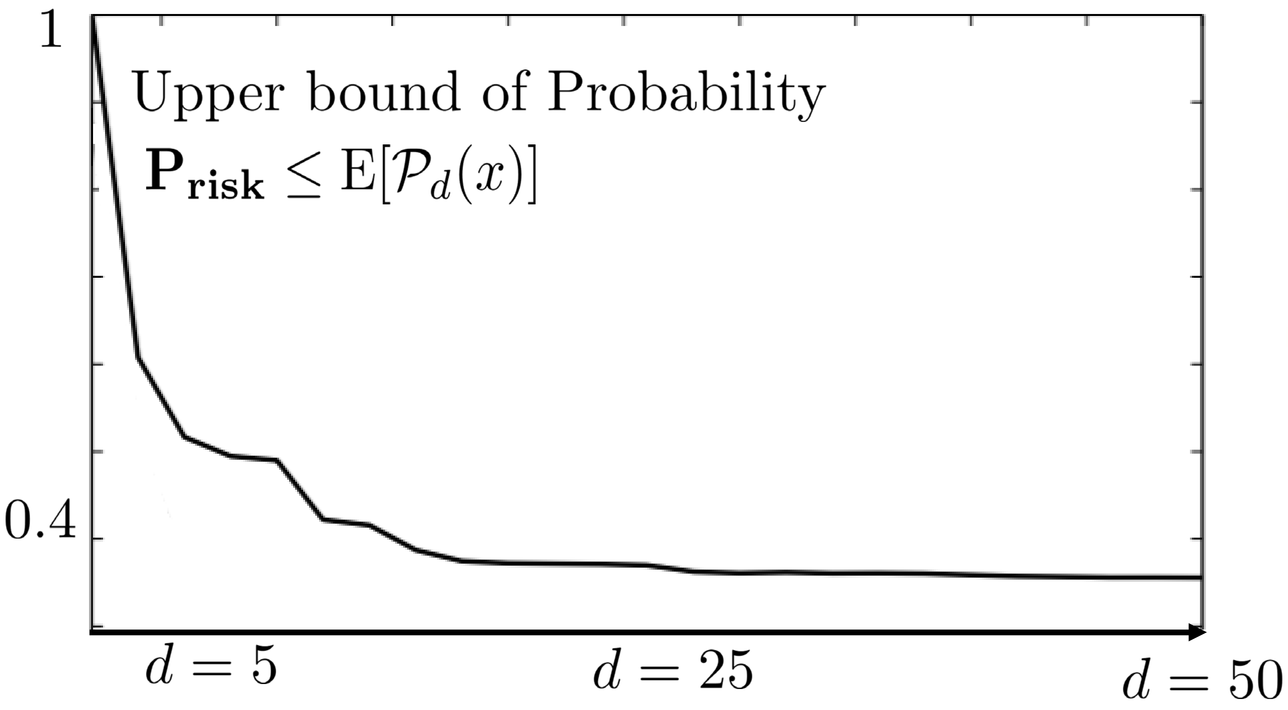
$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x,\omega)pr(x)pr(\omega)dxd\omega \longrightarrow \mathrm{E}[\mathcal{P}_d(x,\omega)] \geq \mathbf{P}_{\mathbf{risk}}$$

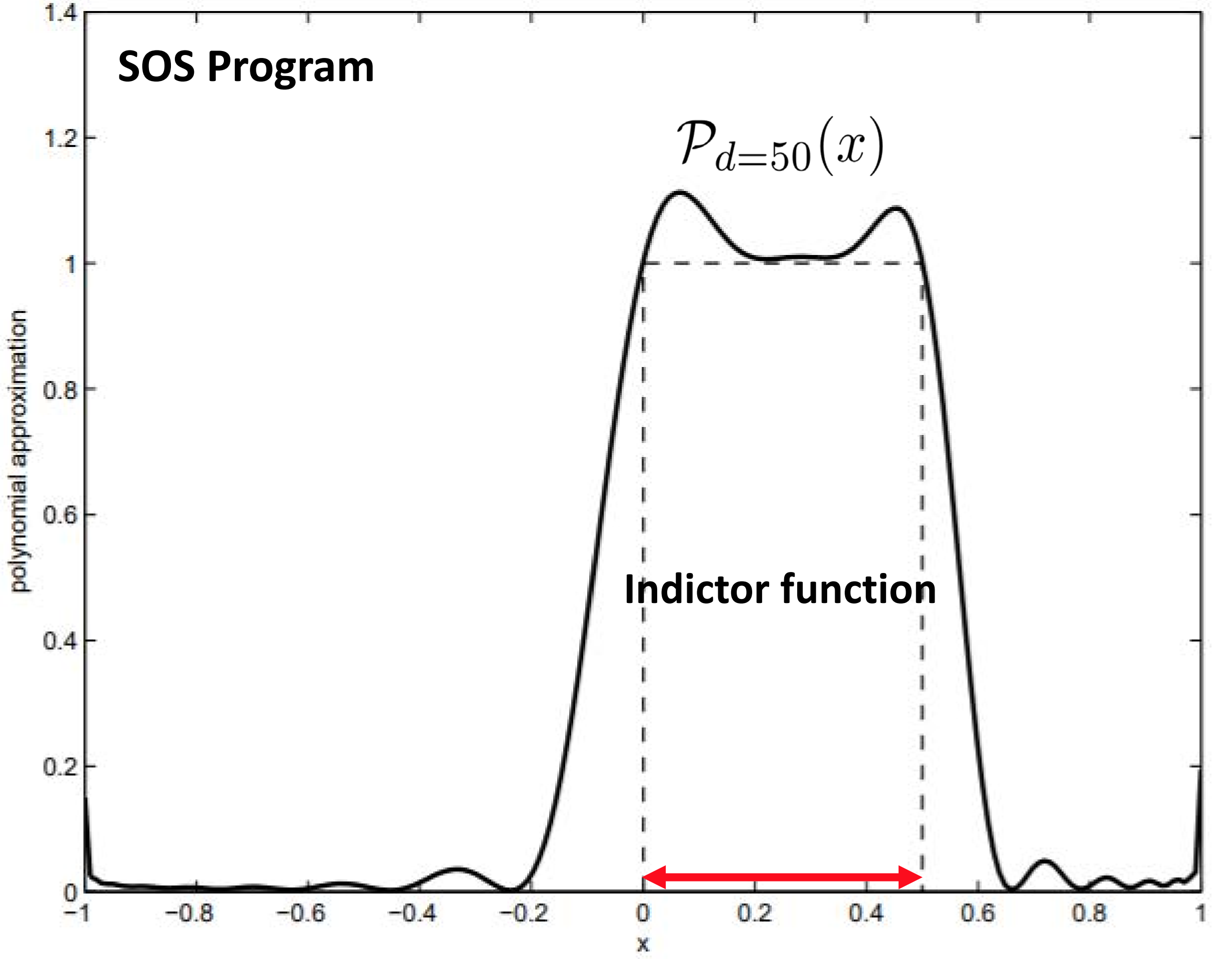
$$\text{subject to} \quad \left.\begin{array}{l} \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ \mathcal{P}_d(x,\omega) \geq 0 \end{array}\right\} \text{SOS Conditions}$$

Example 1

$$x \sim pr(x) : \text{Uniform}([-1,1])$$

$$\text{Probability}_{pr(x)}\{x \in \underbrace{\{x : x(0.5 - x) \geq 0\}}_{\text{Safety constraint}}\}$$





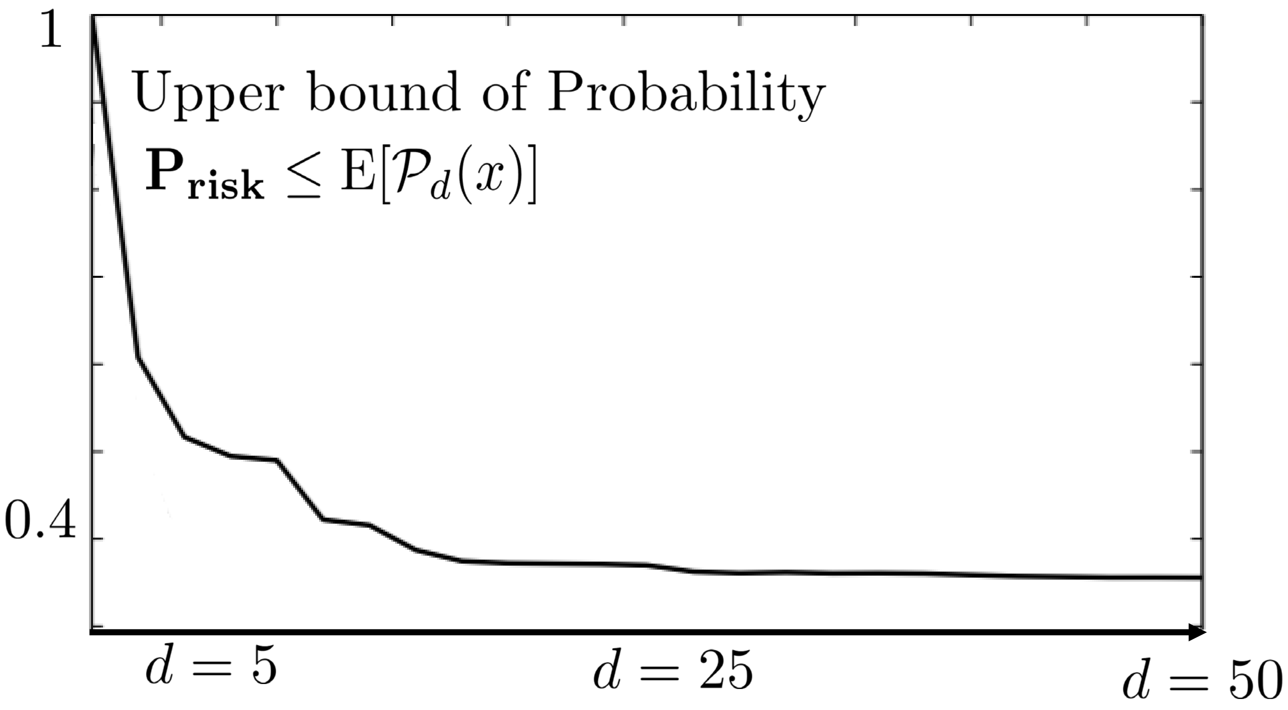
$$\{x : x(0.5 - x) \geq 0\}$$

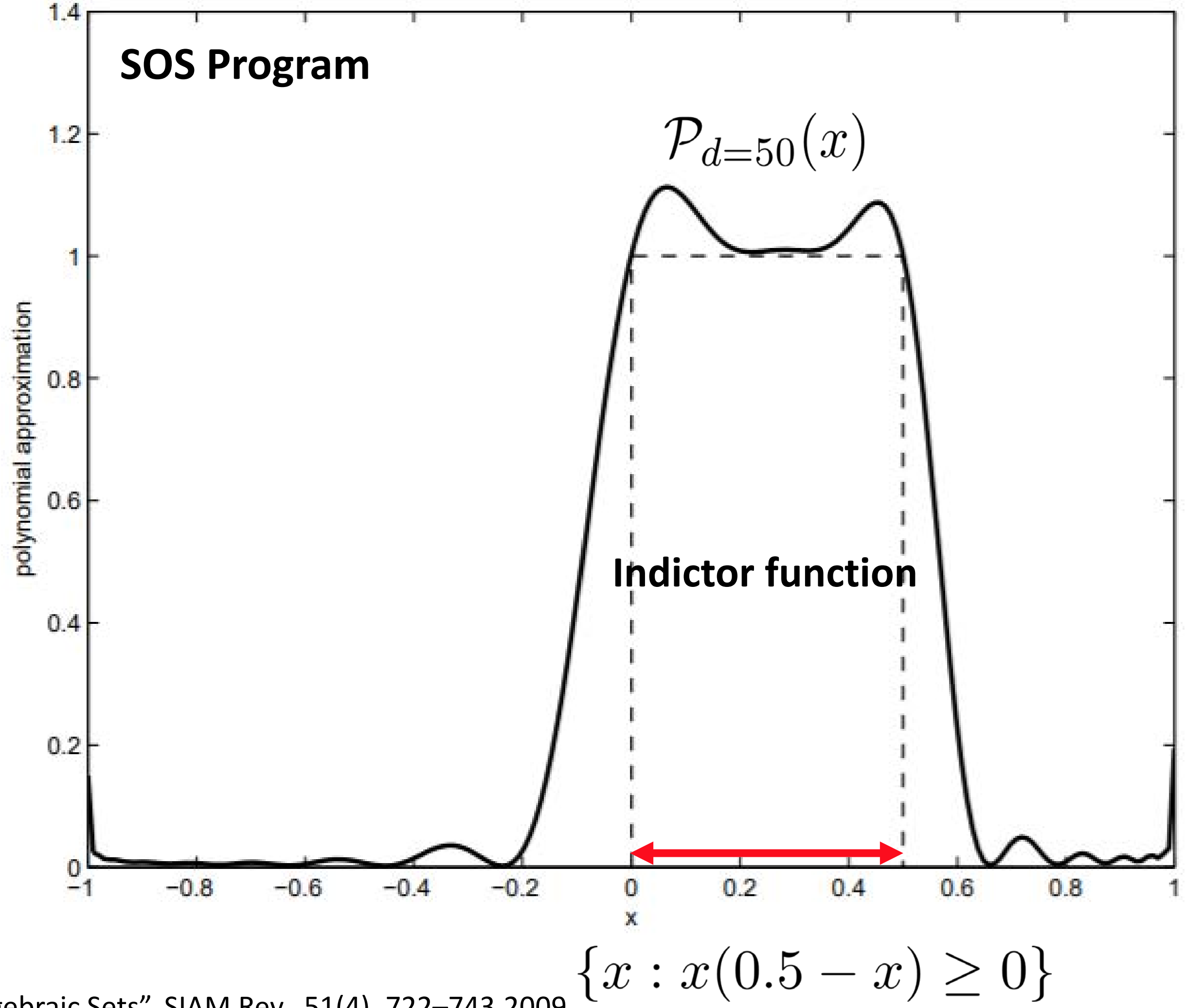
D. Henrion, J. B. Lasserre, and C. Savorgnan “Approximate Volume and Integration for Basic Semialgebraic Sets”, SIAM Rev., 51(4), 722–743,2009.

Example 1

$$x \sim pr(x) : \text{Uniform}([-1, 1])$$

$$\text{Probability}_{pr(x)}\{x \in \underbrace{\{x : x(0.5 - x) \geq 0\}}_{\text{Safety constraint}}\}$$





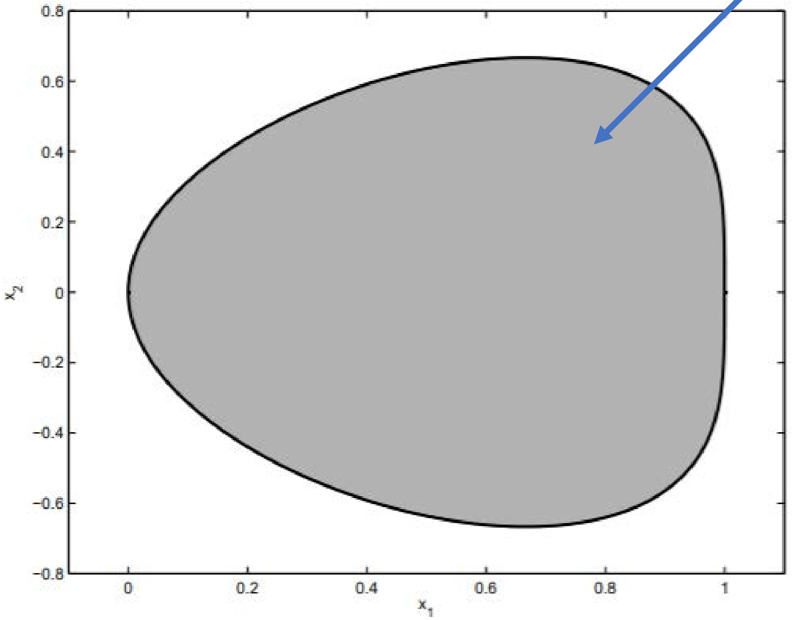
https://github.com/jasour/rarnop19/blob/master/Lecture10_Probabilistic_Safety_Verification/Risk_Estimation/Example_1/Example_1_MomentSDP_upper.m

D. Henrion, J. B. Lasserre, and C. Savorgnan “Approximate Volume and Integration for Basic Semialgebraic Sets”, SIAM Rev., 51(4), 722–743,2009.

Example 2

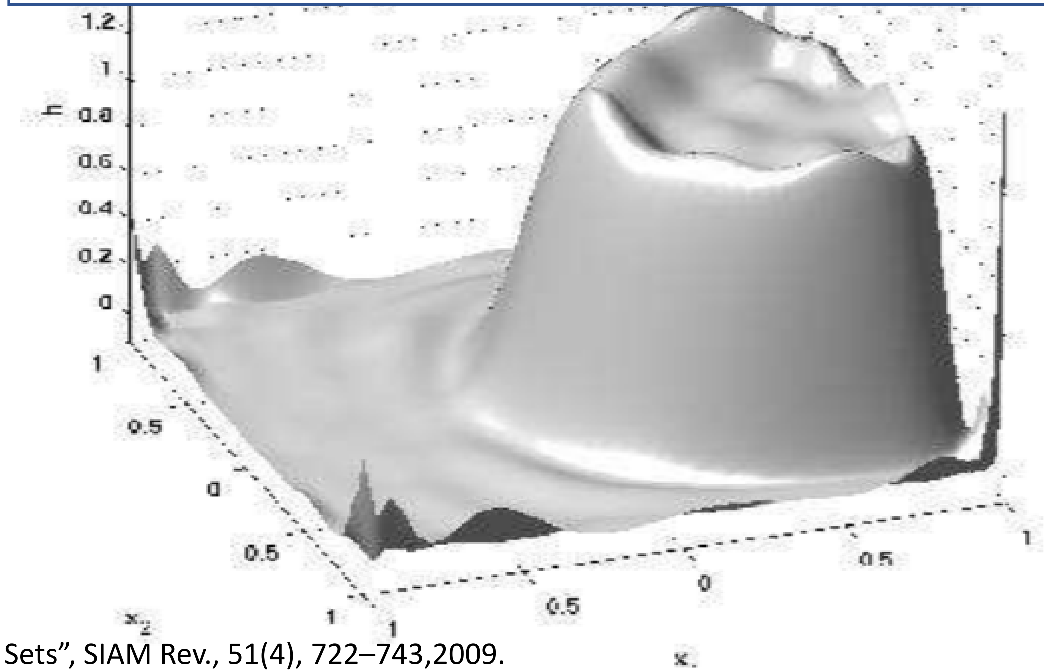
$$(x_1, x_2) \sim pr(x) : \text{Uniform}([-1,1]^2)$$

$$\text{Probability}_{pr(x)}\{(x_1,x_2) \in \underbrace{\{(x_1,x_2) : x_1(x_1^2+x_2^2) - (x_1^4 + x_1^2 x_2^2 + x_2^4) \geq 0\}}_{\text{Safety constraint}}\}$$



SOS Program $\mathbf{B} = [-1,1]^2$

$$\boxed{\mathcal{P}_{d=20}(x) \qquad \mathbf{P}_{\mathbf{risk}} \leq \mathrm{E}[\mathcal{P}_d(x)]}$$

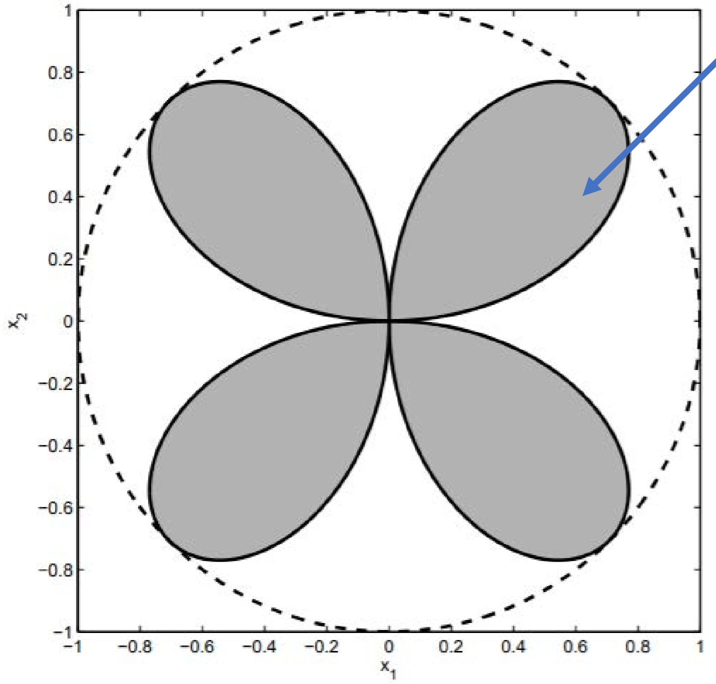


D. Henrion, J. B. Lasserre, and C. Savorgnan "Approximate Volume and Integration for Basic Semialgebraic Sets", SIAM Rev., 51(4), 722–743,2009.

Example 3

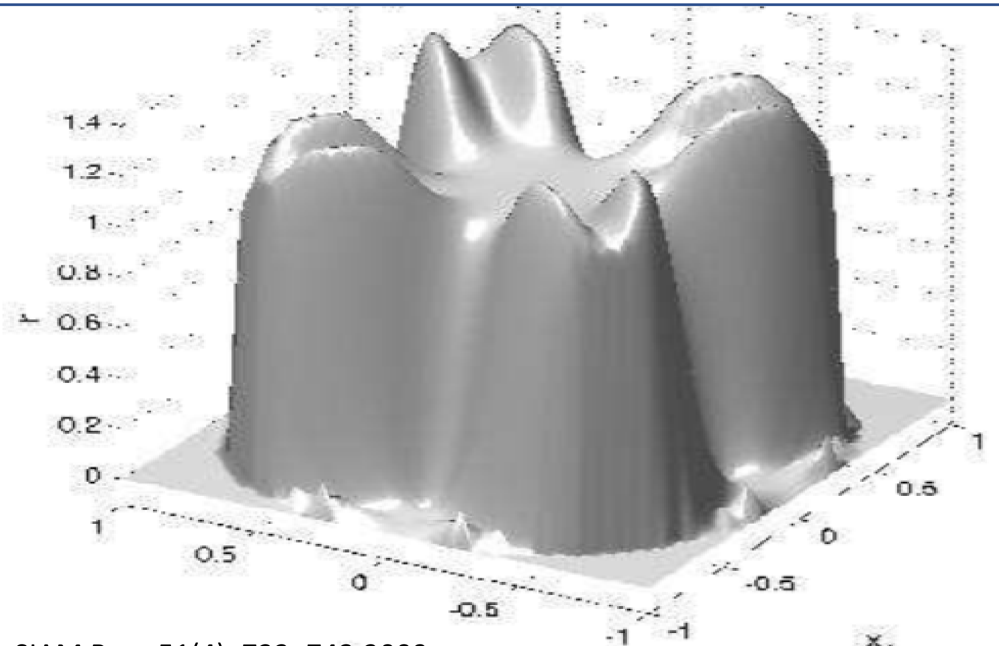
$$(x_1, x_2) \sim pr(x) : \text{Uniform}([-1,1]^2)$$

$$\text{Probability}_{pr(x)}\{(x_1,x_2) \in \underbrace{\{(x_1,x_2) : x_1(x_1^2+x_2^2) - (x_1^4 + x_1^2 x_2^2 + x_2^4) \geq 0\}}_{\text{Safety constraint}}\}$$



SOS Program $\mathbf{B} = [-1,1]^2$

$$\boxed{\mathcal{P}_{d=20}(x) \qquad \mathbf{P_{risk}} \leq \mathrm{E}[\mathcal{P}_d(x)]}$$

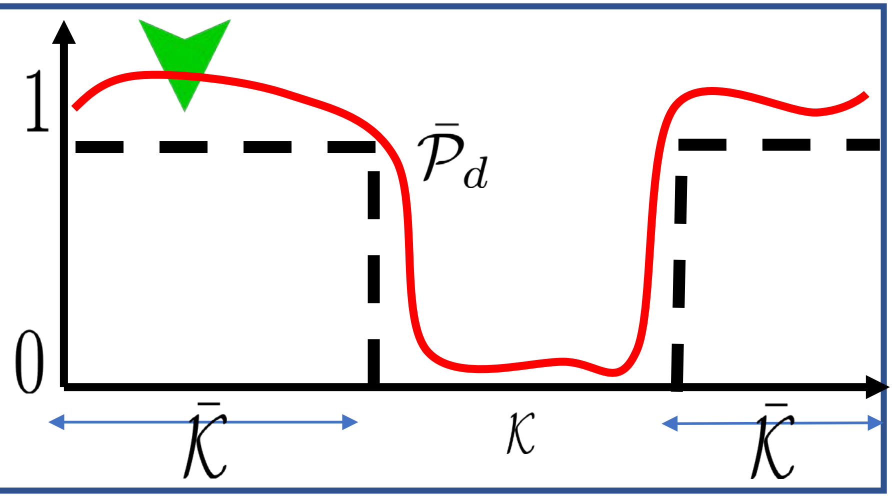


D. Henrion, J. B. Lasserre, and C. Savorgnan “Approximate Volume and Integration for Basic Semialgebraic Sets”, SIAM Rev., 51(4), 722–743, 2009.

Lower Bound Probability Estimation

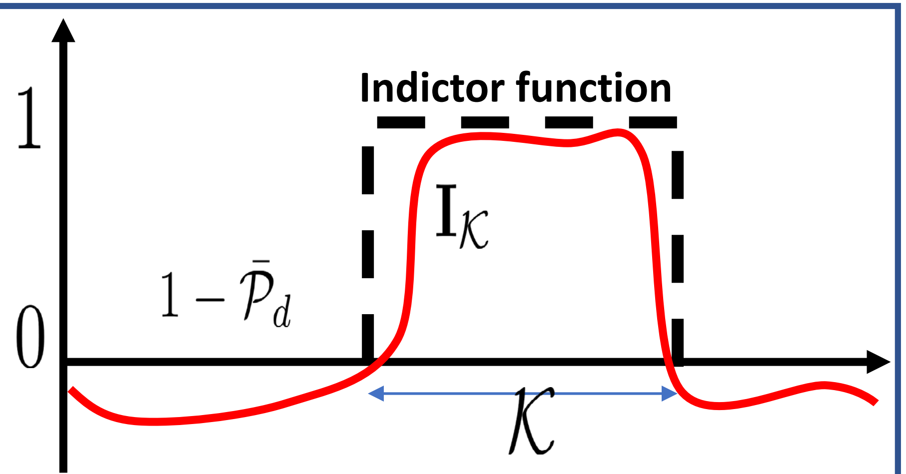
$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$
$$= \int_{\overbrace{\{(x,\omega):\{l_j \le \mathcal{P}_j(x,\omega) \le u_j\}_{j=1}^{\ell}\}}^{\mathcal{K}}} pr(x)pr(\omega)dx_1 \dots dx_n d\omega_1 \dots d\omega_m = \text{E}[\mathbf{I}_{\mathcal{K}}]$$

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{P}}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int_{\mathbf{B}} \bar{\mathcal{P}}_d(x,\omega)dxd\omega$$
$$\text{subject to} \quad \bar{\mathcal{P}}_d(x,\omega) - 1 \ge 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$$
$$\bar{\mathcal{P}}_d(x,\omega) \ge 0$$



Polynomial $1 - \bar{\mathcal{P}}_d(x,\omega)$ is an lower bound approximation of indicator function $\mathbf{I}_{\mathcal{K}}$ and monotonically converges as its degree d increases.

$$1 - \text{E}[\bar{\mathcal{P}}_d(x,\omega)] \le \mathbf{P}_{\mathbf{risk}} \qquad \lim_{d\to\infty} 1 - \text{E}[\bar{\mathcal{P}}_d(x,\omega)] = \mathbf{P}_{\mathbf{risk}}$$



Polynomial based Probability Estimation

- Probability distributions of uncertainties

$$x \in \mathbb{R}^n \sim pr(x) \qquad \omega \in \mathbb{R}^m \sim pr(\omega)$$

- Uncertain safety set

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x,\omega) \leq u_j, j = 1, \ldots, \ell\ \}$$

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

Upper Bound

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x,\omega)pr(x)pr(\omega)dxd\omega \longrightarrow \mathrm{E}[\mathcal{P}_d(x,\omega)] \geq \mathbf{P}_{\mathbf{risk}}$$

$$\left.\begin{aligned} \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}\right\} \text{SOS Conditions}$$

Lower Bound

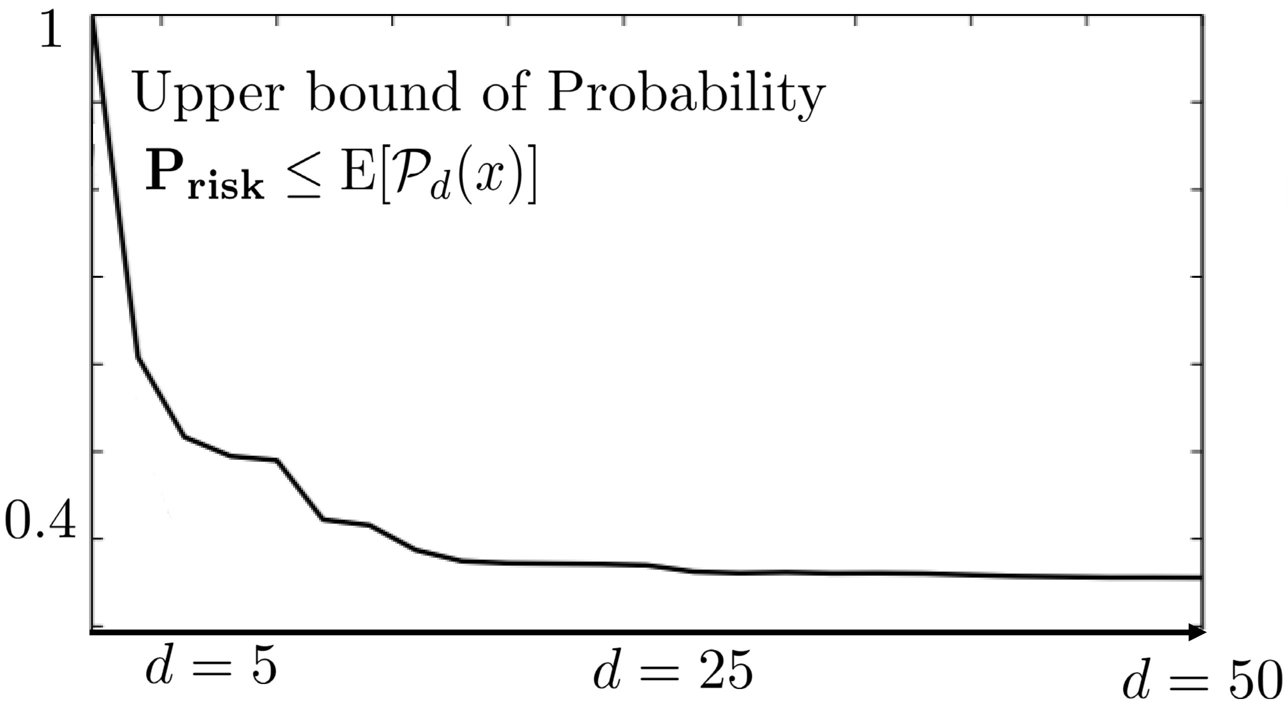
$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{P}}_d(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int_{\mathbf{B}} \bar{\mathcal{P}}_d(x,\omega)pr(x)pr(\omega)dxd\omega \longrightarrow \mathrm{E}[\bar{\mathcal{P}}_d(x,\omega)] \leq \mathbf{P}_{\mathbf{risk}}$$

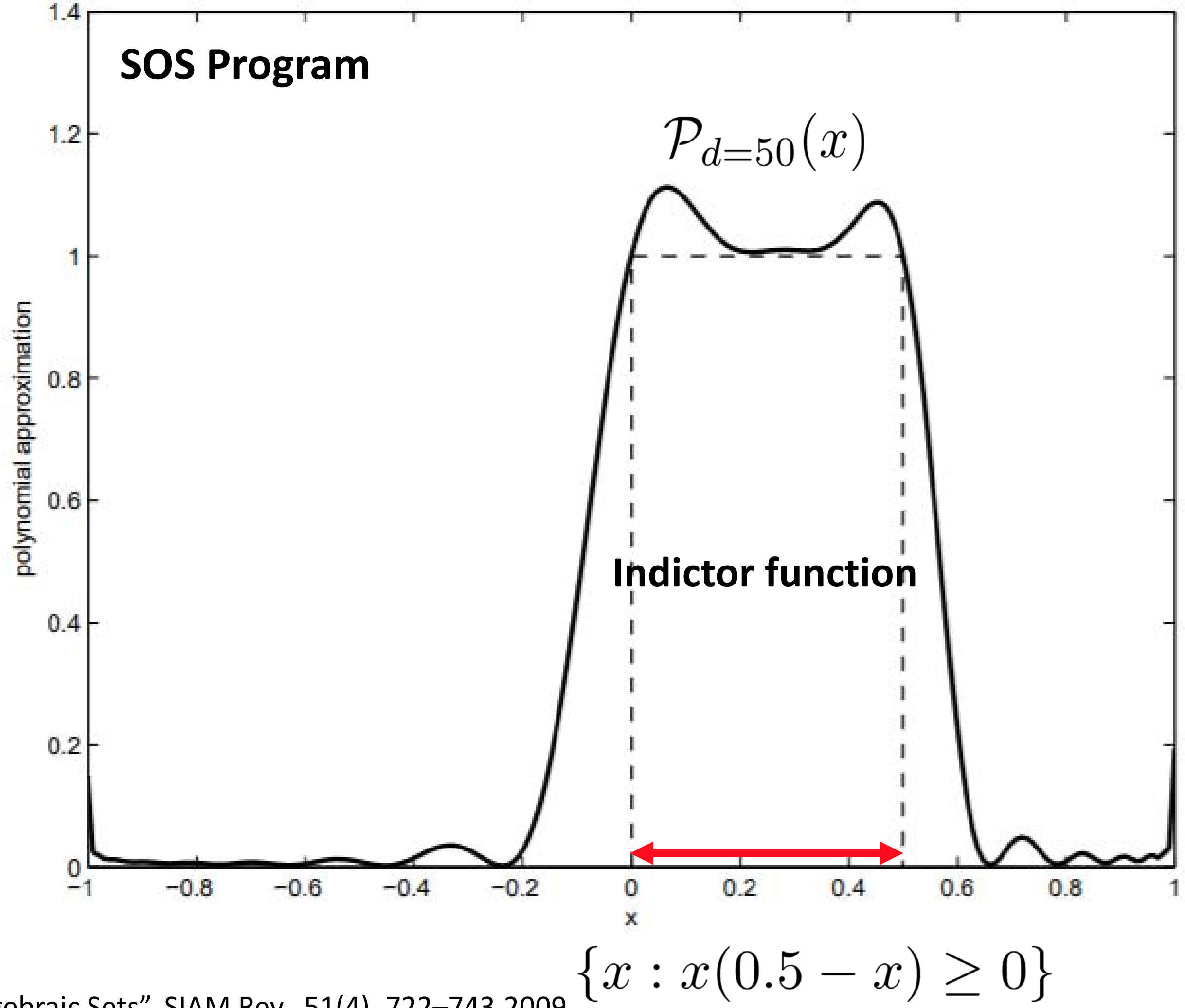
$$\left.\begin{aligned} \text{subject to} \quad & \bar{\mathcal{P}}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}} \\ & \bar{\mathcal{P}}_d(x,\omega) \geq 0 \end{aligned}\right\} \text{SOS Conditions}$$

Example 1

$$x \sim pr(x) : \text{Uniform}([-1,1])$$

$$\text{Probability}_{pr(x)}\{x \in \underbrace{\{x : x(0.5-x) \geq 0\}}_{\text{Safety constraint}}\}$$





D. Henrion, J. B. Lasserre, and C. Savorgnan "Approximate Volume and Integration for Basic Semialgebraic Sets", SIAM Rev., 51(4), 722–743,2009.

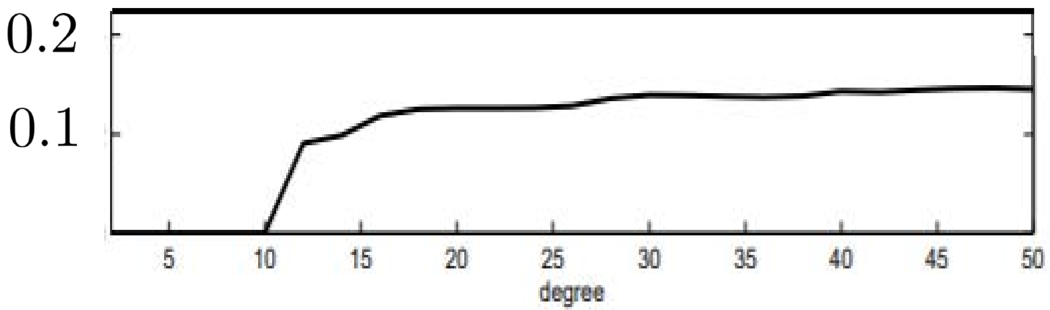
Example 1

$$x \sim pr(x) : \text{Uniform}([-1, 1])$$

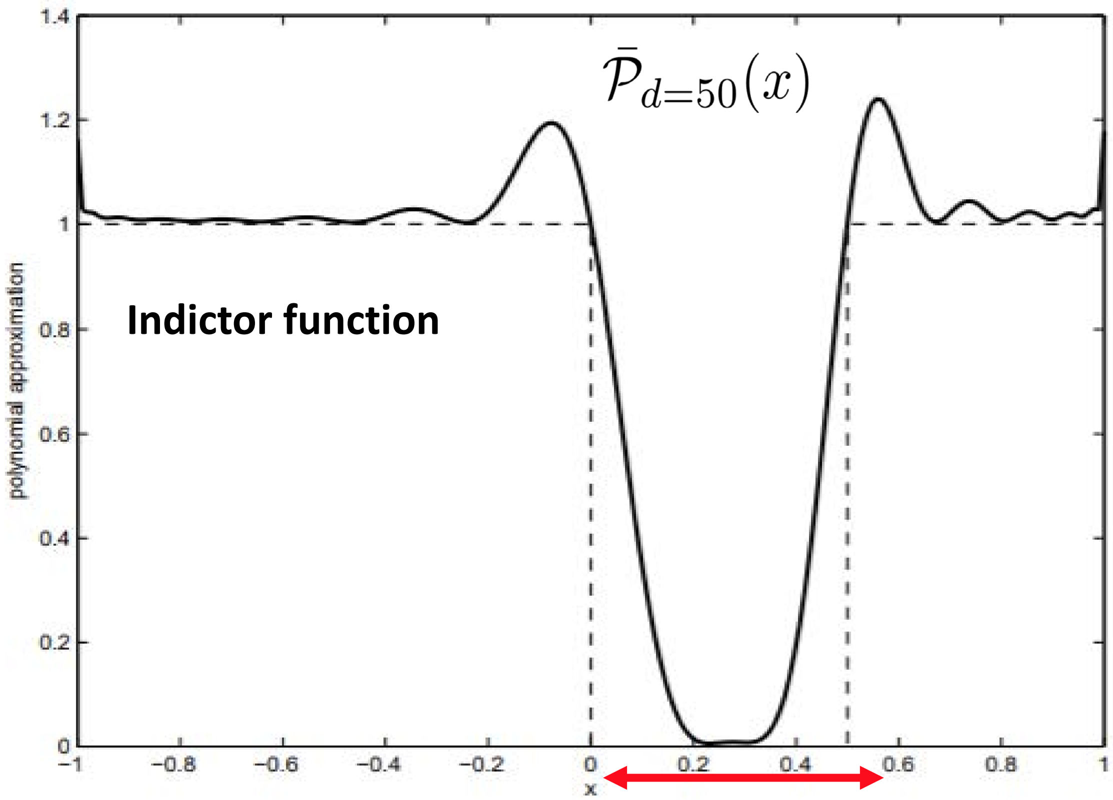
$$\text{Probability}_{pr(x)}\{x \in \underbrace{\{x : x(0.5 - x) \geq 0\}}_{\text{Safety constraint}}\}$$

$$1 - \text{E}[\bar{\mathcal{P}}_d(x)] \leq \mathbf{P_{risk}}$$

Lower bound of Probability



SOS Program



$$\{x : x(0.5 - x) \geq 0\}$$

D. Henrion, J. B. Lasserre, and C. Savorgnan "Approximate Volume and Integration for Basic Semialgebraic Sets", SIAM Rev., 51(4), 722–743,2009.

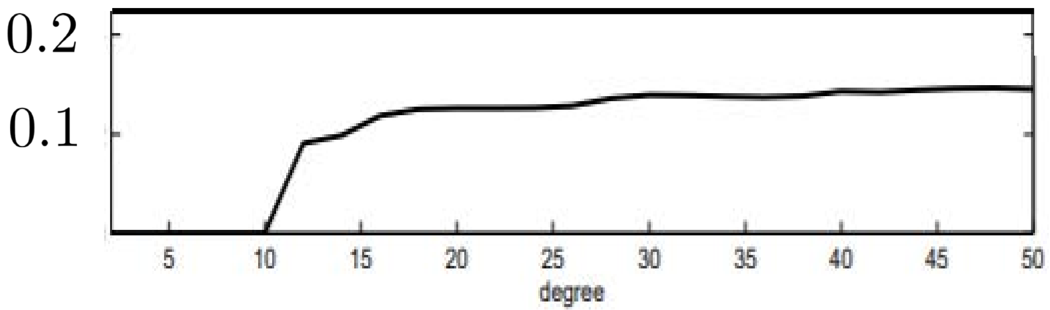
Example 1

$$x \sim pr(x) : \text{Uniform}([-1,1])$$

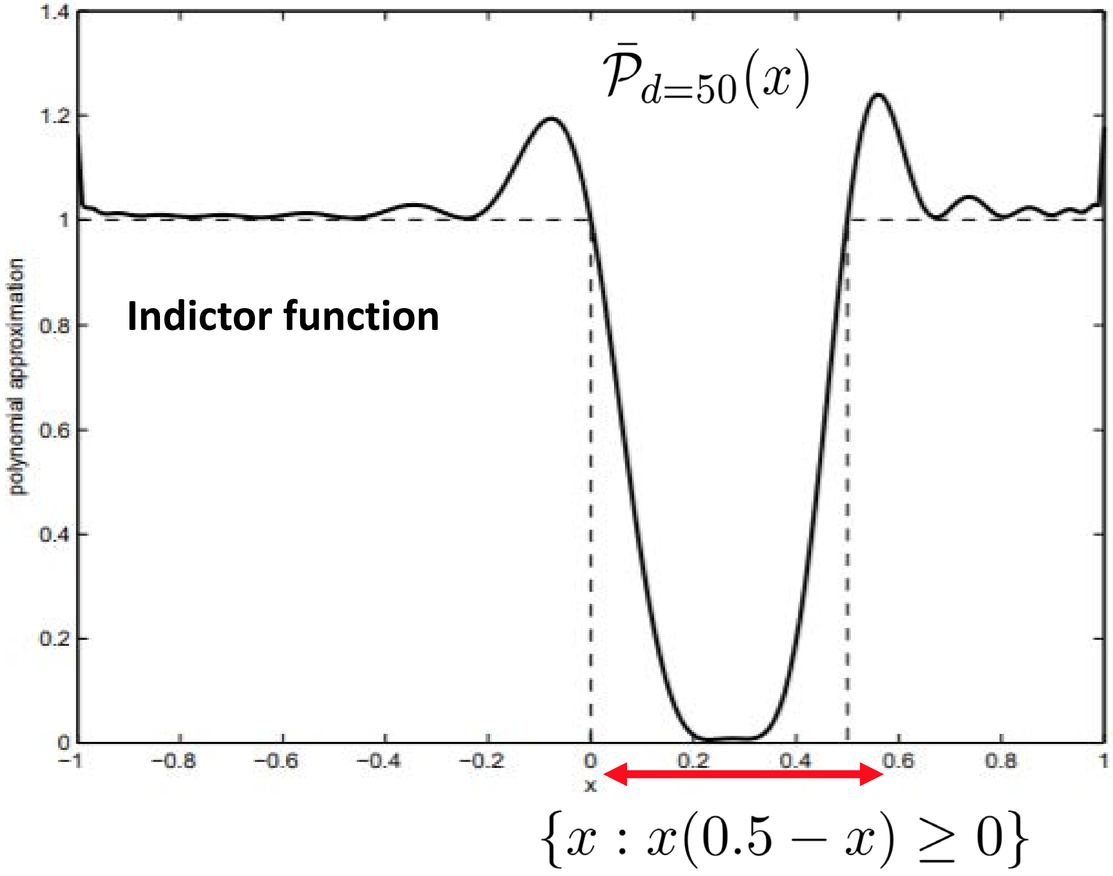
$$\text{Probability}_{pr(x)}\{x \in \underbrace{\{x : x(0.5 - x) \geq 0\}}_{\text{Safety constraint}}\}$$

$$1 - \mathrm{E}[\bar{\mathcal{P}}_d(x)] \leq \mathbf{P_{risk}}$$

Lower bound of Probability



SOS Program

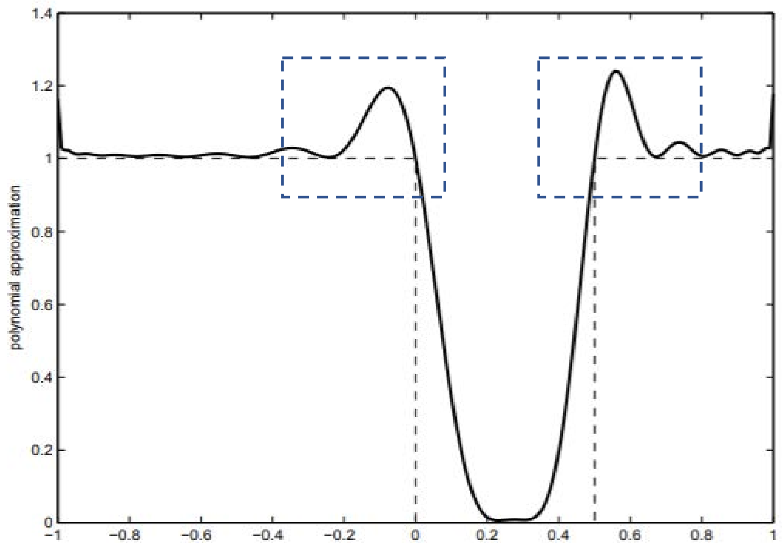


https://github.com/jasour/rarnop19/blob/master/Lecture10_Probabilistic_Safety_Verification/Risk_Estimation/Example_1/Example_1_MomentSDP_lower.m

D. Henrion, J. B. Lasserre, and C. Savorgnan "Approximate Volume and Integration for Basic Semialgebraic Sets", SIAM Rev., 51(4), 722–743,2009.

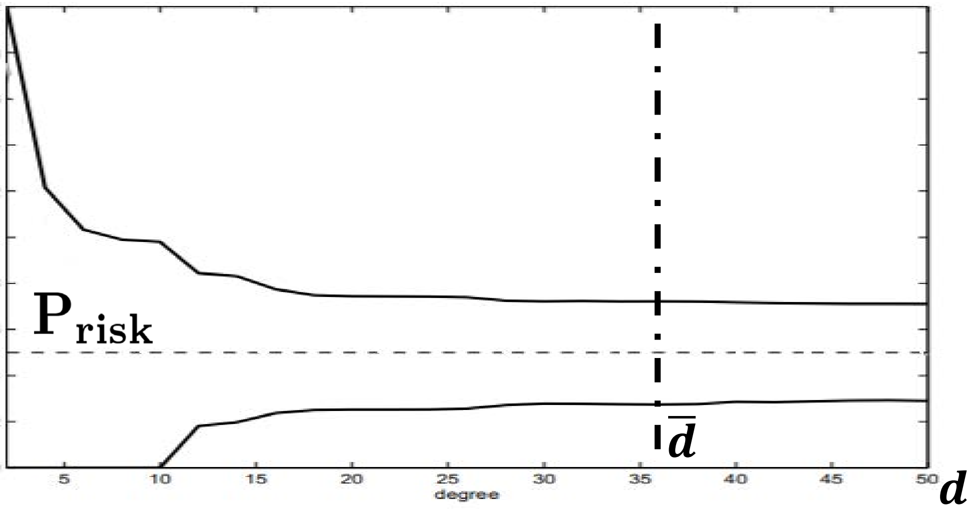
Numerical Improvements

- Obtained polynomial approximation of indicator function **oscillates** near the boundary of the set (Gibbs phenomenon)



- The quality of probability bounds do not really improve for degree greater than $\bar{d}$

$$\mathbf{P_{risk}} \leq \mathrm{E}[\mathcal{P}_d(x)]$$



$$1 - \mathrm{E}[\bar{\mathcal{P}}_d(x)] \leq \mathbf{P_{risk}}$$

$$\text{Probability}_{pr(x)}\{x \in \{x : x(0.5 - x) \geq 0\}\}$$

$$x \sim pr(x) : \text{Uniform}([-1, 1])$$

Numerical Improvements

- Reformulating SOS program in terms of "**Orthogonal polynomial basis**" instead of the **"standard power basis",** Improve the probability bounds.

- **Polynomials in "standard power basis":**

$$p(x) = \sum_\alpha p_\alpha x^\alpha$$

coefficients monomials

$$p(x) = \mathbf{p}B(x)$$

vector of coefficients vector of monomials in x

Monomial (powers of variables):

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

Example: $p(x) = 2x_1^4 + 5x_2^4 - x_1^2 x_2^2 + 2x_1^3 x_2$

- **SOS polynomials in "standard power basis":**

$$\underbrace{Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0}_{\text{PSD Matrix}} \qquad p(x) = B(x)^T Q B(x)$$

Example: $p(x) = 2x_1^4 + 5x_2^4 - x_1^2 x_2^2 + 2x_1^3 x_2 = \begin{bmatrix} x_2^2 \\ x_1 x_2 \\ x_1^2 \end{bmatrix}^T \begin{bmatrix} 5 & 0 & -3 \\ 0 & 5 & 1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_2^2 \\ x_1 x_2 \\ x_1^2 \end{bmatrix}$

Numerical Improvements

- Reformulating SOS program in terms of "**Orthogonal polynomial basis"** instead of the **"standard power basis",**
Improve the probability bounds.

- **Polynomials in "Orthogonal polynomial basis" :**

$$p(x) = \sum_{\alpha} \underbrace{p_{\alpha}}_{\text{coefficients}} \underbrace{\mathrm{B}_{\alpha}(x)}_{\text{Polynomial Basis}}$$

Example: Chebyshev Polynomial Basis

Chebyshev Polynomial degree 0:	Chebyshev Polynomial degree 1:	Chebyshev Polynomial degree α:
$T_0(x) = 1$	$T_1(x) = x$	$T_{\alpha}(x) = 2xT_{\alpha}(x) - T_{\alpha-1}(x)$

Numerical Improvements

- Reformulating SOS program in terms of "**Orthogonal polynomial basis"** instead of the **"standard power basis",** Improve the probability bounds.

- **Polynomials in "Orthogonal polynomial basis" :**

$$p(x) = \sum_{\alpha} p_{\alpha} \mathrm{B}_{\alpha}(x)$$

coefficients   Polynomial Basis

Example: Chebyshev Polynomial Basis

Chebyshev Polynomial degree 0:

$$T_0(x) = 1$$

Chebyshev Polynomial degree 1:

$$T_1(x) = x$$

Chebyshev Polynomial degree α:

$$T_{\alpha}(x) = 2xT_{\alpha}(x) - T_{\alpha-1}(x)$$

$$\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= 2x^2 - 1\\
T_3(x) &= 4x^3 - 3x\\
T_4(x) &= 8x^4 - 8x^2 + 1\\
T_5(x) &= 16x^5 - 20x^3 + 5x\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1\\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x\\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1\\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x\\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1\\
T_{11}(x) &= 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x
\end{aligned}$$

Numerical Improvements

- Reformulating SOS program in terms of "**Orthogonal polynomial basis"** instead of the **"standard power basis",** Improve the probability bounds.

- **Polynomials in "Orthogonal polynomial basis" :**

$$p(x) = \sum_{\alpha} p_{\alpha} \mathrm{B}_{\alpha}(x)$$

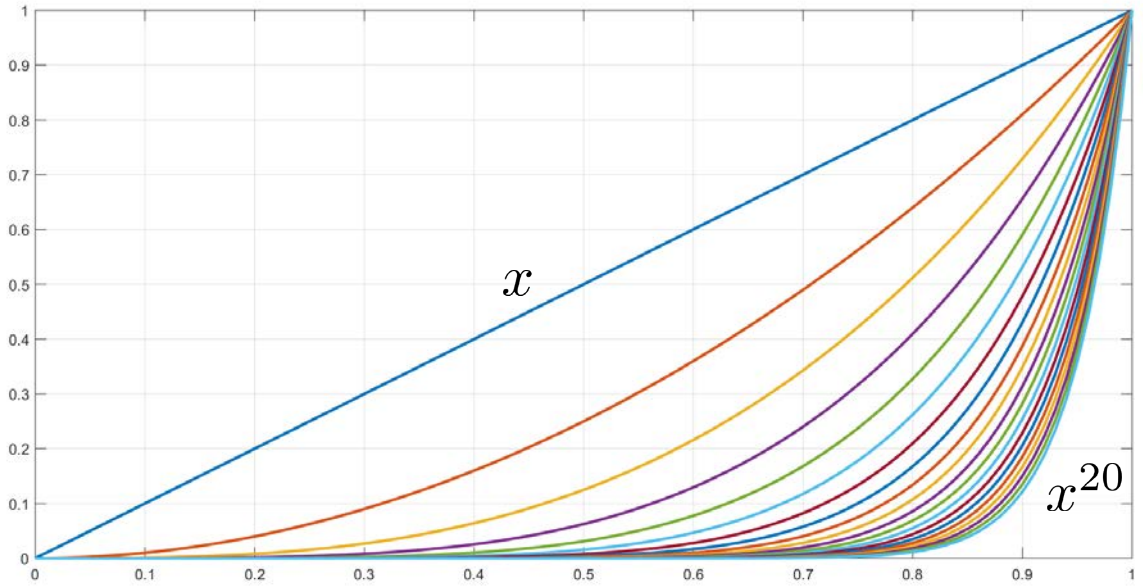
coefficients Polynomial Basis

Example: Chebyshev Polynomial Basis

Chebyshev Polynomial degree 0:	Chebyshev Polynomial degree 1:	Chebyshev Polynomial degree α:
$T_0(x) = 1$	$T_1(x) = x$	$T_\alpha(x) = 2xT_\alpha(x) - T_{\alpha-1}(x)$

Example: $p(x) = \underbrace{8x^4 - 8x^2 + 1 + 4x}_{\text{Monomial Basis}} = \underbrace{T_4(x) + 4T_1(x)}_{\text{Chebyshev Polynomial Basis}}$

- Chebyshev Polynomials are orthogonal $\int T_n(x)T_m(x)dx = 0 \quad \text{for } n \neq m$



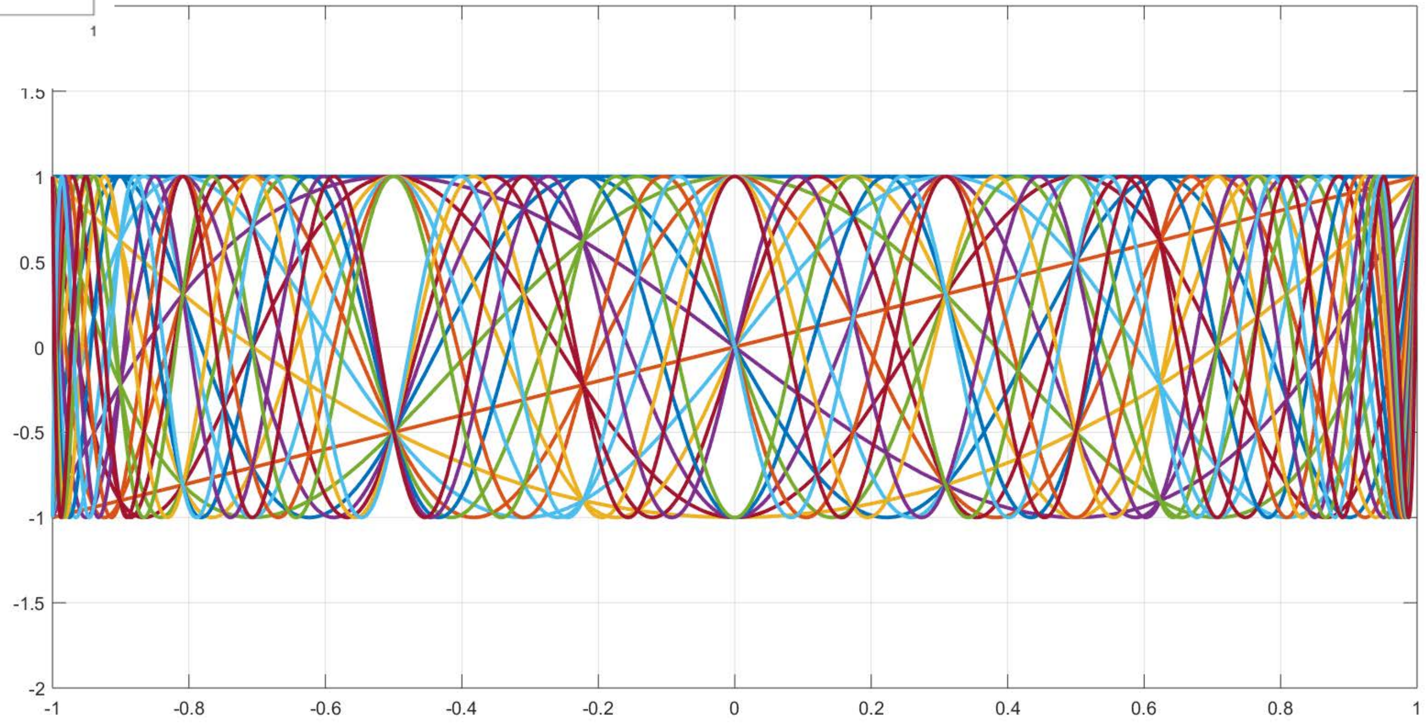
Monomial Basis $x^{\alpha}, \ \alpha = 1, ..., 20$

- As the power increases, monomial basis converge in the interval [0 1]

- Hence, adding the higher order monomials (increasing the degree of polynomial indicator function) may not improve the probability bounds.

Chebyshev Polynomial Basis

$$T_{\alpha}(x), \ \alpha = 1, ..., 20$$



Numerical Improvements

- Reformulating SOS program in terms of "**Orthogonal polynomial basis"** instead of the **"standard power basis",** Improve the probability bounds.

- **Polynomials in "Orthogonal polynomial basis" :**

$$p(x) = \sum_\alpha p_\alpha \mathrm{B}_\alpha(x)$$

coefficients Polynomial Basis

Example: Chebyshev Polynomial Basis

Chebyshev Polynomial degree 0:	Chebyshev Polynomial degree 1:	Chebyshev Polynomial degree α:
$T_0(x) = 1$	$T_1(x) = x$	$T_\alpha(x) = 2xT_\alpha(x) - T_{\alpha-1}(x)$

- **SOS polynomials in "Orthogonal polynomial basis" :**

$$\underbrace{Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0}_{\text{PSD Matrix}} \qquad p(x) = B(x)^T Q B(x)$$

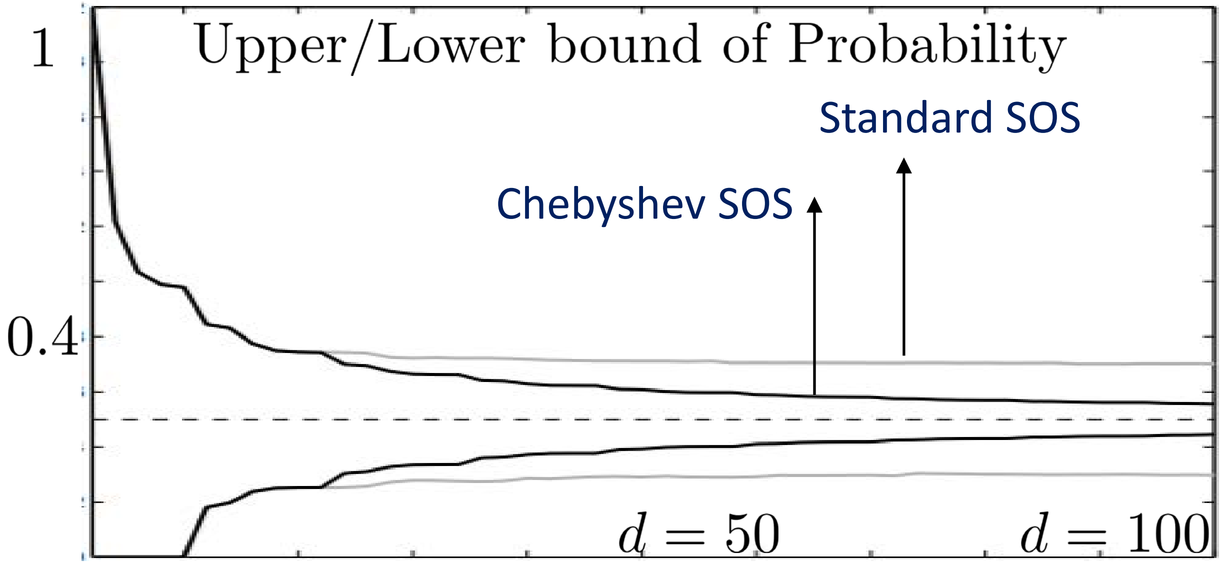
Vector of Polynomial Basis, e.g., Chebyshev basis

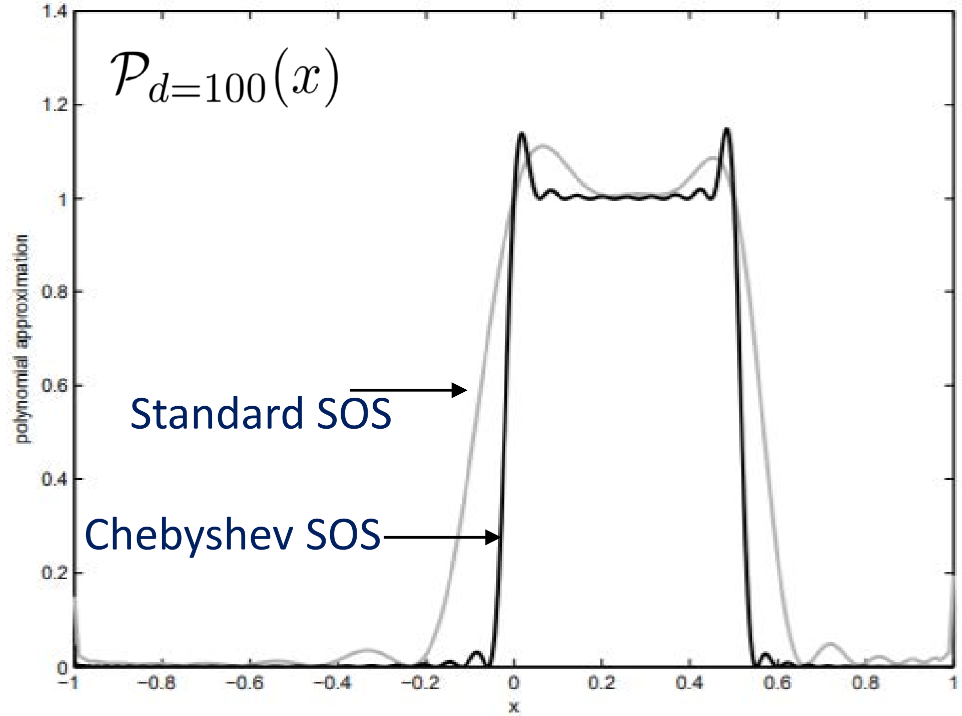
Example 1

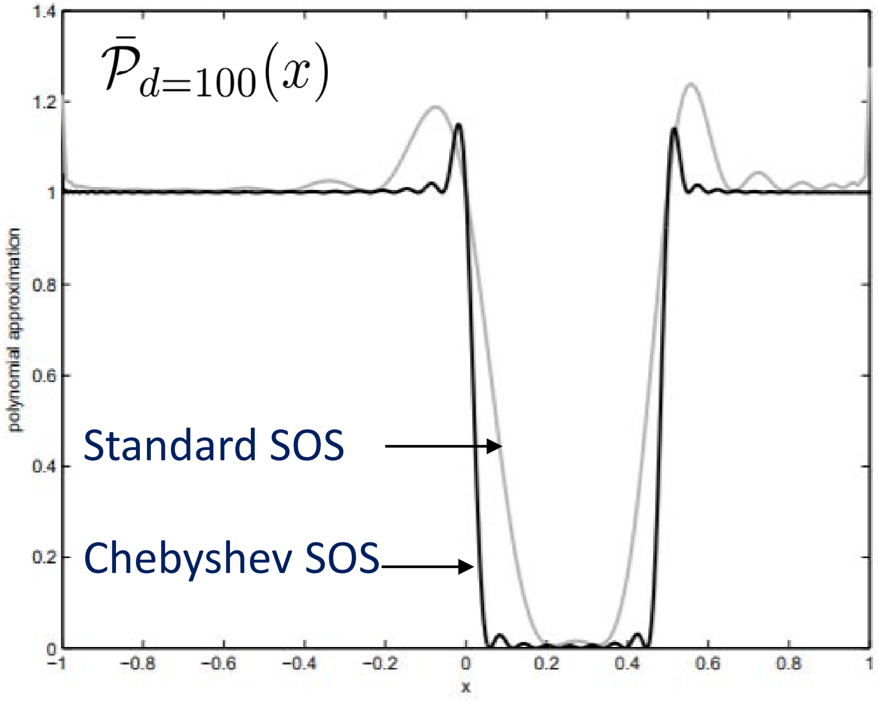
Improved results using Chebyshev SOS program

$$x \sim pr(x) : \text{Uniform}([-1,1])$$

$$\text{Probability}_{pr(x)}\{x \in \underbrace{\{x : x(0.5-x) \geq 0\}}_{\text{Safety constraint}}\}$$







Moment Program
for
Upper/Lower Bound Probability Estimation

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x \rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b \rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) pr(x) pr(\omega) dx d\omega \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x \rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b \rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \qquad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

$\mu : \mu_x \times \mu_\omega$ given probability measure of uncertainties

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x \rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b \rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

measure space

$$\begin{aligned} \underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \mu \\ & supp(\bar{\mu}) \subset \mathcal{K} \end{aligned}$$

continuous function space

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

$\mu : \mu_x \times \mu_\omega$ given probability measure of uncertainties

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

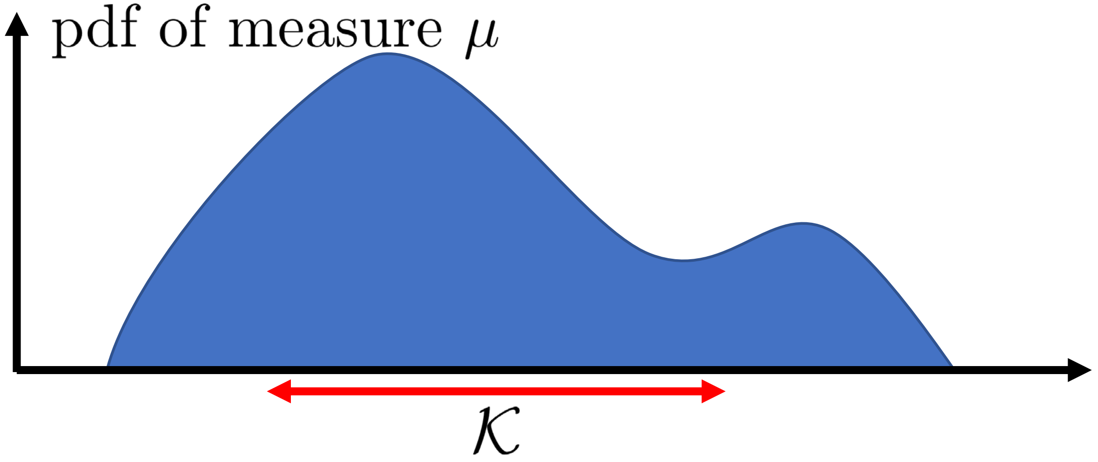
measure space

$$\begin{aligned} \underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \mu \\ & supp(\bar{\mu}) \subset \mathcal{K} \end{aligned}$$

continuous function space

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

$\mu : \mu_x \times \mu_\omega$ given probability measure of uncertainties



Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

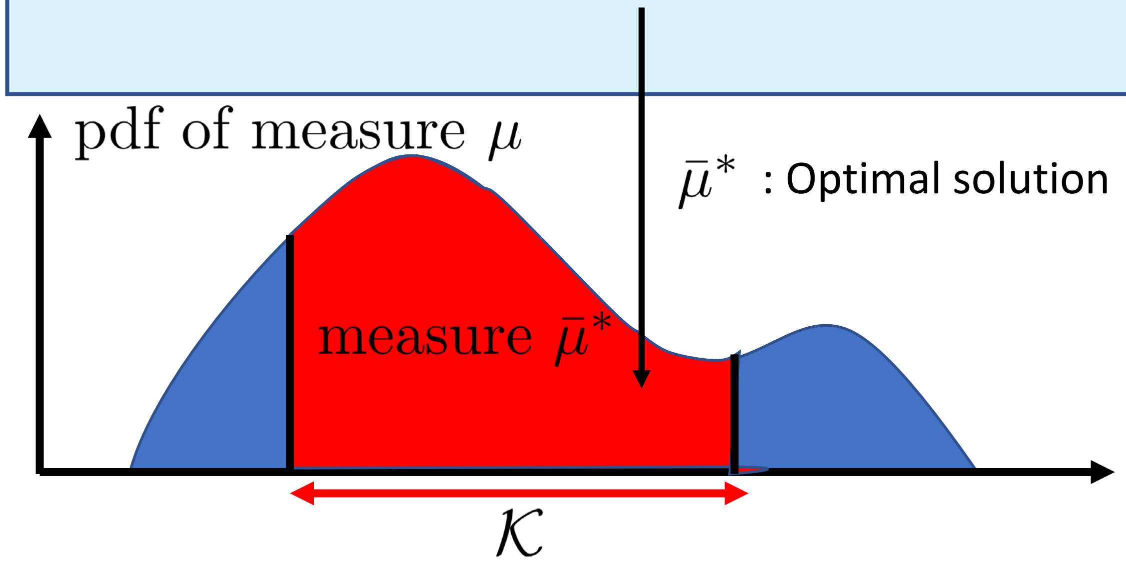
measure space

$$\begin{aligned} \underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \mu \\ & supp(\bar{\mu}) \subset \mathcal{K} \end{aligned}$$

continuous function space

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

$\mu : \mu_x \times \mu_\omega$ given probability measure of uncertainties



Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

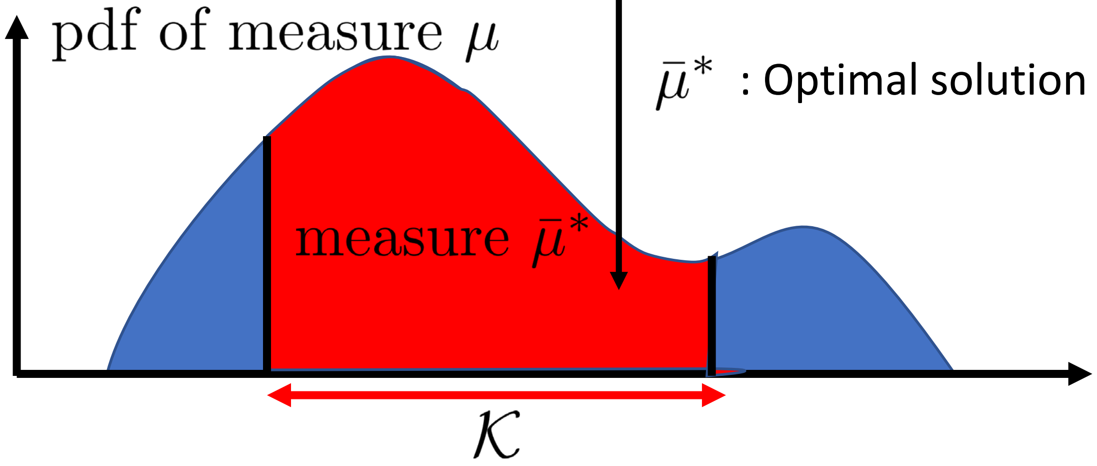
measure space

$$\begin{aligned} \underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \mu \\ & supp(\bar{\mu}) \subset \mathcal{K} \end{aligned}$$

continuous function space

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

$\mu : \mu_x \times \mu_\omega$ given probability measure of uncertainties



$\bar{\mu}^*$: Optimal solution

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

$$= \int_{\underbrace{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\}}_{\mathcal{K}}} pr(x)pr(\omega)dxd\omega$$

$$= \int_{\{(x,\omega):\{l_j \leq \mathcal{P}_j(x,\omega) \leq u_j\}_{j=1}^{\ell}\}} d\mu = \int d\bar{\mu}$$

Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x\rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

measure space

$$\begin{aligned} \underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \mu \\ & supp(\bar{\mu}) \subset \mathcal{K} \end{aligned}$$

DUAL

continuous function space

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

$\mu : \mu_x \times \mu_\omega$ given probability measure of uncertainties

moment space

$$\begin{aligned} \underset{\bar{y}}{\text{maximize}} \quad & \bar{y}_0 \\ \text{subject to} \quad & M_d(\bar{y}) \preccurlyeq M_d(y) \\ & M_d(\mathcal{P}_j \bar{y}) \end{aligned}$$

Moment and Localizing Matrix

DUAL

polynomial space

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega) \in \mathbb{R}[x,\omega]}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

Dual of moment SDP

moment space

$$\begin{aligned} \underset{\bar{y}}{\text{maximize}} \quad & \bar{y}_0 \\ \text{subject to} \quad & M_d(\bar{y}) \preccurlyeq M_d(y) \\ & M_d(\mathcal{P}_j \bar{y}) \end{aligned}$$

Moment and Localizing Matrix

DUAL

polynomial space

$$\begin{aligned} \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}[x,\omega]}{\text{minimize}} \quad & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega)d\mu \\ \text{subject to} \quad & \mathcal{P}_d(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 \end{aligned}$$

Similar to SOS program, we can describe the moments in terms of orthogonal basis, e.g., Chebyshev polynomials.

Standard Moment $\quad y_\alpha = \mathrm{E}[x^\alpha]$

Chebyshev Moment $\quad y_\alpha = \mathrm{E}[T_\alpha]$

Dual of moment SDP

moment space

$$\begin{array}{ll} \underset{\bar{y}}{\text{maximize}} & \bar{y}_0 \\ \text{subject to} & M_d(\bar{y}) \preccurlyeq M_d(y) \\ & M_d(\mathcal{P}_j \bar{y}) \end{array}$$

Moment and Localizing Matrix

DUAL

polynomial space

$$\begin{array}{lll} \underset{\mathcal{P}_d(x,\omega)\in\mathbb{R}[x,\omega]}{\text{minimize}} & \int_{\mathbf{B}} \mathcal{P}_d(x,\omega) d\mu & \\ \text{subject to} & \mathcal{P}_d(x,\omega) - 1 \geq 0 & \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{P}_d(x,\omega) \geq 0 & \end{array}$$

Similar to SOS program, we can describe the moments in terms of orthogonal basis, e.g., Chebyshev polynomials.

Standard Moment $\quad y_\alpha = \mathrm{E}[x^\alpha]$

Chebyshev Moment $\quad y_\alpha = \mathrm{E}[T_\alpha]$

Example: Orthogonal moment based chance optimization

https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_5_Orthogonal_Moments_ChanceOpt

Modified SOS/Moment Program
for
Probability Estimation in High Dimensions

- Probability distributions of uncertainties

$$x \in \mathbb{R}^n \sim pr(x) \qquad \omega \in \mathbb{R}^m \sim pr(\omega)$$

- Uncertain safety set

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \le \mathcal{P}_j(x,\omega) \le u_j, j = 1,\dots,\ell\ \}$$

$$\mathbf{P}^*_{\mathbf{risk}} := \mathrm{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

➢ Described SOS program looks for a Multivariate polynomial Approximation

- (n+m)-dimensional polynomial
- Multivariate SOS Optimization

➢ Modified Method looks for a Univariate Polynomial

- SOS program that looks for a univariate polynomial
- Univariate SOS Optimization

- Probability distributions of uncertainties

$$x \in \mathbb{R}^n \sim pr(x) \qquad \omega \in \mathbb{R}^m \sim pr(\omega)$$

- Uncertain safety set

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x,\omega) \leq u_j, j = 1, \ldots, \ell\}$$

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{pr(x),pr(\omega)}\{x \in \chi(\omega)\}$$

➢ Described SOS program looks for a Multivariate polynomial Approximation

- (n+m)-dimensional polynomial
- Multivariate SOS Optimization

$$\mathbb{R}^{n+m}$$

$$\Downarrow$$

➢ Modified Method looks for a Univariate Polynomial

- SOS program that looks for a univariate polynomial
- Univariate SOS Optimization

$$\mathbb{R}$$

- A. Jasour, A. Hofmann, B. C. Williams, "Moment-Sum-Of-Squares Approach for Fast Risk Estimation in Uncertain Environments", IEEE Conference on Decision and Control, 2018.

1) Unsafe Set Involving One Polynomial

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_1 \leq \mathcal{P}(x, \omega) \leq u_1 \}$$

2) Unsafe Set Involving Multiple Polynomials

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x, \omega) \leq u_j, j = 1, \ldots, \ell \}$$

Unsafe Set Involving One Polynomial

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_1 \leq \mathcal{P}(x,\omega) \leq u_1 \}$$

Unsafe Set Involving One Polynomial

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_1 \leq \mathcal{P}(x,\omega) \leq u_1 \}$$

- Define random variable $z \in \mathbb{R}$

$$z = \mathcal{P}(x,\omega)$$

Unsafe Set Involving One Polynomial

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_1 \leq \mathcal{P}(x,\omega) \leq u_1 \}$$

- Define random variable $z \in \mathbb{R}$

$$z = \mathcal{P}(x,\omega)$$

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_z}\{l_1 \leq z \leq u_1\}$$

$$\mathbb{R}^{n+m}$$

$$\mathbb{R}$$

Unsafe Set Involving One Polynomial

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_z}\{l_1 \leq z \leq u_1\} \boxed{= E[\mathbf{I}_{[l_1,u_1]}]}$$

Unsafe Set Involving One Polynomial

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_z}\{l_1 \leq z \leq u_1\} \boxed{= E[\mathbf{I}_{[l_1,u_1]}]}$$

$\mathcal{P}_d(z) \in \mathbb{R}_d[z]$ polynomial approximation of Indicator function of $\mathcal{K} = [l_1, u_1]$

$\bar{\mathcal{P}}_d(z) \in \mathbb{R}_d[z]$ polynomial approximation of Indicator function of $\bar{\mathcal{K}}$

$$1 - E[\bar{\mathcal{P}}_d(z)] \leq \mathbf{P}^*_{\mathbf{risk}} \leq E[\mathcal{P}_d(z)]$$

$$\mathbf{P}^*_{\mathbf{risk}} = \lim_{d\to\infty} E[\mathcal{P}_d(z)] = \lim_{d\to\infty} 1 - E[\bar{\mathcal{P}}_d(z)]$$

Unsafe Set Involving One Polynomial

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_z}\{l_1 \le z \le u_1\} \boxed{= E[\mathbf{I}_{[l_1,u_1]}]}$$

- Hence, to calculate the upper/lower bound of probability we need to

- Find the polynomial indicator function of the interval $[l_1, u_1]$,i.e., $\bar{\mathcal{P}}_d(z), \mathcal{P}_d(z)$
- Moment of new random variable z (to calculated the expected values)

$$1 - E[\bar{\mathcal{P}}_d(z)] \le \mathbf{P}^*_{\mathbf{risk}} \le E[\mathcal{P}_d(z)]$$

Unsafe Set Involving One Polynomial

- Moments of z

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_1 \leq \mathcal{P}(x,\omega) \leq u_1 \}$$

$$\boxed{z = \mathcal{P}(x,\omega)}$$

$$m_\alpha^z = E[z^\alpha] = E[\mathcal{P}^\alpha(x,\omega)] = \sum_{i,j} a_{ij} m_i^x m_j^\omega$$

Moments of z

Moments of x

Moments of ω

Illustrative Example

- $x \sim \mu_x = U[-0.5, 0.5]$
- $q \sim \mu_q = Beta(3-\sqrt{2}, 3+\sqrt{2})$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{U,Beta}\{-0.4 \leq 0.5(x-q) \leq 0\}$$

Illustrative Example

- $x \sim \mu_x = U[-0.5, 0.5]$
- $q \sim \mu_q = Beta(3-\sqrt{2}, 3+\sqrt{2})$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{U,Beta}\{-0.4 \leq 0.5(x-q) \leq 0\}$$

➢ Let $z = 0.5(x-q)$

Illustrative Example

- $x \sim \mu_x = U[-0.5, 0.5]$
- $q \sim \mu_q = Beta(3-\sqrt{2}, 3+\sqrt{2})$

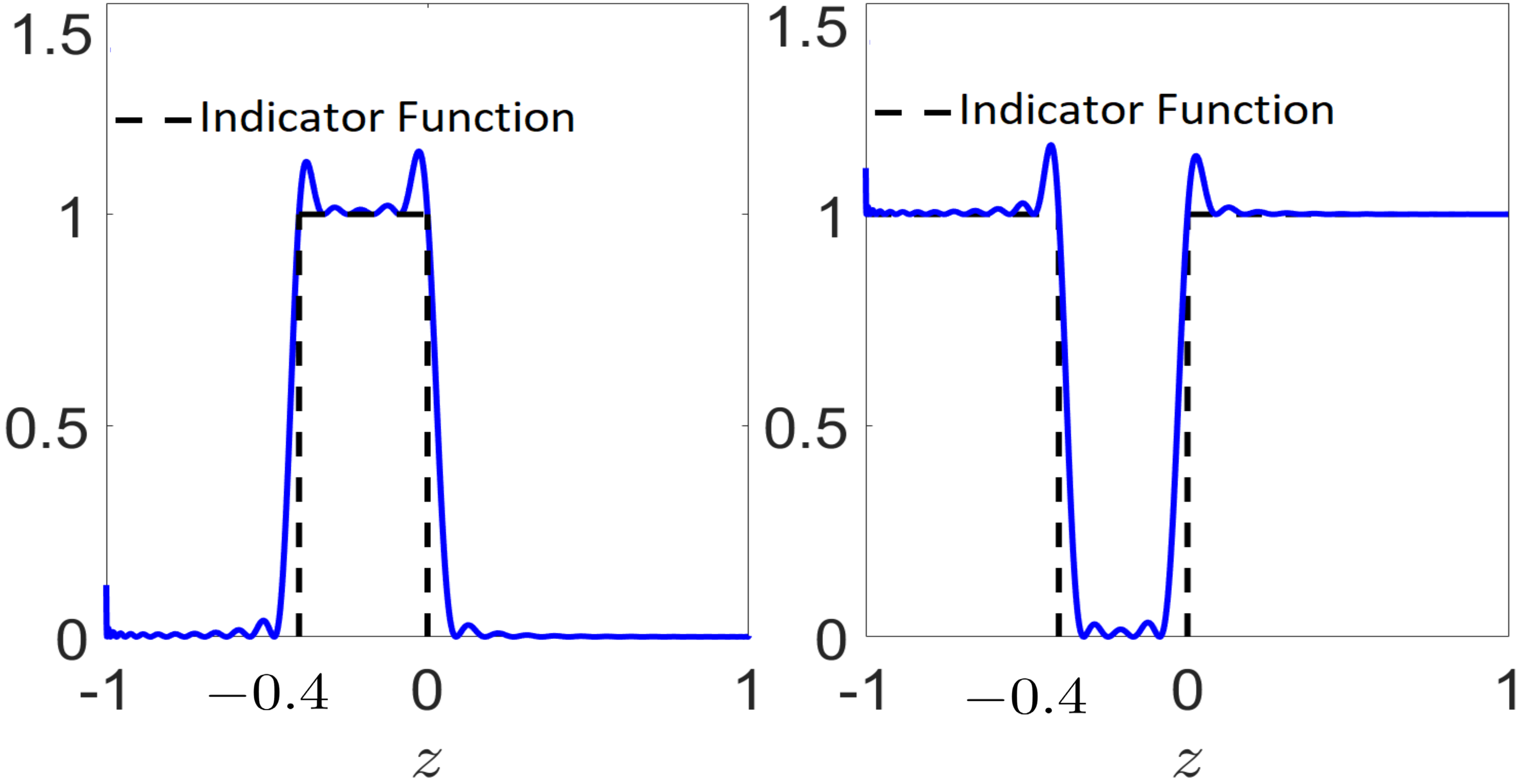
$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{U,Beta}\{-0.4 \leq 0.5(x-q) \leq 0\}$$

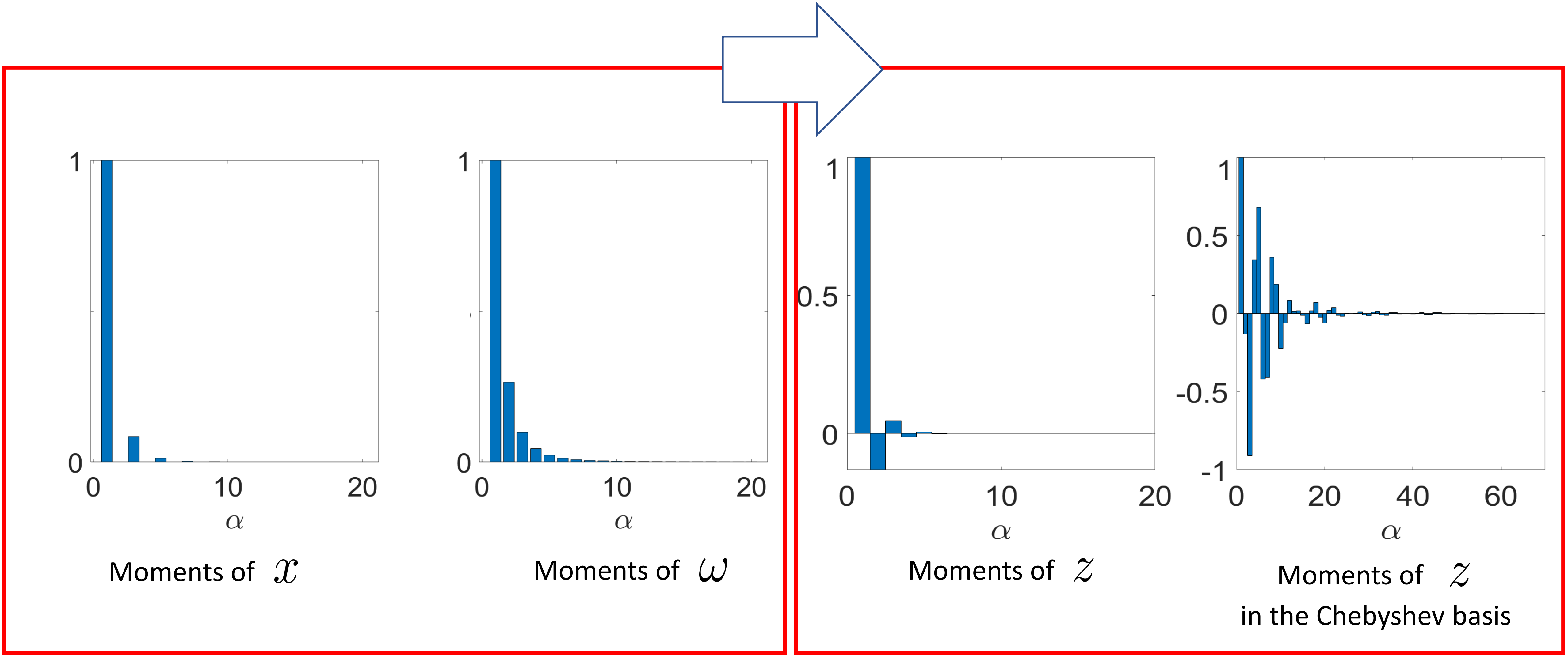
➢ Let $z = 0.5(x-q)$

$$\text{Probability}_{U,Beta}\{-0.4 \leq 0.5(x-q) \leq 0\} = \text{Probability}_{\mu_z}\{-0.4 \leq z \leq 0\}$$

$$\mathbb{R}^2 \Rightarrow \mathbb{R}$$

Chebyshev based polynomial approximations of the indicator functions of order *d*=66





1
0
0
10
20
α
Moments of x
1
0
0
10
20
α
Moments of ω
1
0.5
0
0
10
20
α
Moments of z
1
0.5
0
-0.5
-1
0
20
40
60
α
Moments of z
in the Chebyshev basis

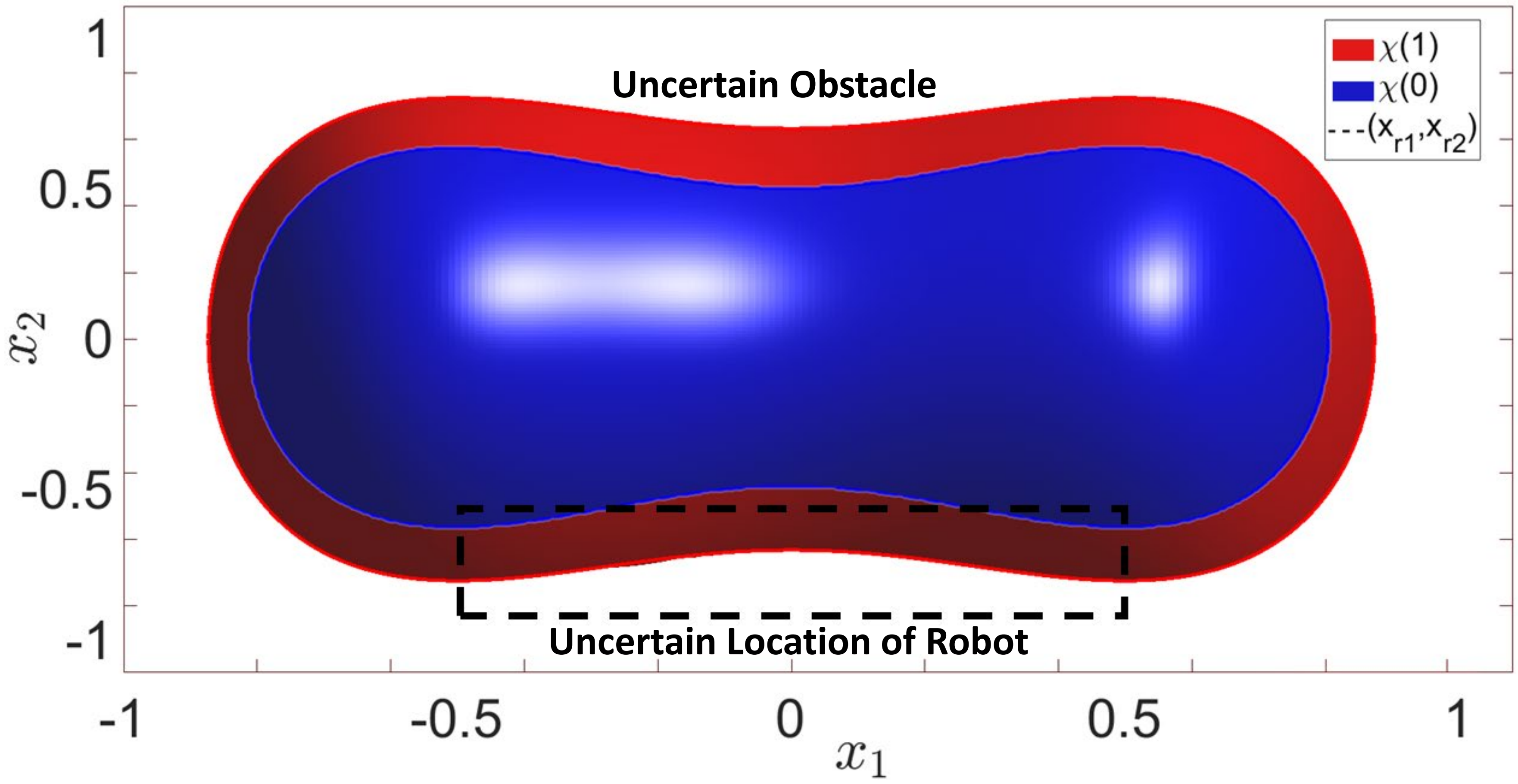
$$1 - E[\bar{\mathcal{P}}_d(z)] \leq \mathbf{P}^*_{\mathbf{risk}} \leq E[\mathcal{P}_d(z)]$$

$$1 - \sum_{i=0}^{d} \bar{c}_{T_i} m^z_{T_i} \leq \mathbf{P}^*_{\mathbf{risk}} \leq \sum_{i=0}^{d} c_{T_i} m^z_{T_i}$$

Polynomial order	d	20	30	40	50	60	66
Upper Bound	p_u	0.92	0.879	0.859	0.822	0.804	0.798
Lower Bound	p_l	0.401	0.485	0.511	0.562	0.586	0.591

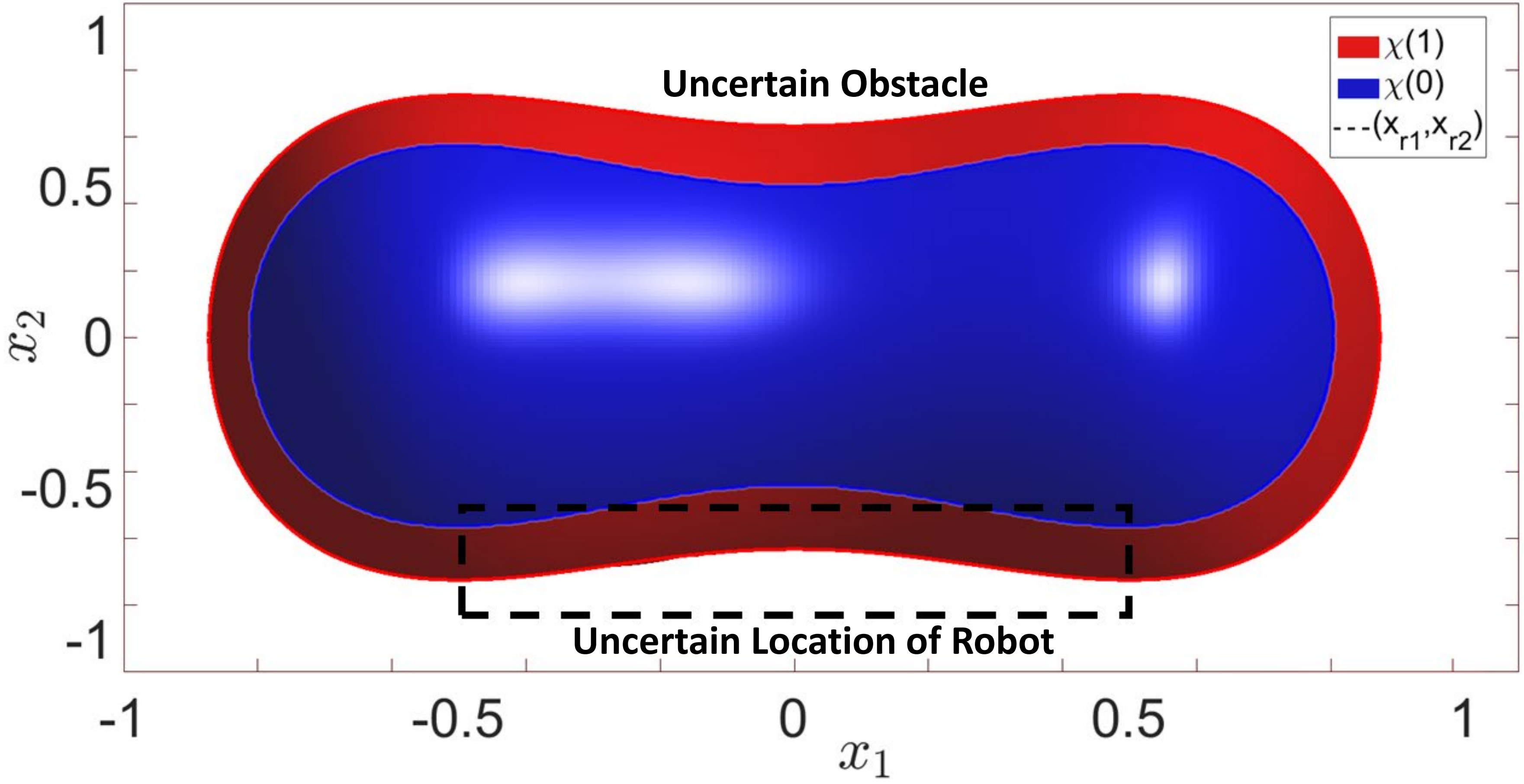
$$\mathbf{P}^*_{\mathbf{risk}} = 0.7$$

Example: Uncertain Obstacle



$$\chi(\omega) = \{x \in \mathbb{R}^2 : -0.1 \leq -x_1^4 + 0.5(x_1^2 - x_2^2) + 0.1\omega \leq 0.2\}$$

- A. Jasour, A. Hofmann, B. C. Williams, "Moment-Sum-Of-Squares Approach for Fast Risk Estimation in Uncertain Environments", IEEE Conference on Decision and Control, 2018.

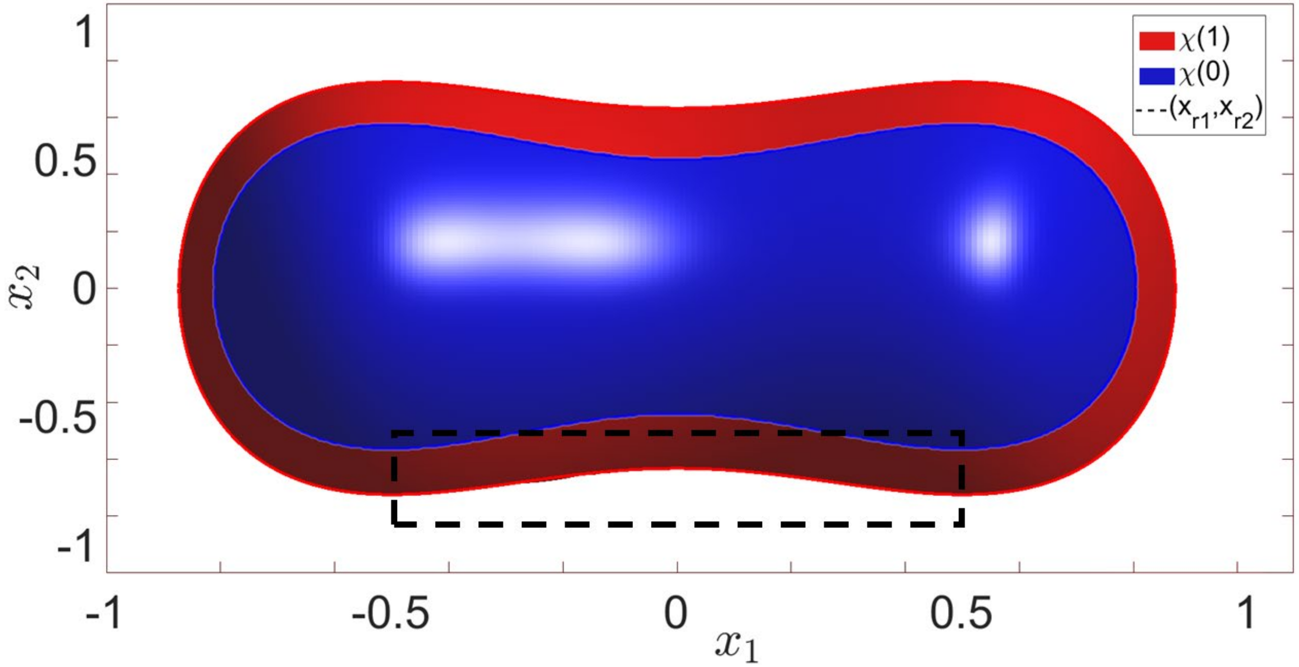


$$x_{r1} \sim U[-0.5, 0.5]$$

$$x_{r2} \sim U[-0.8, -0.5]$$

$$\omega \sim Beta(4, 4)$$

$$\chi(\omega) = \{x \in \mathbb{R}^2 : -0.1 \leq -x_1^4 + 0.5(x_1^2 - x_2^2) + 0.1\omega \leq 0.2\}$$



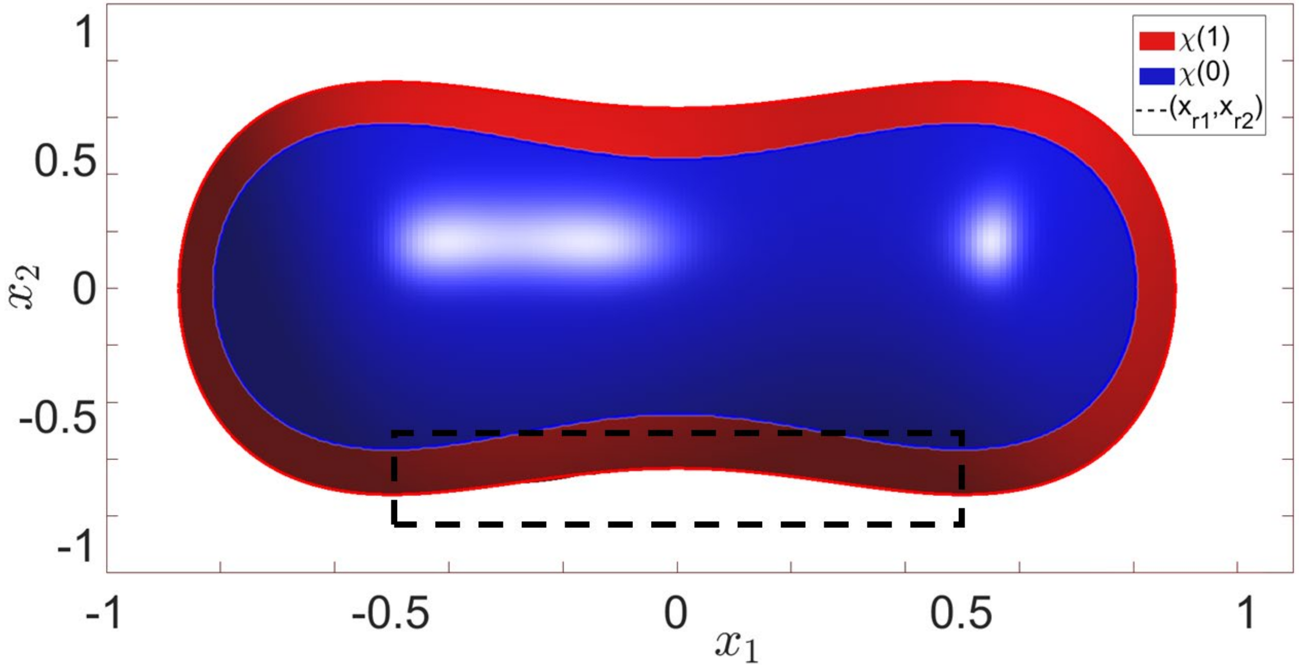
$$x_{r1} \sim U[-0.5, 0.5]$$

$$x_{r2} \sim U[-0.8, -0.5]$$

$$\omega \sim Beta(4,4)$$

$$\chi(\omega) = \{x \in \mathbb{R}^2 : -0.1 \leq -x_1^4 + 0.5(x_1^2 - x_2^2) + 0.1\omega \leq 0.2\}$$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_{x_{r1}}, \mu_{x_{r2}}, \mu_\omega}\{-0.1 \leq -x_{r1}^4 + 0.5(x_{r1}^2 - x_{r2}^2) + 0.1\omega \leq 0.2\}$$



$$x_{r1} \sim U[-0.5, 0.5]$$

$$x_{r2} \sim U[-0.8, -0.5]$$

$$\omega \sim Beta(4, 4)$$

$$\chi(\omega) = \{x \in \mathbb{R}^2 : -0.1 \leq -x_1^4 + 0.5(x_1^2 - x_2^2) + 0.1\omega \leq 0.2\}$$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_{x_{r1}}, \mu_{x_{r2}}, \mu_\omega}\{-0.1 \leq \boxed{-x_{r1}^4 + 0.5(x_{r1}^2 - x_{r2}^2) + 0.1\omega} \leq 0.2\}$$

$$\boxed{\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_z}\{-0.1 \leq z \leq 0.2\}}$$

Results:

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_{x_{r1}},\mu_{x_{r2}},\mu_\omega}\{-0.1 \leq -x_{r1}^4 + 0.5(x_{r1}^2 - x_{r2}^2) + 0.1\omega \leq 0.2\}$$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_z}\{-0.1 \leq z \leq 0.2\}$$

	multivariate SOS				proposed univariate SOS	
Polynomial order	d	10	20	30	d	88
Upper Bound	p_u	0.54	0.50	0.495	p_u	0.48
Computation time	$t_u(s)$	≈2.6	≈76	≈3689	$t_u(s)$	≈17
Lower Bound	p_l	0.13	0.15	0.161	p_l	0.169
Computation time	$t_l(s)$	≈4.5	≈70	≈3156	$t_l(s)$	≈15

$$\mathbf{P}^*_{\mathbf{risk}} = 0.32$$

Results:

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_{x_{r1}},\mu_{x_{r2}},\mu_q}\{-0.1 \leq -x_{r1}^4 + 0.5(x_{r1}^2 - x_{r2}^2) + 0.1\omega \leq 0.2\}$$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_z}\{-0.1 \leq z \leq 0.2\}$$

	multivariate SOS				proposed univariate SOS	
Polynomial order	d	10	20	30	d	88
Upper Bound	p_u	0.54	0.50	0.495	p_u	0.48
Computation time	$t_u(s)$	≈ 2.6	≈ 76	≈ 3689	$t_u(s)$	≈ 17
Lower Bound	p_l	0.13	0.15	0.161	p_l	0.169
Computation time	$t_l(s)$	≈ 4.5	≈ 70	≈ 3156	$t_l(s)$	≈ 15

$$\mathbf{P}^*_{\mathbf{risk}} = 0.32$$

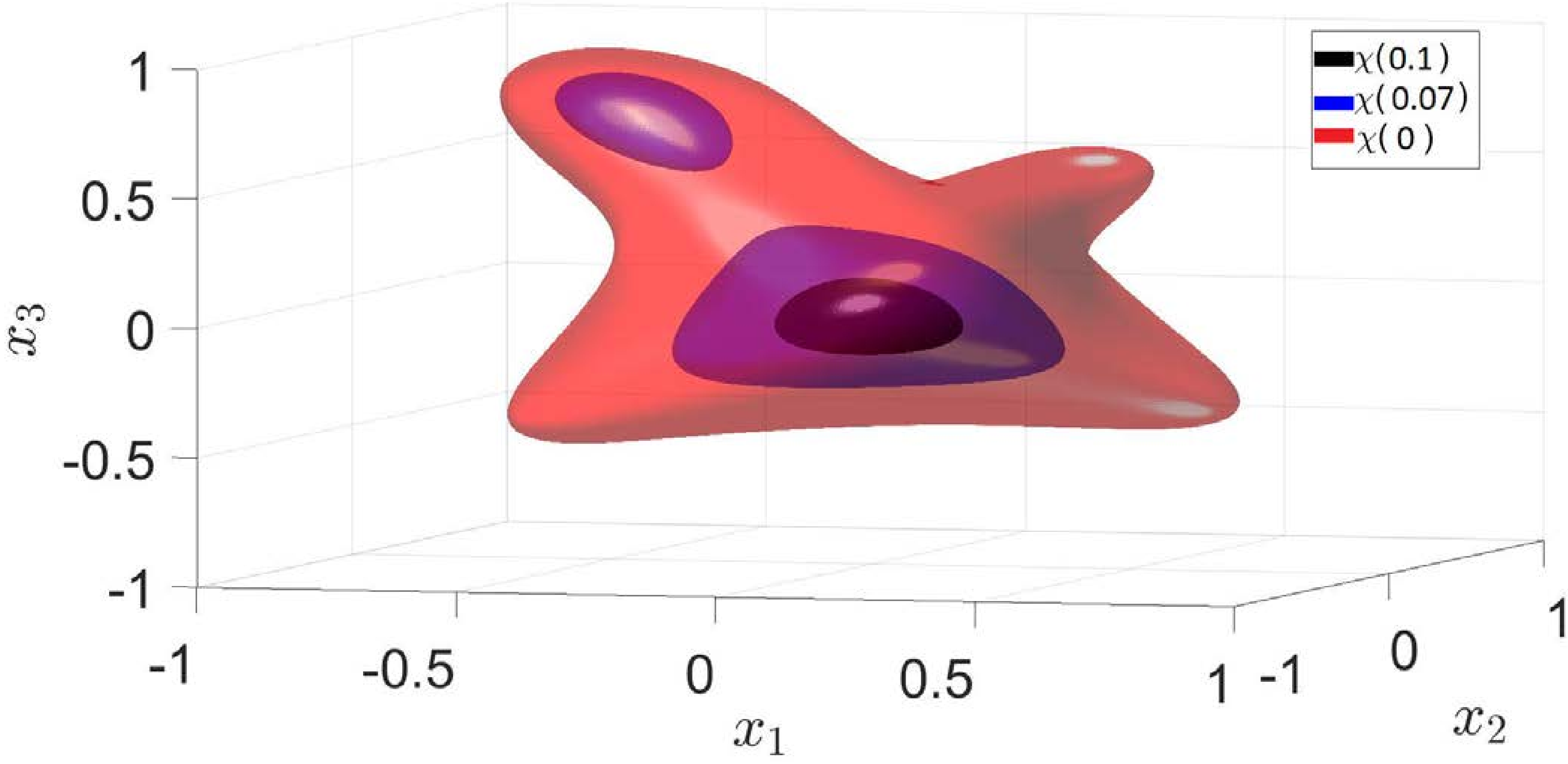
Example: 3D Random Pattern

$$\chi(\omega) = \{x \in \mathbb{R}^3 : 0.84 \le \mathcal{P}(x_1, x_2, x_3, \omega) \le 1\},$$

$\mathcal{P} = 0.9487722614 - 0.0022x_1 - 0.0042x_2 - 0.0457x_3 - 0.3877x_1^2 + 0.0405x_1x_2 - 0.3105x_2^2 - 0.0537x_1x_3 - 0.0179x_2x_3 - 0.4094x_3^2 - 0.1059x_1^3 - 0.0212x_1^2x_2 + 0.0906x_1x_2^2 - 0.0543x_2^3 + 0.1451x_1^2x_3 - 1.8302x_1x_2x_3 + 0.1135x_2^2x_3 - 0.1096x_1x_3^2 + 0.1205x_2x_3^2 + 0.3407x_3^3 - 0.3285x_1^4 - 0.1338x_1^3x_2 + 0.4847x_1^2x_2^2 + 0.1127x_1x_2^3 - 0.3495x_2^4 + 0.0394x_1^3x_3 + 0.0149x_1^2x_2x_3 - 0.0051x_1x_2^2x_3 - 0.0594x_2^3x_3 + 0.5418x_1^2x_3^2 - 0.0659x_1x_2x_3^2 + 0.4840x_2^2x_3^2 + 0.0085x_1x_3^3 + 0.0657x_2x_3^3 - 0.3076x_3^4 + 0.1268x_1^5 + 0.0058x_1^4x_2 - 0.1012x_1^3x_2^2 + 0.0070x_1^2x_2^3 + 0.0053x_1x_2^4 + 0.0718x_2^5 - 0.0226x_1^4x_3 + 0.7338x_1^3x_2x_3 - 0.0716x_1^2x_2^2x_3 + 0.7226x_1x_2^3x_3 - 0.2075x_2^4x_3 + 0.0378x_1^3x_3^2 - 0.0139x_1^2x_2x_3^2 + 0.0224x_1x_2^2x_3^2 - 0.0566x_2^3x_3^2 - 0.0773x_1^2x_3^3 + 0.7345x_1x_2x_3^3 + 0.0955x_2^2x_3^3 + 0.0399x_1x_3^4 - 0.0653x_2x_3^4 - 0.3173x_3^5 - \omega$

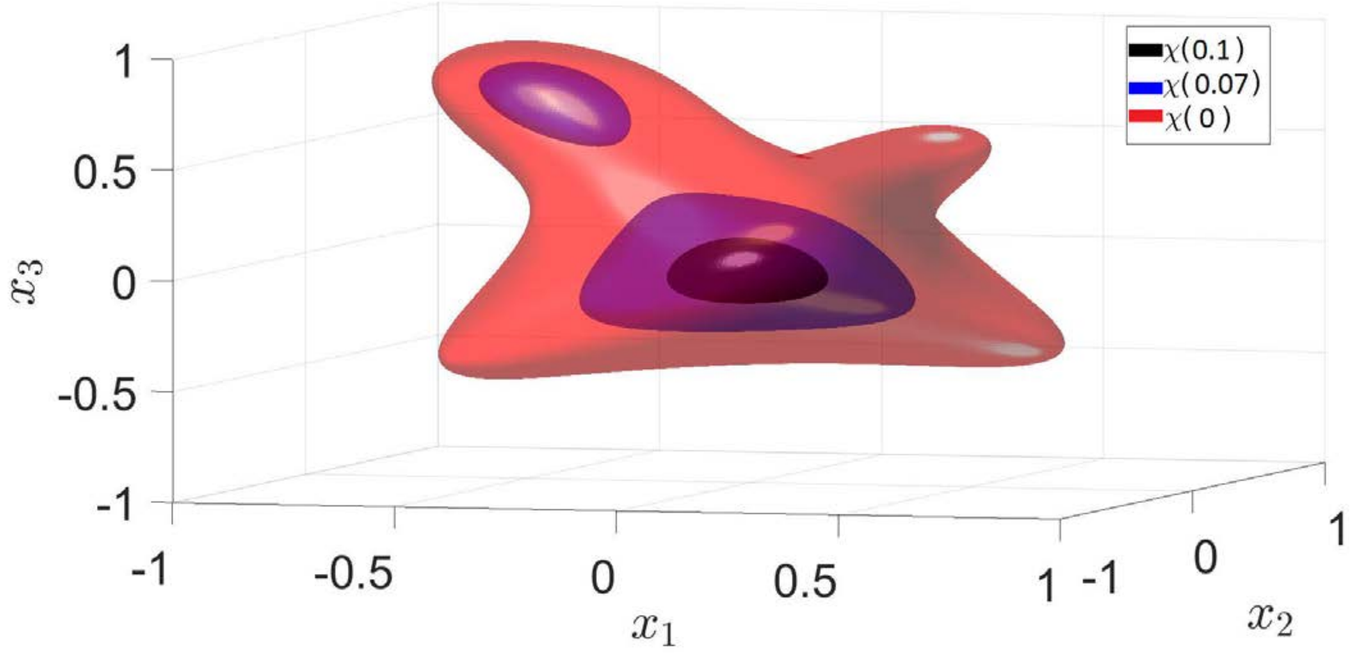
- $\omega \sim U[0, 0.1]$

- A. Jasour, A. Hofmann, B. C. Williams, "Moment-Sum-Of-Squares Approach for Fast Risk Estimation in Uncertain Environments", IEEE Conference on Decision and Control, 2018.



- $\omega \sim U[0, 0.1]$

$$\chi(\omega) = \{x \in \mathbb{R}^3 : 0.84 \leq \mathcal{P}(x_1, x_2, x_3, \omega) \leq 1\},$$



3D Uncertain Obstacle

- $\omega \sim U[0, 0.1]$

$$\chi(\omega) = \{x \in \mathbb{R}^3 : 0.84 \leq \mathcal{P}(x_1, x_2, x_3, \omega) \leq 1\},$$

Uncertain Location of Robot

- $x_1,\ x_2,\ x_3 \sim U[-0.4, 0.4]$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \mu_\omega}\{0.84 \leq \mathcal{P}(x_1, x_2, x_3, \omega) \leq 1\}$$

$$\boxed{\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_z}\{0.84 \leq z \leq 1\}}$$

Results:

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_{x_1},\mu_{x_2},\mu_{x_3},\mu_\omega}\{0.84 \leq \mathcal{P}(x_1, x_2, x_3, \omega) \leq 1\}$$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_z}\{0.84 \leq z \leq 1\}$$

	multivariate SOS				proposed univariate SOS	
Polynomial order	d	10	20	30	d	48
Upper Bound	p_u	0.81	0.78	–	p_u	0.77
Computation time	$t_u(s)$	≈12	≈7459	–	$t_u(s)$	≈5
Lower Bound	p_l	0.189	0.239	–	p_l	0.25
Computation time	$t_l(s)$	≈11	≈6657	–	$t_l(s)$	≈5

$$\mathbf{P}^*_{\mathbf{risk}} = 0.519$$

Results:

$$\mathbf{P}^*_{\textbf{risk}} = \text{Probability}_{\mu_{x_1},\mu_{x_2},\mu_{x_3},\mu_\omega}\{0.84 \le \mathcal{P}(x_1, x_2, x_3, \omega) \le 1\}$$

$$\mathbf{P}^*_{\textbf{risk}} = \text{Probability}_{\mu_z}\{0.84 \le z \le 1\}$$

	multivariate SOS				proposed univariate SOS	
Polynomial order	d	10	20	30	d	48
Upper Bound	p_u	0.81	0.78	–	p_u	0.77
Computation time	$t_u(s)$	≈12	≈7459	–	$t_u(s)$	≈5
Lower Bound	p_l	0.189	0.239	–	p_l	0.25
Computation time	$t_l(s)$	≈11	≈6657	–	$t_l(s)$	≈5

$$\mathbf{P}^*_{\textbf{risk}} = 0.519$$

Results:

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \mu_\omega} \{0.84 \leq \mathcal{P}(x_1, x_2, x_3, \omega) \leq 1\}$$

$$\mathbf{P}^*_{\mathbf{risk}} = \text{Probability}_{\mu_z} \{0.84 \leq z \leq 1\}$$

	multivariate SOS				proposed univariate SOS	
Polynomial order	d	10	20	30	d	48
Upper Bound	p_u	0.81	0.78	–	p_u	0.77
Computation time	$t_u(s)$	≈12	≈7459	–	$t_u(s)$	≈5
Lower Bound	p_l	0.189	0.239	–	p_l	0.25
Computation time	$t_l(s)$	≈11	≈6657	–	$t_l(s)$	≈5

$$\mathbf{P}^*_{\mathbf{risk}} = 0.519$$

2) Unsafe Set Involving Multiple Polynomials

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x, \omega) \leq u_j, j = 1, \ldots, \ell \}$$

2) Unsafe Set Involving Multiple Polynomials

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x, \omega) \leq u_j, j = 1, \ldots, \ell \}$$

- Let $Z = [z_1, ..., z_\ell], \quad z_j = \mathcal{P}_j(x, \omega), \; j = 1, ..., \ell$

2) Unsafe Set Involving Multiple Polynomials

$$\chi(\omega) := \{x \in \mathbb{R}^n : l_j \leq \mathcal{P}_j(x,\omega) \leq u_j, j = 1, \ldots, \ell \}$$

- Let $Z = [z_1, ..., z_\ell], \quad z_j = \mathcal{P}_j(x,\omega), \; j = 1, ..., \ell$

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_Z}\{Z \in [l_1,u_1] \times [l_2,u_2] ... \times [l_\ell,u_\ell]\}$$

$$\mathbb{R}^{n+m} \qquad \mathbb{R}^{\ell}$$

Unsafe Set Involving Multiple Polynomials

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_Z}\{Z \in [l_1, u_1] \times [l_2, u_2] ... \times [l_\ell, u_\ell]\}$$

$\mathcal{P}_{\mathcal{K}_j}(z_j)$ Upper bound polynomial approximation of interval $\mathcal{K}_j = [l_j, u_j]$ obtained by solving univariate SOS optimization.

Unsafe Set Involving Multiple Polynomials

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_Z}\{Z \in [l_1, u_1] \times [l_2, u_2] ... \times [l_\ell, u_\ell]\}$$

$\mathcal{P}_{\mathcal{K}_j}(z_j)$ Upper bound polynomial approximation of interval $\mathcal{K}_j = [l_j, u_j]$ obtained by solving univariate SOS optimization.

$\prod_{j=1}^{\ell} \mathcal{P}_{\mathcal{K}_j}(z_j)$ Upper bound polynomial approximation of hypercube $[l_1, u_1] \times [l_2, u_2] ... \times [l_\ell, u_\ell]$

Unsafe Set Involving Multiple Polynomials

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_Z}\{Z \in [l_1,u_1] \times [l_2,u_2] ... \times [l_\ell,u_\ell]\}$$

$\mathcal{P}_{\mathcal{K}_j}(z_j)$ Upper bound polynomial approximation of interval $\mathcal{K}_j = [l_j, u_j]$ obtained by solving univariate SOS optimization.

$\prod_{j=1}^{\ell} \mathcal{P}_{\mathcal{K}_j}(z_j)$ Upper bound polynomial approximation of hypercube $[l_1,u_1] \times [l_2,u_2] ... \times [l_\ell,u_\ell]$

Moments of random vector Z:

$$m^Z_{\alpha_1,\ldots,\alpha_\ell} = E[\prod_{j=1}^{\ell} z_j^{\alpha_j}] = E[\prod_{j=1}^{\ell} \mathcal{P}_j^{\alpha_j}(x,q)] = \sum_{i,j} a_{ij} m_i^x m_j^q$$

Unsafe Set Involving Multiple Polynomials

$$\mathbf{P}^*_{\mathbf{risk}} := \text{Probability}_{\mu_x,\mu_\omega}\{x \in \chi(\omega)\} = \text{Probability}_{\mu_Z}\{Z \in [l_1,u_1] \times [l_2,u_2]... \times [l_\ell,u_\ell]\}$$

$\mathcal{P}_{\mathcal{K}_j}(z_j)$ Upper bound polynomial approximation of interval $\mathcal{K}_j = [l_j, u_j]$ obtained by solving univariate SOS optimization.

$\prod_{j=1}^{\ell} \mathcal{P}_{\mathcal{K}_j}(z_j)$ Upper bound polynomial approximation of hypercube $[l_1,u_1] \times [l_2,u_2]... \times [l_\ell,u_\ell]$

Moments of random vector Z:

$$m^Z_{\alpha_1,\ldots,\alpha_\ell} = E[\prod_{j=1}^{\ell} z_j^{\alpha_j}] = E[\prod_{j=1}^{\ell} \mathcal{P}_j^{\alpha_j}(x,q)] = \sum_{i,j} a_{ij} m_i^x m_j^q$$

Upper Bound Estimation:

$$\mathbf{P}^*_{\mathbf{risk}} \leq E[\prod_{j=1}^{\ell} \mathcal{P}_{\mathcal{K}_j}(z_j)]$$

Polynomial based Probability Bound

- Sum-of-Squares Program for Upper/Lower Bound Probability Estimation
- Moment Program for Upper/Lower Bound Probability Estimation (Dual Optimization)
- Modified SOS/Moment Program for Probability Estimation in High Dimensions

Probabilistic Safety Verification:

1) Uncertainty (Moment) propagation through nonlinear uncertain dynamics

2) Risk estimation in presence of nonlinear safety constraints

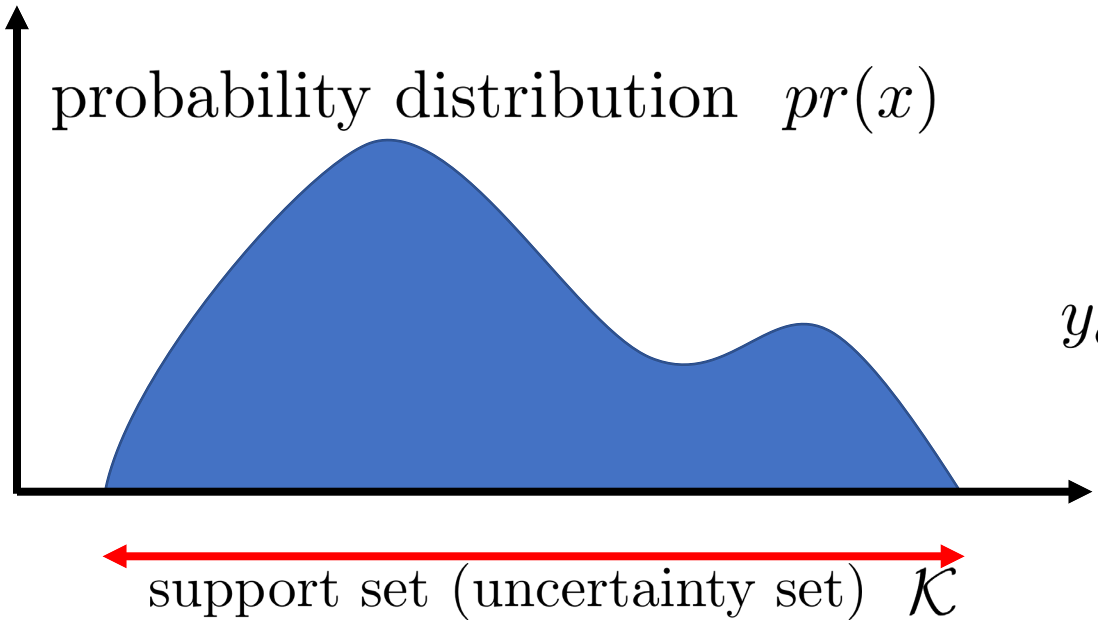
3) Uncertainty set construction from the moment information

4) Probability density function construction from the moment information

Uncertainty Set from the moment information

Input: Finite sequence of the moments $y = [y_0, \ldots, y_d]$

Output: Uncertainty set $\mathcal{K}$, i.e., support of probability distribution whose moments are $y = [y_0, \ldots, y_d]$



$$y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha pr(x) dx$$

Uncertainty Set from the moment information

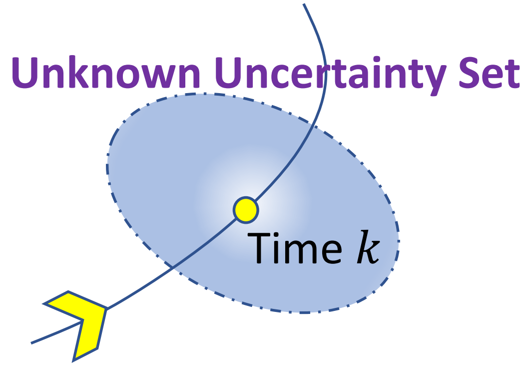
Main Idea: We aim at finding **polynomial** approximations of **indicator function** of the support set.

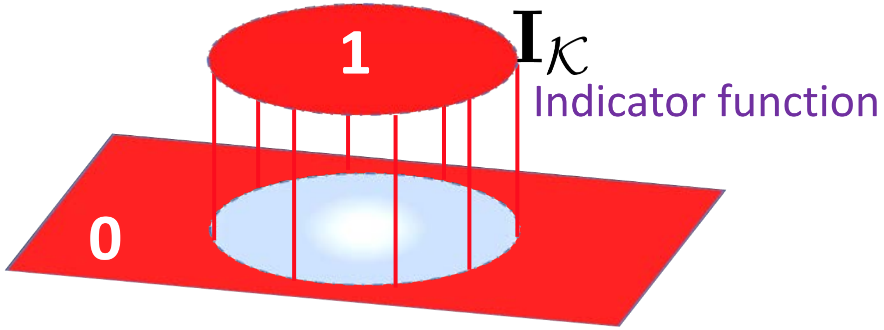
$$\mathbf{I}_{\mathcal{K}} = \begin{cases} 1 & x \in \mathcal{K} \\ 0 & \text{Otherwise} \end{cases}$$

Uncertainty Set: $\mathcal{K} = \{x \in \mathbb{R}^n : \mathbf{I}_{\mathcal{K}}(x) = 1\}$

- Polynomial order of d
- Upper bound Polynomial approximation of Indicator function

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$





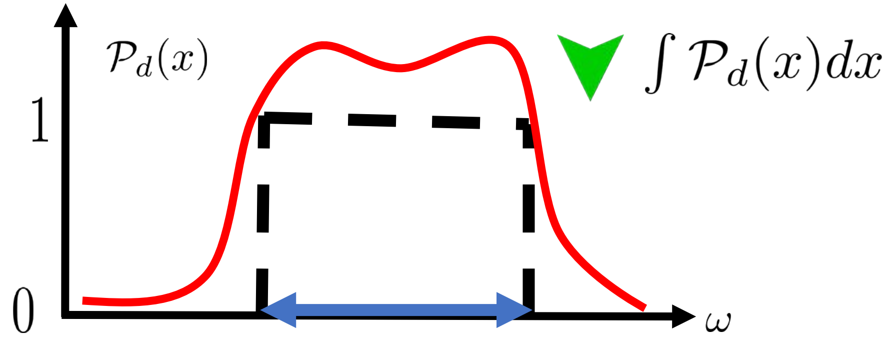
Uncertainty Set: $\mathcal{K} = \{x \in \mathbb{R}^n : \mathbf{I}_{\mathcal{K}}(x) = 1\}$

Polynomial approximations of **indicator function** of the support set.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$

$$\text{subject to} \quad \mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \mathcal{K}$$

$$\mathcal{P}_d(x) \geq 0$$



B : simple box that contains safety set

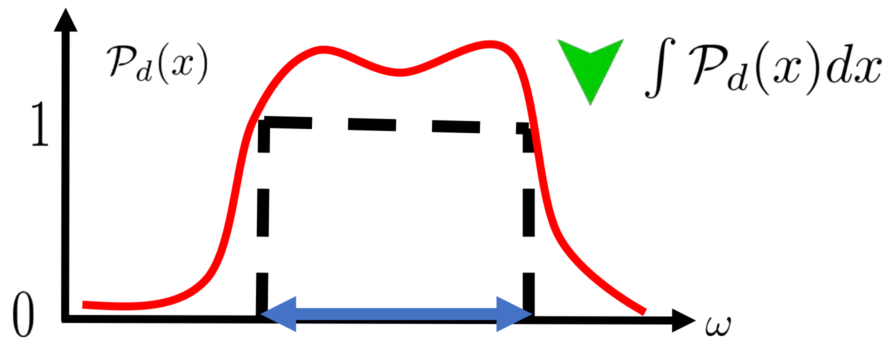
Assumption: After rescaling of polynomials $\mathrm{B} = [-1,1]^n$

$$\text{polynomial } \mathcal{P}(x) = \text{Upper bound of } \mathbf{I}_{\mathcal{K}}(x)$$

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

Polynomial approximations of **indicator function** of the support set.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$
$$\text{subject to} \quad \mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \mathcal{K}$$
$$\mathcal{P}_d(x) \geq 0$$



polynomial $\mathcal{P}(x)$ = Upper bound of $\mathbf{I}_{\mathcal{K}}(x)$

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

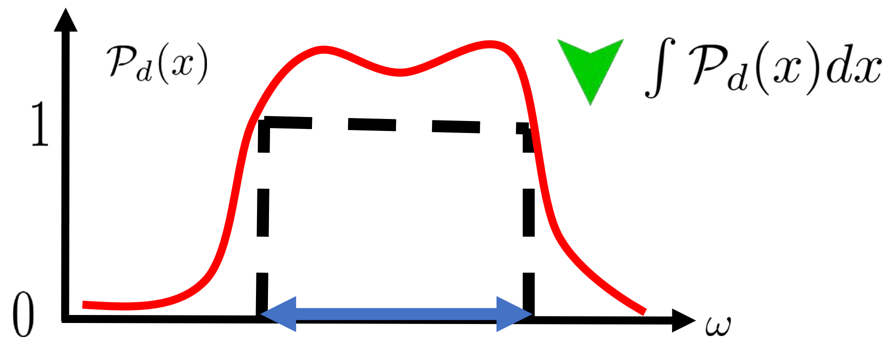
- We don't have the information of the uncertainty set $\mathcal{K}$

Polynomial approximations of **indicator function** of the support set.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$

$$\text{subject to} \quad \mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \mathcal{K}$$

$$\mathcal{P}_d(x) \geq 0$$



polynomial $\mathcal{P}(x) =$ Upper bound of $\mathbf{I}_{\mathcal{K}}(x)$

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

- We don't have the information of the uncertainty set $\mathcal{K}$
- We will replace the condition $\mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \mathcal{K}$ with other constraint in terms of the given moments

Polynomial approximations of **indicator function** of the support set.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$
$$\text{subject to} \quad \mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \mathcal{K}$$
$$\mathcal{P}_d(x) \geq 0$$

- Moments of distribution supported on the set $\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$ should satisfy the **localizing matrix** condition

- Probability distribution $pr(x)$

Supported on $\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) - 1 \geq 0\}$ $\Longleftrightarrow$ $\mathbf{M}_d\left((\mathcal{P}_d(x) - 1)y\right) \succcurlyeq 0$

Given moments

moments $y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha pr(x)dx$

- Localizing matrix in terms of moments up to order d

Polynomial approximations of **indicator function** of the support set.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$
$$\text{subject to} \quad \boxed{\mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \mathcal{K}}$$
$$\mathcal{P}_d(x) \geq 0$$

$$\Downarrow$$

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$
$$\text{subject to} \quad \boxed{\mathbf{M}_d\left((\mathcal{P}_d(x) - 1)y\right) \succcurlyeq 0}$$
$$\mathcal{P}_d(x) \geq 0$$

Polynomial approximations of **indicator function** of the support set.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$
$$\text{subject to} \quad \boxed{\mathcal{P}_d(x) - 1 \geq 0 \quad \forall x \in \mathcal{K}}$$
$$\mathcal{P}_d(x) \geq 0$$

An upper bound approximation

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$
$$\text{subject to} \quad \boxed{\mathbf{M}_d\left((\mathcal{P}_d(x) - 1)y\right) \succcurlyeq 0}$$
$$\mathcal{P}_d(x) \geq 0$$

Not necessarily an
Upper bound approximation

Polynomial approximations of **indicator function** of the support set.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$

Linear function in terms of coefficients of polynomial

$$\text{subject to} \quad \mathbf{M}_d\left((\mathcal{P}_d(x)-1)y\right) \succcurlyeq 0$$

LMI in terms of given moments and coefficients of polynomial

$$\mathcal{P}_d(x) \geq 0$$

$$\text{SOS}(\mathcal{P}_d(x))$$

LMI in terms of coefficients of polynomial

Uncertainty Set:

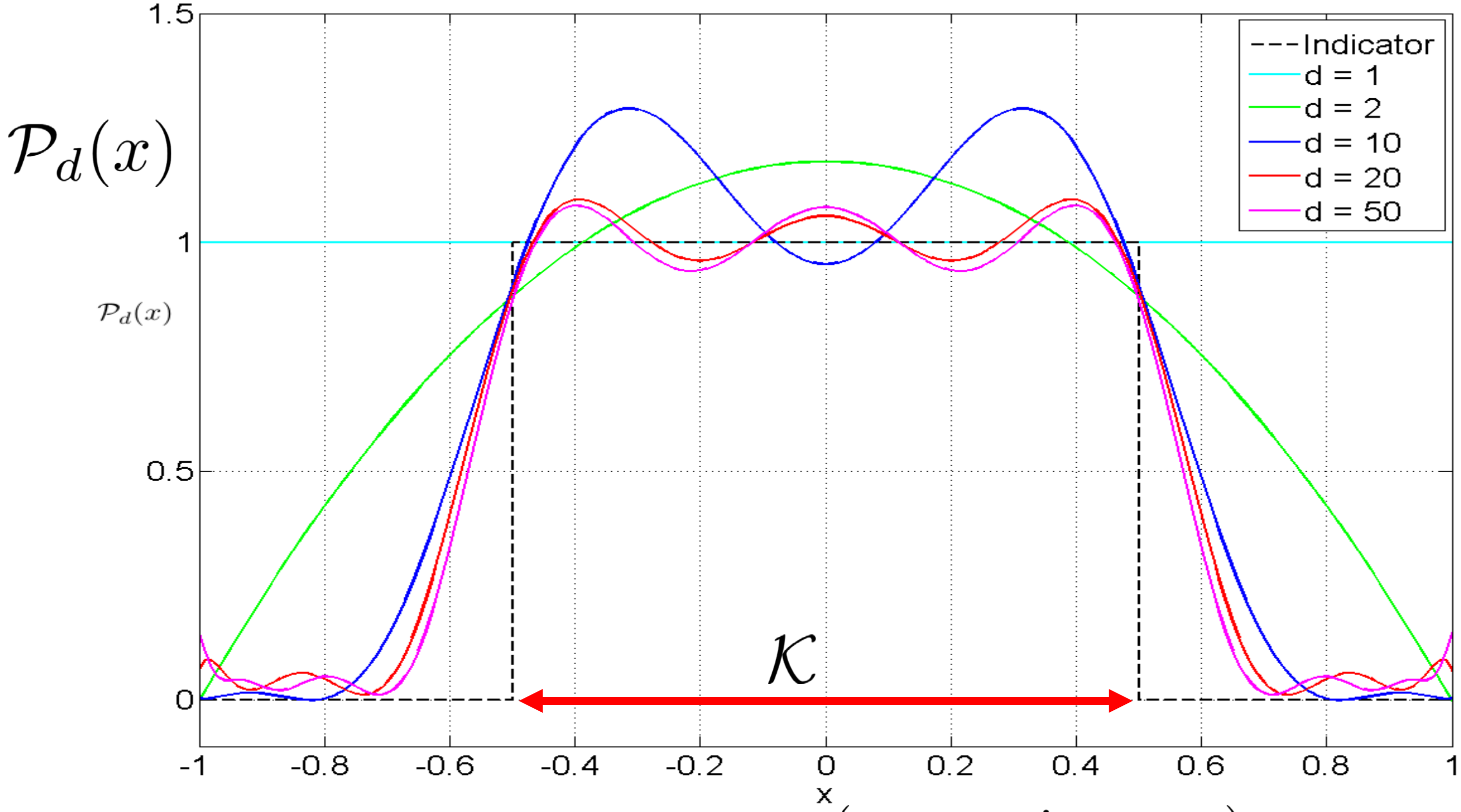
$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

Theorem:

$$\text{Lim}_{d\to\infty}\mathcal{P}_d(x) \to \mathbf{I}_{\mathcal{K}} \qquad \text{Lim}_{d\to\infty}\mathcal{K}_d \to \mathcal{K}$$

Example: Moments of Uniform distribution

Input: Finite sequence of the moments $y_\alpha = \frac{0.5^{\alpha+1} - (-0.5)^{\alpha+1}}{(\alpha+1)}$ $\quad \alpha = 0, \dots d$

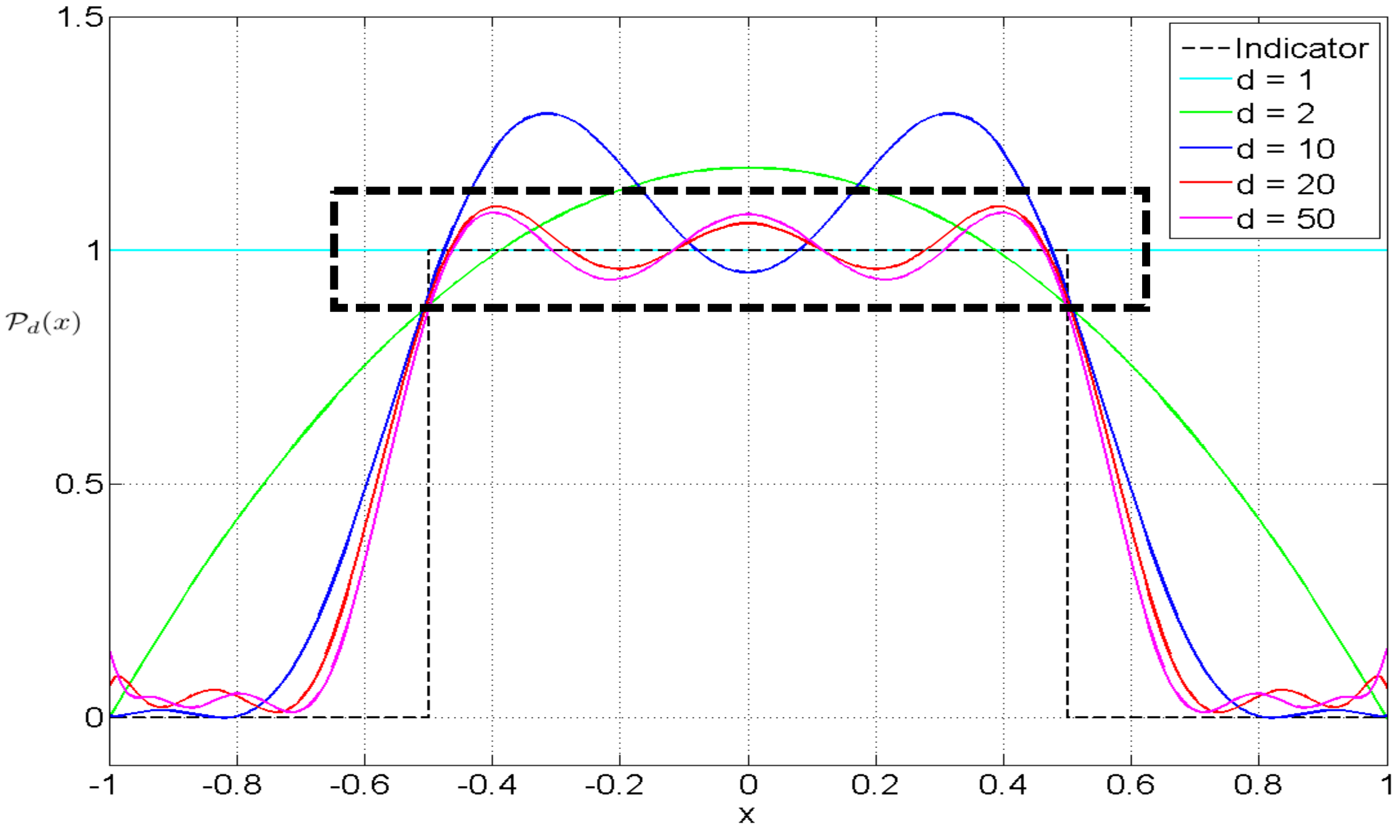


support set (uncertainty set)

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) - 1 \geq 0\}$$

https://github.com/jasour/rarnop19/blob/master/Lecture10_Probabilistic_Safety_Verification/Uncertainty_Set_Construction/Example_1_Uniform/Example_Uniform.m

- To minimize the regions inside the support where $\mathcal{P}_d(x)$ is below 1, we maximize the values of $\mathcal{P}_d(x)$ inside the support set while still trying to bring its values as low as possible everywhere else.



- To minimize the regions inside the support where $\mathcal{P}_d(x)$ is below 1, we maximize the values of $\mathcal{P}_d(x)$ inside the support set while still trying to bring its values as low as possible everywhere else.

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx$$
$$\text{subject to} \quad \mathbf{M}_d\left((\mathcal{P}_d(x)-1)y\right) \succcurlyeq 0$$
$$\mathcal{P}_d(x) \geq 0$$

Uncertainty Set:

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx - w_h h$$
$$\text{subject to} \quad \mathbf{M}_d\left((\mathcal{P}_d(x)-h)y\right) \succcurlyeq 0$$
$$\mathcal{P}_d(x) \geq 0$$
$$1 \leq h \leq 1+\Delta$$

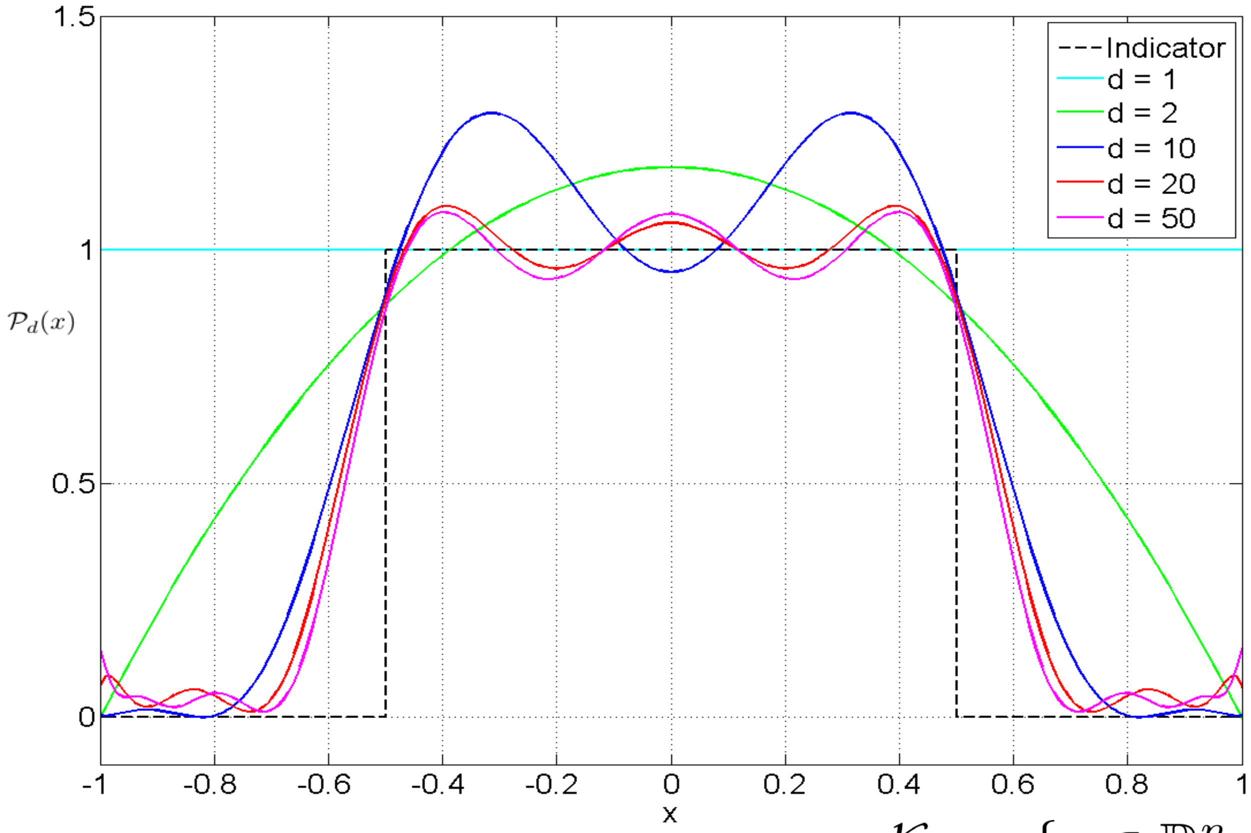
Uncertainty Set:

$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$$

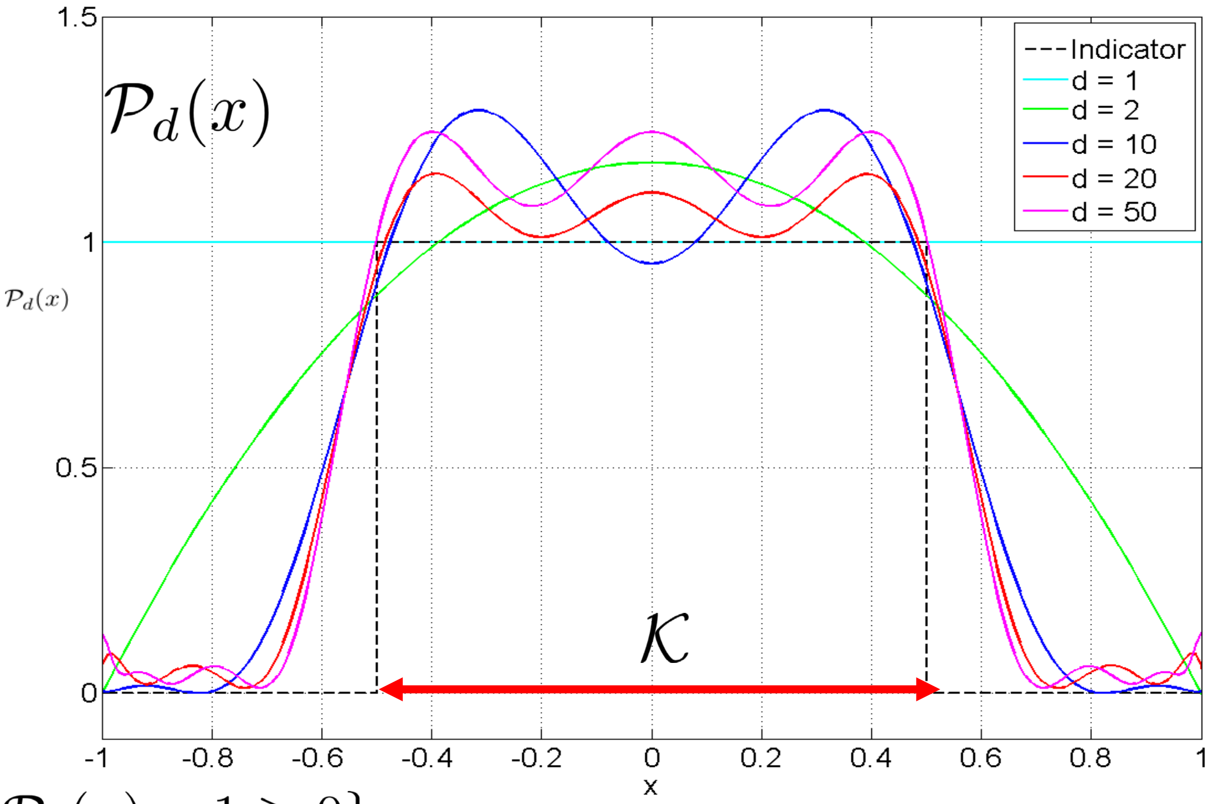
w_h, Δ : Positive design parameters

Example: Moments of Uniform distribution

Input: Finite sequence of the moments $y_\alpha = \frac{0.5^{\alpha+1} - (-0.5)^{\alpha+1}}{(\alpha+1)}$ $\quad \alpha = 0, \ldots d$



Improved SOS $\quad w_h = 1.2, \ \Delta = 0.2$



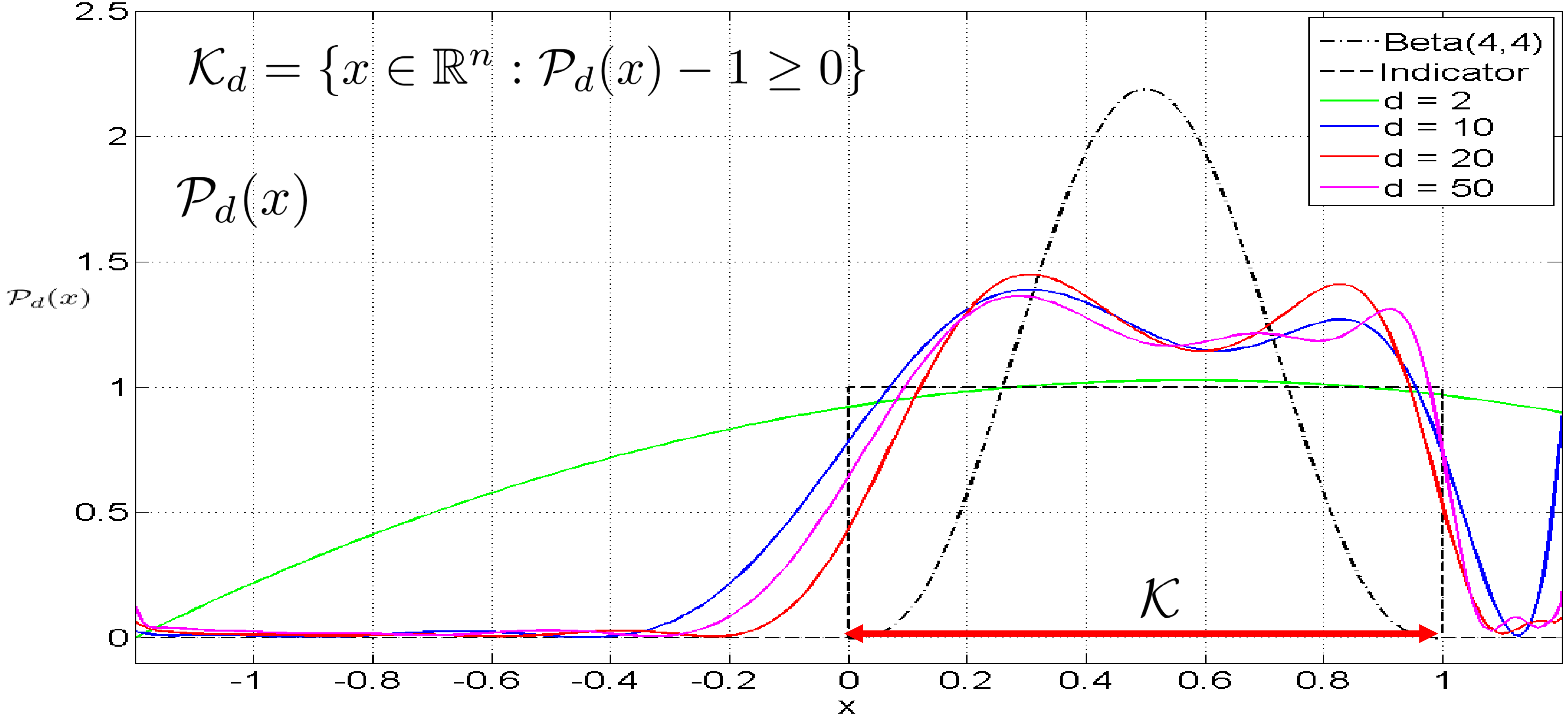
$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) - 1 \geq 0\}$$

https://github.com/jasour/rarnop19/blob/master/Lecture10_Probabilistic_Safety_Verification/Uncertainty_Set_Construction/Example_1_Uniform/Example_uniform_Improved1.m

Example: Moments of Beta distribution

Input: Finite sequence of the moments $y_\alpha = \frac{4+\alpha-1}{8+\alpha-1} y_{\alpha-1}, y_0 = 1 \qquad \alpha = 0, \ldots d$

Improved SOS $w_h = 1.2, \;\; \Delta = 0.2$



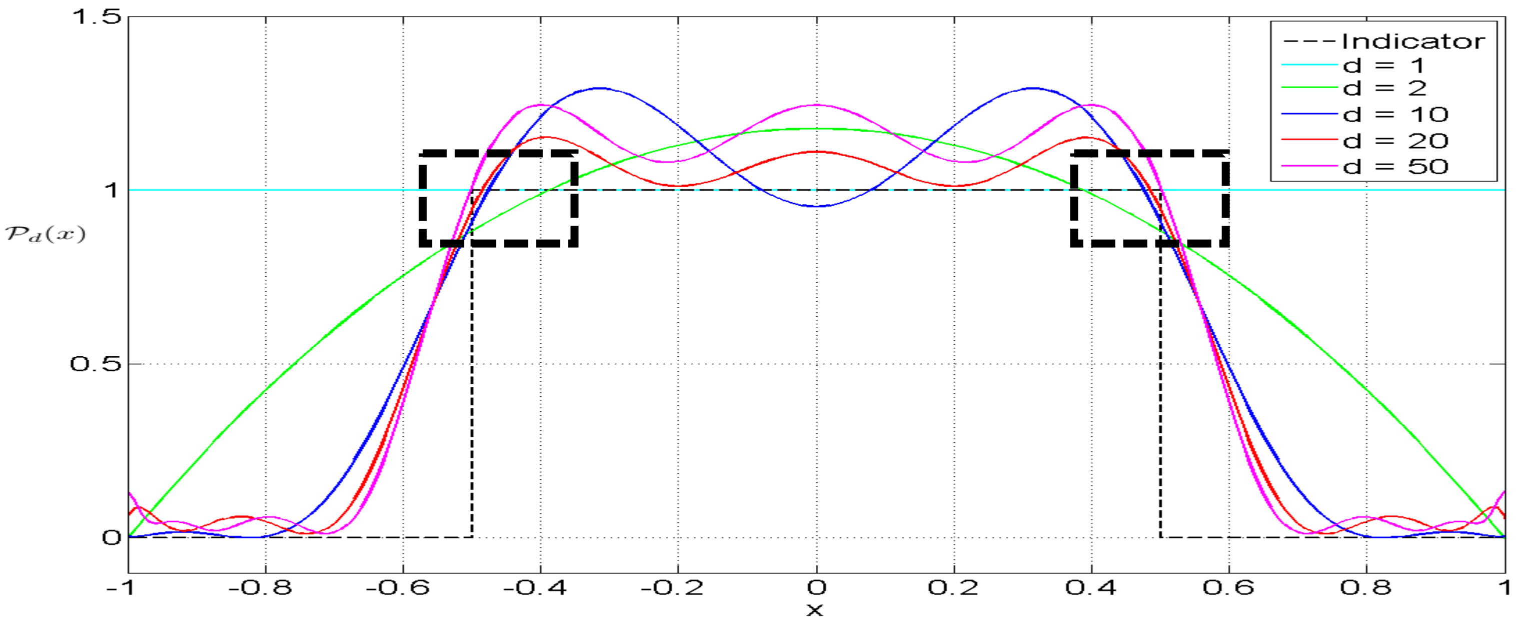
https://github.com/jasour/rarnop19/blob/master/Lecture10_Probabilistic_Safety_Verification/Uncertainty_Set_Construction/Example_2_Beta/Example_Beta.m

Uncertainty Set of Uniform distribution

Input: Finite sequence of the moments of uniform probability distribution $y = [y_0, \ldots, y_d]$

Output: Uncertainty set $\mathcal{K}$, i.e., support of uniform probability distribution whose moments are $y = [y_0, \ldots, y_d]$

- We will use the property of uniform distributions to improve the results of SOS program



$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) - 1 \geq 0\}$$

- On the boundary

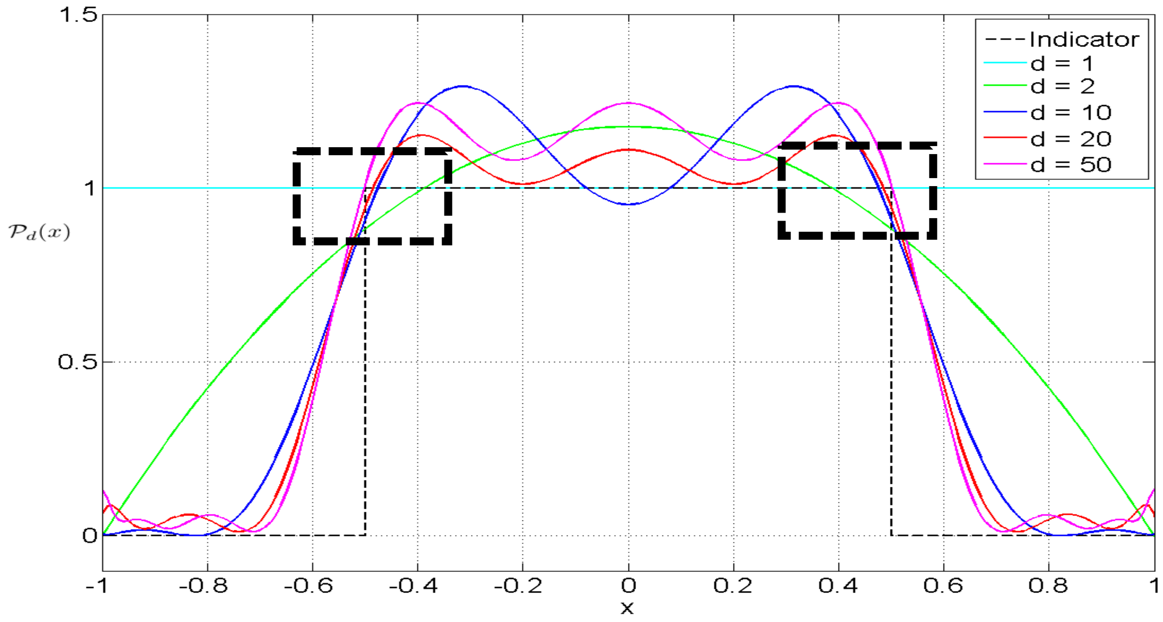
$$\mathcal{P}_d(x) - 1 = 0$$

Uncertainty Set of Uniform distribution

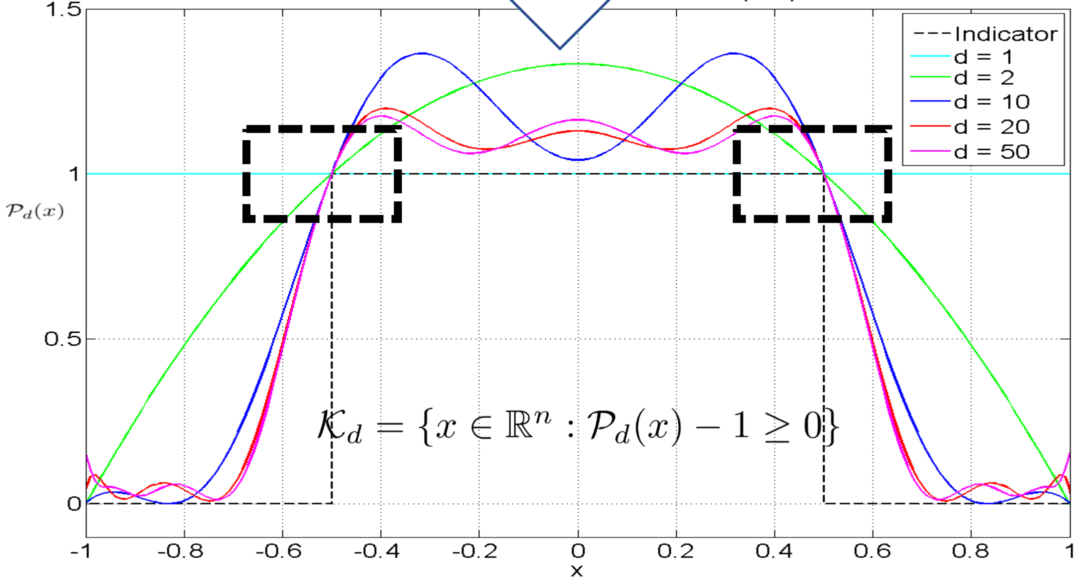
Input: Finite sequence of the moments of uniform probability distribution $y = [y_0, \ldots, y_d]$

Output: Uncertainty set $\mathcal{K}$, i.e., support of uniform probability distribution whose moments are $y = [y_0, \ldots, y_d]$

- We will use the property of uniform distributions to improve the results of SOS program



- On the boundary $\mathcal{P}_d(x) - 1 = 0$



Uncertainty Set of Uniform distribution

Let:

$y = [y_0, ..., y_d]$: Finite sequence of the moments of uniform probability distribution

$\partial\mathcal{K}$: Boundary of the support set $\mathcal{K}$

$\mathbf{M}_d(y)$: Moment Matrix

Define: scaled moment matrix $\bar{\mathbf{M}}_d(y) = \mathbf{N} .* \mathbf{M}_d(y)$

Weight matrix   Element-wise product

$\mathbf{M}_d(y) = \mathrm{E}[B_d(x)B_d^T(x)]$ → Elements of the matrix: $\mathrm{E}[x^\alpha x^\beta] = y_{\alpha+\beta}$

$\bar{\mathbf{M}}_d(y) = \mathbf{N} .* \mathbf{M}_d(y)$ → Elements of the matrix: $\frac{n+|\alpha|+|\beta|}{n+|\alpha|}\mathrm{E}[x^\alpha x^\beta]$

Dimension of the random variable

Uncertainty Set of Uniform distribution

$$\bar{\mathbf{M}}_d\,(y) = \mathbf{N}.*\mathbf{M}_d\,(y)$$

$\mathbf{M}_d(y) = \mathrm{E}[B_d(x)B_d^T(x)]$ $\longrightarrow$ Entries are: $\mathrm{E}[x^\alpha x^\beta] = y_{\alpha+\beta}$

$\bar{\mathbf{M}}_d(y) = \mathbf{N}.*\mathbf{M}_d\,(y)$ $\longrightarrow$ Entries are: $\frac{n+|\alpha|+|\beta|}{n+|\alpha|}\mathrm{E}[x^\alpha x^\beta]$

$$\mathbf{M}_2(y) = \mathrm{E}\left[\begin{bmatrix} x_1^0x_2^0 \\ x_1^1x_2^0 \\ x_1^0x_2^1 \end{bmatrix}\begin{bmatrix} x_1^0x_2^0 \\ x_1^1x_2^0 \\ x_1^0x_2^1 \end{bmatrix}^T\right] = \begin{bmatrix} \mathrm{E}[(x_1^0x_2^0)(x_1^0x_2^0)] & \mathrm{E}[(x_1^0x_2^0)(x_1^1x_2^0)] & \mathrm{E}[(x_1^0x_2^0)(x_1^0x_2^1)] \\ \mathrm{E}[(x_1^1x_2^0)(x_1^1x_2^0)] & \mathrm{E}[(x_1^1x_2^0)(x_1^1x_2^0)] & \mathrm{E}[(x_1^1x_2^0)(x_1^1x_2^1)] \\ \mathrm{E}[(x_1^0x_2^1)(x_1^0x_2^1)] & \mathrm{E}[(x_1^0x_2^1)(x_1^1x_2^1)] & \mathrm{E}[(x_1^0x_2^1)(x_1^0x_2^1)] \end{bmatrix} = \begin{bmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix}$$

$$\bar{\mathbf{M}}_2(y) = \begin{bmatrix} \frac{1+0+0}{1+0}\mathrm{E}[(\overbrace{x_1^0x_2^0}^{|\alpha|=0})(\overbrace{x_1^0x_2^0}^{|\beta|=0})] & \frac{1+0+1}{1+0}\mathrm{E}[(x_1^0x_2^0)(x_1^1x_2^0)] & \frac{1+0+1}{1+0}\mathrm{E}[(\overbrace{x_1^0x_2^0}^{|\alpha|=0})(\overbrace{x_1^0x_2^1}^{|\beta|=1})] \\ \frac{1+1+0}{1+1}\mathrm{E}[(x_1^1x_2^0)(x_1^1x_2^0)] & \frac{1+1+1}{1+1}\mathrm{E}[(x_1^1x_2^0)(x_1^1x_2^0)] & \frac{1+1+1}{1+1}\mathrm{E}[(x_1^1x_2^0)(x_1^1x_2^1)] \\ \frac{1+1+0}{1+1}\mathrm{E}[(x_1^0x_2^1)(x_1^0x_2^1)] & \frac{1+1+1}{1+1}\mathrm{E}[(x_1^0x_2^1)(x_1^1x_2^1)] & \frac{1+1+1}{1+1}\mathrm{E}[(x_1^0x_2^1)(x_1^0x_2^1)] \end{bmatrix} = \begin{bmatrix} y_{00} & 2y_{10} & 2y_{01} \\ y_{10} & \frac{3}{2}y_{20} & \frac{3}{2}y_{11} \\ y_{01} & \frac{3}{2}y_{11} & \frac{3}{2}y_{02} \end{bmatrix}$$

Uncertainty Set of Uniform distribution

Let: $\mathbf{p}$ be a vector such that $\bar{\mathbf{M}}_d(y)\mathbf{p} = 0$

- Polynomial $\mathcal{P}(x) = \sum \mathbf{p}_\alpha x^\alpha$ whose vector of coefficients is $\mathbf{p}$ vanishes on the boundary of support set.

$$\mathcal{P}(x) = \sum \mathbf{p}_\alpha x^\alpha = 0 \quad \forall\ x \in \partial\mathcal{K}$$

Uncertainty Set of Uniform distribution

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{P}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int_{\mathbf{B}} \mathcal{P}_d(x)dx - w_h h + \boxed{w_M||\bar{\mathbf{M}}_d(y)\underbrace{(\mathbf{p}-1)}_{\text{Coefficients of the polynomial } \mathcal{P}_d(x)-1}||_2}$$

$$\begin{aligned} \text{subject to} \quad & \mathbf{M}_d\left((\mathcal{P}_d(x)-h)y\right) \succcurlyeq 0 \\ & \mathcal{P}_d(x) \geq 0 \\ & 1 \leq h \leq 1+\Delta \end{aligned}$$

w_M, w_h, Δ : Positive design parameters

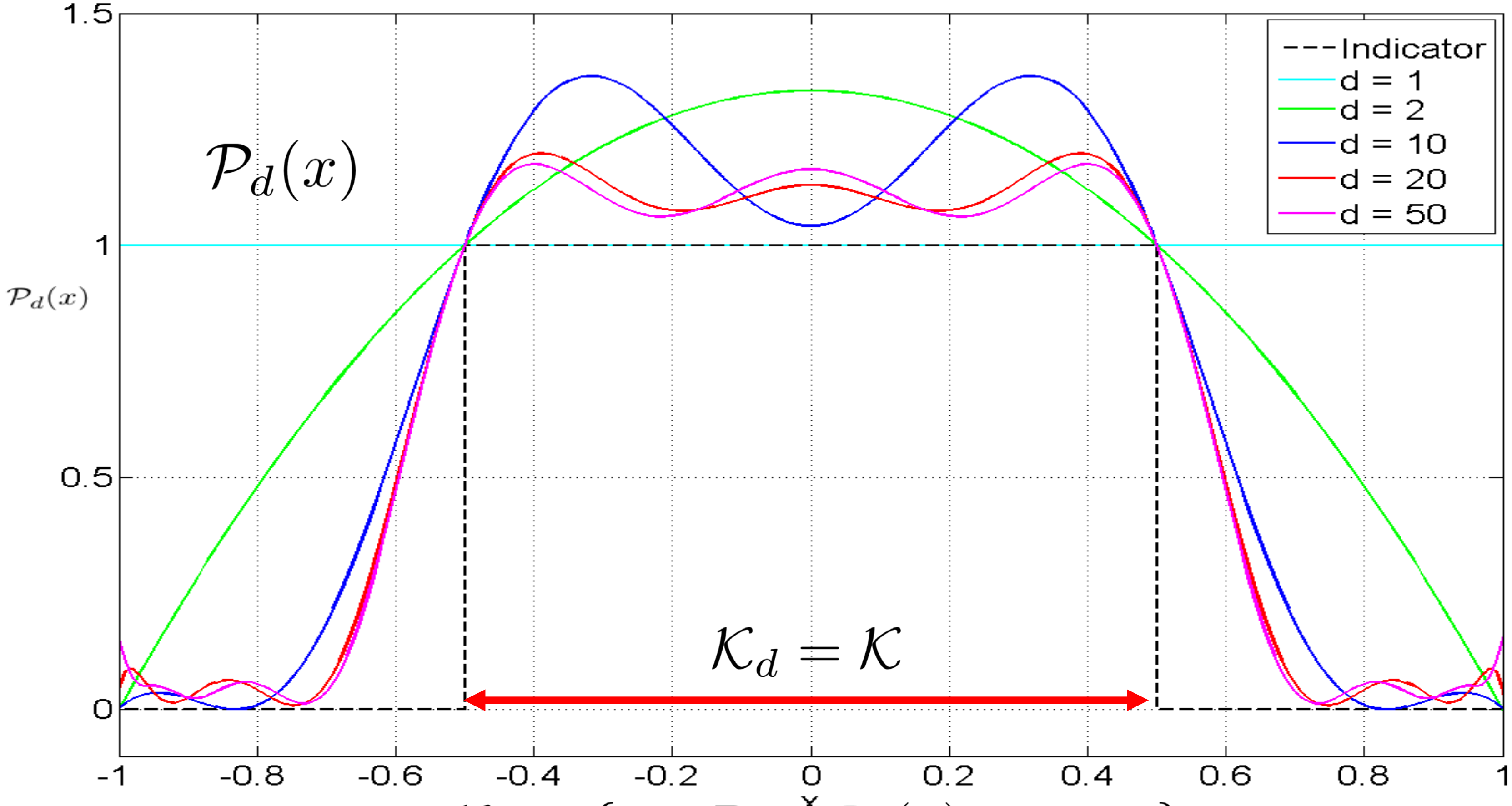
Uncertainty Set: $\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) \geq 1\}$

- We aim at "pushing" the coefficients of the polynomial $\mathcal{P}_d(x) - 1$as close as possible to the null space of $\bar{\mathbf{M}}_d(y)$ by minimizing the term $||\bar{\mathbf{M}}_d(y)(\mathbf{p}-1)||_2$
- In this case obtained polynomial $\mathcal{P}_d(x)$ becomes close to 1 at the boundary of support set while we still aim at having $\mathcal{P}_d(x)$ larger than 1 inside the support set.

Example: Moments of Uniform distribution

Input: Finite sequence of the moments $y_\alpha = \frac{0.5^{\alpha+1} - (-0.5)^{\alpha+1}}{(\alpha+1)}$ $\quad \alpha = 0, \dots d$

Improved SOS $\quad w_M = 10, w_h = 1.2, \ \Delta = 0.2$



$$\mathcal{K}_d = \{x \in \mathbb{R}^n : \mathcal{P}_d(x) - 1 \geq 0\}$$

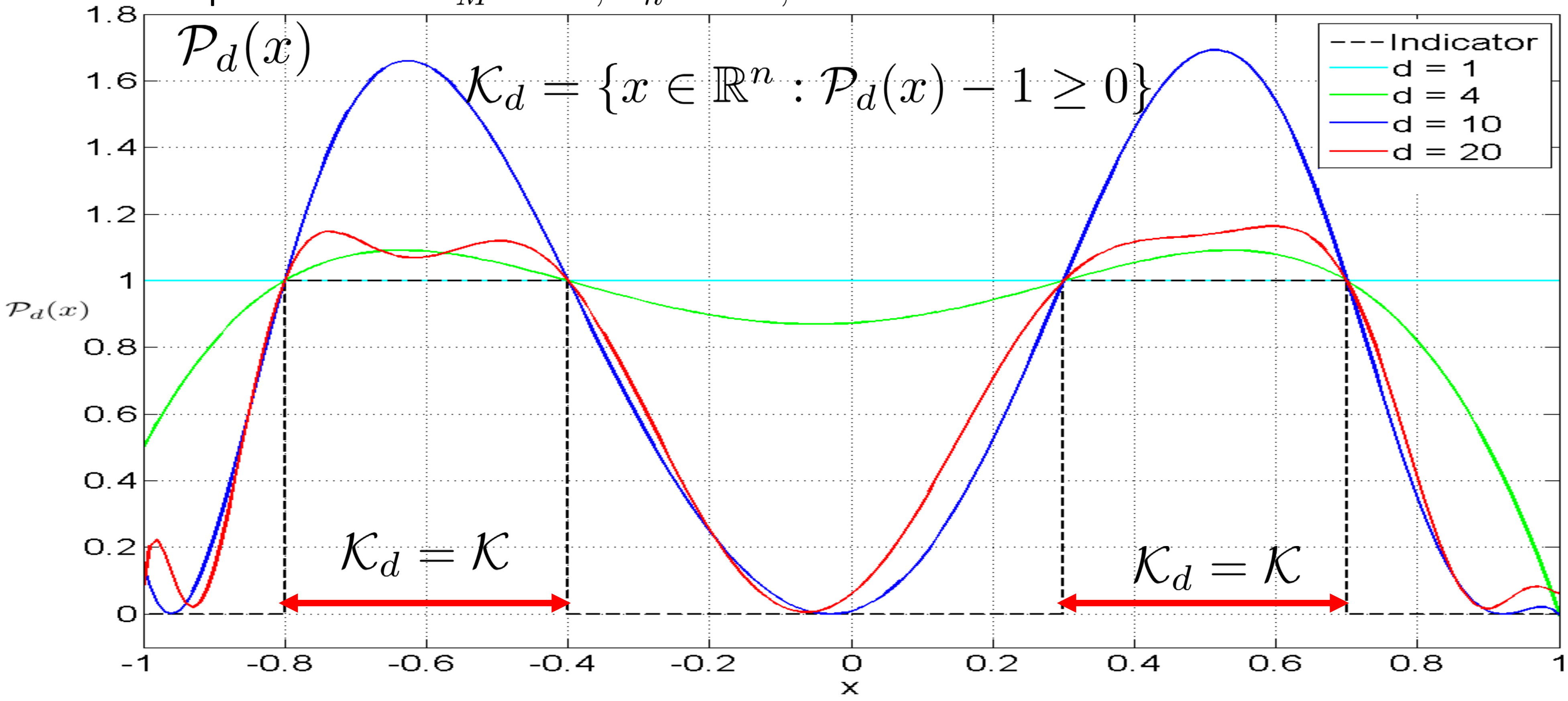
https://github.com/jasour/rarnop19/blob/master/Lecture10_Probabilistic_Safety_Verification/Uncertainty_Set_Construction/Example_1_Uniform/Example_Uniform_Improved2.m

Example: Moments of Union of Uniforms distributions

Input: Finite sequence of the moments $y_\alpha = \frac{(-0.4)^{\alpha+1}-(-0.8)^{\alpha+1}}{(0.4)(\alpha+1)} + \frac{0.7^{\alpha+1}-(0.3)^{\alpha+1}}{(0.4)(\alpha+1)} \quad \alpha = 0, \ldots d$

- y_α are moments of Uniform probability over the union of the sets $[-0.8, -0.4]$ and $[0.3, 0.7]$

Improved SOS $w_M = 10, w_h = 1.2,\ \Delta = 0.2$



https://github.com/jasour/rarnop19/blob/master/Lecture10_Probabilistic_Safety_Verification/Uncertainty_Set_Construction/Example_1_Uniform/Example_Uniform_Union.m

Probabilistic Safety Verification:

1) Uncertainty (Moment) propagation through nonlinear uncertain dynamics

2) Risk estimation in presence of nonlinear safety constraints

3) Uncertainty set construction from the moment information

4) Probability density function construction from the moment information

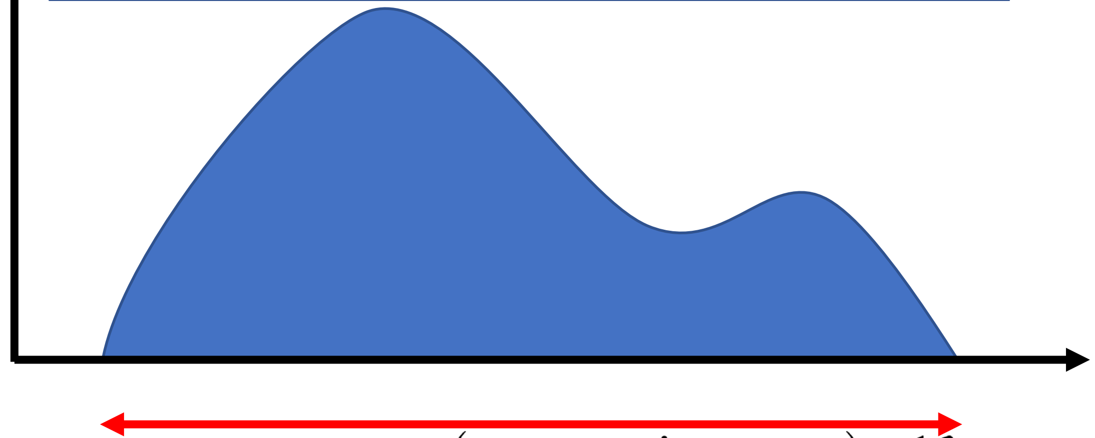
Probability Density Function from the moment information

Input: Finite sequence of the moments $y = [y_0, \ldots, y_d]$

Output: Polynomial approximation of Probability density function (pdf)

$$y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha pr(x)dx$$

probability distribution $pr(x)$



support set (uncertainty set) $\mathcal{K}$

$p_d(x)$: polynomial approximation of pr(x)

$$y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha p_d(x)dx$$

Probability Density Function from the moment information

Main Idea: We aim at finding **polynomial** approximations of **the probability density function.**

$$pr(x): \quad \text{probability distribution function (pdf)}$$

$$p_d(x): \quad \text{polynomial approximation of pdf}$$

$$\mathbf{P^{*d}} = \underset{p_d(x) \in \mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx$$

$$\mathbf{B} \subset \mathbb{R}^n: \quad \text{simple box}$$

- In the unconstrained optimization $pr(x)$ is unknown.
- We will use the given moments to represent $pr(x)$

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x) \in \mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx$$

$$\int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx = \boxed{\int_{\mathbf{B}} p_d^2(x) dx} - 2 \boxed{\int_{\mathbf{B}} p_d(x) pr(x) dx} + \boxed{\int_{\mathbf{B}} pr^2(x) dx}$$

- We rewrite each term in terms of the moments.

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx$$

$$\int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx = \underbrace{\boxed{\int_{\mathbf{B}} p_d^2(x)dx}}_{\mathbf{p}^T \mathbf{M}_d(y_{leb})\mathbf{p}} - 2\int_{\mathbf{B}} p_d(x)pr(x)dx + \int_{\mathbf{B}} pr^2(x)dx$$

Unknown Coefficients vector of polynomial $p_d(x)$

Moment matrix of Lebesgue measure $\int_{\mathbf{B}} dx$
Known Moments y_{leb}

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} (pr(x) - p_d(x))^2\, dx$$

$$\int_{\mathbf{B}} (pr(x) - p_d(x))^2\, dx = \boxed{\int_{\mathbf{B}} p_d^2(x)dx} - 2\int_{\mathbf{B}} p_d(x)pr(x)dx + \int_{\mathbf{B}} pr^2(x)dx$$

$$\mathbf{p}^T\mathbf{M}_d(y_{leb})\mathbf{p}$$

Unknown Coefficients vector of polynomial $p_d(x)$

Moment matrix of Lebesgue measure $\int_{\mathbf{B}} dx$
Known Moments y_{leb}

Lecture 4, page 158:

$$\int p_d^2(x)dx = \mathrm{E}[p_d^2(x)] = \mathrm{E}[(\mathbf{p}^T B_d(x))(\mathbf{p}^T B_d(x))^T] = \mathrm{E}[\mathbf{p}^T B_d(x) B_d^T(x)\mathbf{p}] = \mathbf{p}^T \mathrm{E}[B_d(x)B_d^T(x)]\mathbf{p} = \mathbf{p}^T\mathbf{M}_d(y_{leb})\mathbf{p}$$

Expected value with respect to Lebesgue measure

Polynomial of order d:
Coefficients vector and monomials
$$p_d(x) = \mathbf{p}^T B_d(x)$$

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x) \in \mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} \left(pr(x) - p_d(x)\right)^2 dx$$

$$\int_{\mathbf{B}} \left(pr(x) - p_d(x)\right)^2 dx = \underbrace{\boxed{\int_{\mathbf{B}} p_d^2(x) dx}}_{\mathbf{p}^T \mathbf{M}_d(y_{leb}) \mathbf{p}} - \underbrace{\boxed{2 \int_{\mathbf{B}} p_d(x) pr(x) dx}}_{2\mathbf{p}^T y} + \int_{\mathbf{B}} pr^2(x) dx$$

given moment vector of $pr(x)$

Probability Density Function from the moment information

$$\mathbf{P^{*d}} = \underset{p_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} (pr(x) - p_d(x))^2\, dx$$

$$\int_{\mathbf{B}} (pr(x) - p_d(x))^2\, dx = \underbrace{\int_{\mathbf{B}} p_d^2(x)dx}_{\mathbf{p}^T\mathbf{M}_d(y_{leb})\mathbf{p}} - \underbrace{2\int_{\mathbf{B}} p_d(x)pr(x)dx}_{2\mathbf{p}^T y} + \int_{\mathbf{B}} pr^2(x)dx$$

given moment vector of $pr(x)$

$$\int p_d(x)pr(x)dx = \int \mathbf{p}^T B_d(x)pr(x)dx = \mathbf{p}^T y$$

$$p_d(x) = \mathbf{p}^T B_d(x)$$

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx$$

$$\int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx = \underbrace{\int_{\mathbf{B}} p_d^2(x)dx}_{\mathbf{p}^T\mathbf{M}_d(y_{leb})\mathbf{p}} - \underbrace{2\int_{\mathbf{B}} p_d(x)pr(x)dx}_{2\mathbf{p}^T y} + \underbrace{\int_{\mathbf{B}} pr^2(x)dx}_{\text{constant}}$$

Unknown parameter

It does not affect the optimization and can be ignored

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2 = \int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx$$

$$\int_{\mathbf{B}} (pr(x) - p_d(x))^2 \, dx = \underbrace{\boxed{\int_{\mathbf{B}} p_d^2(x)dx}}_{\mathbf{p}^T\mathbf{M}_d(y_{leb})\mathbf{p}} - \underbrace{\boxed{2\int_{\mathbf{B}} p_d(x)pr(x)dx}}_{2\mathbf{p}^T y} + \underbrace{\boxed{\int_{\mathbf{B}} pr^2(x)dx}}_{\text{constant}}$$

Unknown parameter

It does not affect the optimization and can be ignored

$$\mathbf{P}^{*\mathbf{d}} = \underset{\mathbf{p}}{\text{minimize}} \quad \mathbf{p}^T\mathbf{M}_d(y_{leb})\mathbf{p} - 2\mathbf{p}^T y$$

y : Given moment vector of $pr(x)$

y_{leb} : Known moment of Lebesgue measure $\int_{\mathbf{B}} dx$

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{\mathbf{p}}{\text{minimize}} \quad \mathbf{p}^T \mathbf{M}_d(y_{leb})\mathbf{p} - 2\mathbf{p}^T y$$

y : Given moment vector of $pr(x)$

y_{leb} : Known moment of Lebesgue measure $\int_{\mathbf{B}} dx$

- Unique solution of Convex Quadratic Program $\mathbf{p}^* = \mathbf{M}_d(y_{leb})^{-1} y$

$\mathbf{M}_d(y_{leb})$: Moment matrix of Lebesgue measure is nonsingular for all d

- Proposition 1: Didier Henrion , Jean B. Lasserre , Martin Mevissen, "Mean Squared Error Minimization for Inverse Moment Problems" Applied Mathematics and Optimization archive Volume 70 Issue 1, August 2014 Pages 83-110

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{\mathbf{p}}{\text{minimize}} \quad \mathbf{p}^T \mathbf{M}_d(y_{leb})\mathbf{p} - 2\mathbf{p}^T y$$

y : Given moment vector of $pr(x)$

y_{leb} : Known moment of Lebesgue measure $\int_{\mathbf{B}} dx$

- Unique solution of Convex Quadratic Program $\mathbf{p}^* = \mathbf{M}_d(y_{leb})^{-1} y$

$\mathbf{M}_d(y_{leb})$: Moment matrix of Lebesgue measure is nonsingular for all d

- Obtained polynomial pdf $p_d(x)$ with coefficients $\mathbf{p}^*$ shares the same moments (up to order d) as the pdf $pr(x)$

Probability Density Function from the moment information

$$\mathbf{P}^{*\mathbf{d}} = \underset{\mathbf{p}}{\text{minimize}} \quad \mathbf{p}^T \mathbf{M}_d(y_{leb})\mathbf{p} - 2\mathbf{p}^T y$$

y : Given moment vector of $pr(x)$

y_{leb} : Known moment of Lebesgue measure $\int_{\mathbf{B}} dx$

- Unique solution of Convex Quadratic Program $\mathbf{p}^* = \mathbf{M}_d(y_{leb})^{-1} y$

$\mathbf{M}_d(y_{leb})$: Moment matrix of Lebesgue measure is nonsingular for all d

- Obtained polynomial pdf $p_d(x)$ with coefficients $\mathbf{p}^*$ shares the same moments (up to order d) as the pdf $pr(x)$

From the obtained solution: $\mathbf{M}_d(y_{leb})\mathbf{p}^* = y$

$$y = \mathbf{M}_d(y_{leb})\mathbf{p}^* = \mathrm{E}[B_d(x)B_d^T(x)]\mathbf{p}^* = \mathrm{E}[B_d(x)B_d^T(x)\mathbf{p}^*] = \mathrm{E}[B_d(x)p_d(x)]$$

Given moments of pdf $pr(x)$

Expected value with respect to Lebesgue measure $\int_{\mathbf{B}} dx$

$$= \int B_d(x)p_d(x)dx = \mathrm{E}_{p_d(x)}[B_d(x)] \Longrightarrow$$

Moment vector up to order d of pdf $p_d(x)$

Expected value with respect to probability measure $\int p_d(x)dx$

Probability Density Function from the moment information

Input: Finite sequence of the moments $y = [y_0, \ldots, y_d]$

Output: Probability density function (pdf) $pr(x)$

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x) \in \mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2$$

$p_d(x)$: Polynomial approximation of $pr(x)$

$$p_d(x) = \mathbf{p}^T B_d(x)$$

$$\mathbf{p} = \mathbf{M}_d(y_{leb})^{-1} y$$

coefficients

Moment matrix of Lebesgue measure $\int_{\mathbf{B}} dx$
Known Moments y_{leb}

$$\text{Lim}_{d \to \infty} ||pr(x) - p_d(x)||_2$$

- Proposition 2: Didier Henrion , Jean B. Lasserre , Martin Mevissen, “Mean Squared Error Minimization for Inverse Moment Problems” Applied Mathematics and Optimization archive Volume 70 Issue 1, August 2014 Pages 83-110

Probability Density Function from the moment information

Input: Finite sequence of the moments $y = [y_0, \dots, y_d]$

Output: Probability density function (pdf) $pr(x)$

$$p_d(x) = \left(\mathbf{M}_d(y_{leb})^{-1} y\right)^T B_d(x)$$ Polynomial approximation of $pr(x)$

- Obtained $p_d(x)$ is guaranteed to share the same moments with $pr(x)$.
- Obtained $p_d(x)$ is NOT guaranteed to yield a nonnegative approximation.
  (Hence, obtain $p_d(x)$ is not a probability density function)

Probability Density Function from the moment information

Input: Finite sequence of the moments $y = [y_0, \ldots, y_d]$

Output: Probability density function (pdf) $pr(x)$

$$p_d(x) = \left(\mathbf{M}_d(y_{leb})^{-1} y\right)^T B_d(x)$$ Polynomial approximation of $pr(x)$

- Obtained $p_d(x)$ is guaranteed to share the same moments with $pr(x)$.
- Obtained $p_d(x)$ is NOT guaranteed to yield a nonnegative approximation. (Hence, obtain $p_d(x)$ is not a probability density function)
- We can add nonnegativity condition to the optimization problem.

$$\mathbf{P}^{*\mathbf{d}} = \underset{p_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad ||pr(x) - p_d(x)||_2^2$$
$$\text{subject to} \quad p_d(x) \geq 0$$

$\Rightarrow$

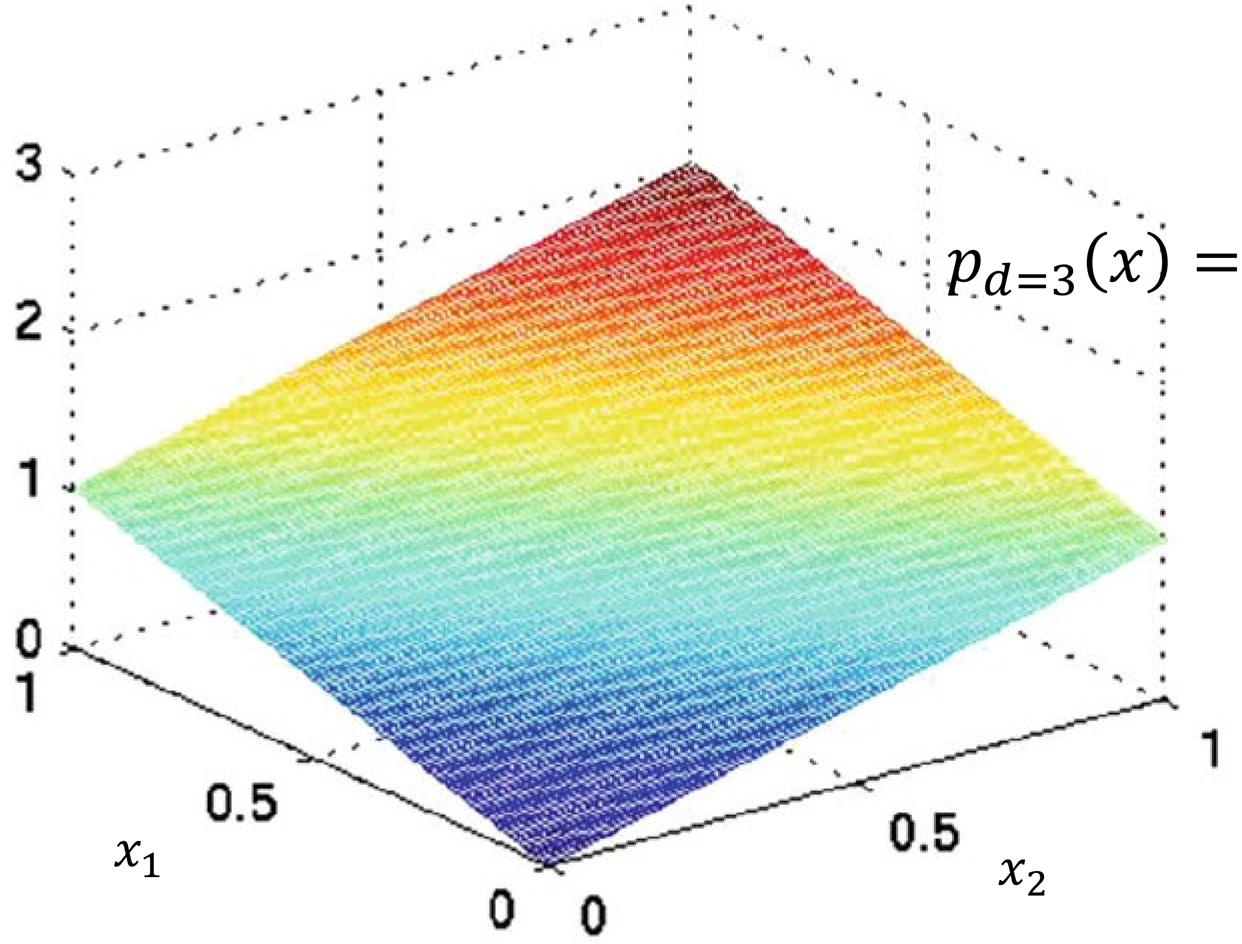
$$\mathbf{P}^{*\mathbf{d}} = \underset{\mathbf{p}}{\text{minimize}} \quad \mathbf{p}^T \mathbf{M}_d(y_{leb})\mathbf{p} - 2\mathbf{p}^T y$$ Convex Quadratic

$$\text{subject to} \quad p_d(x) \geq 0 \longrightarrow$$ SOS($p_d(x)$)

Example 1

Input: Finite sequence of the moments $y_{\alpha_1 \alpha_2} = \frac{1}{(\alpha_1+1)(\alpha_2+2)} + \frac{1}{(\alpha_1+2)(\alpha_2+1)}$

- y_α are moments of density $pr(x_1, x_2) = x_1 + x_2$



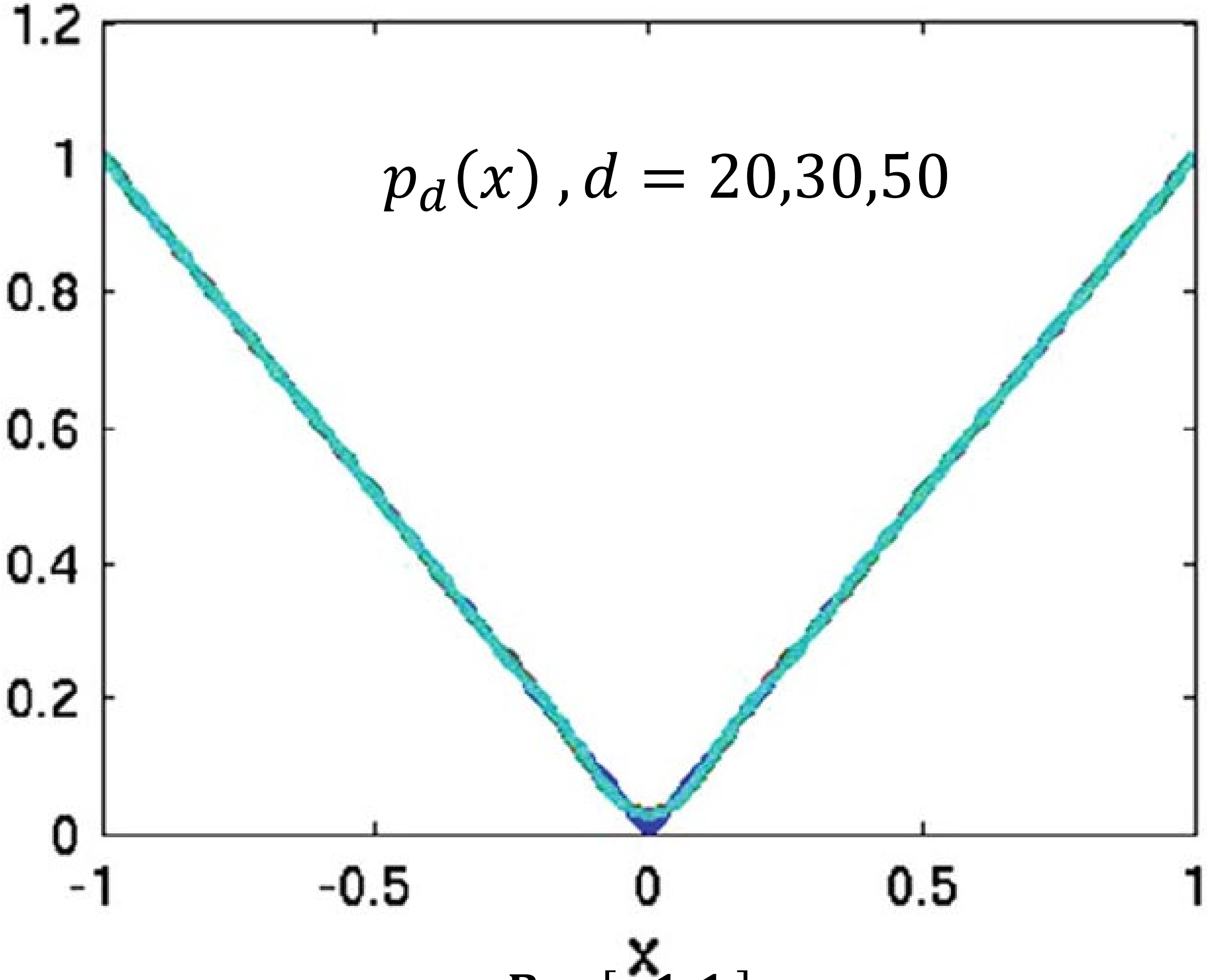
$p_{d=3}(x) = x_1 + x_2 = pr(x_1, x_2)$

$\mathbf{B} = [-1, 1]^2$

Example 2

Input: Finite sequence of the moments $y_\alpha = \frac{1+(-1)^\alpha}{d+2}$

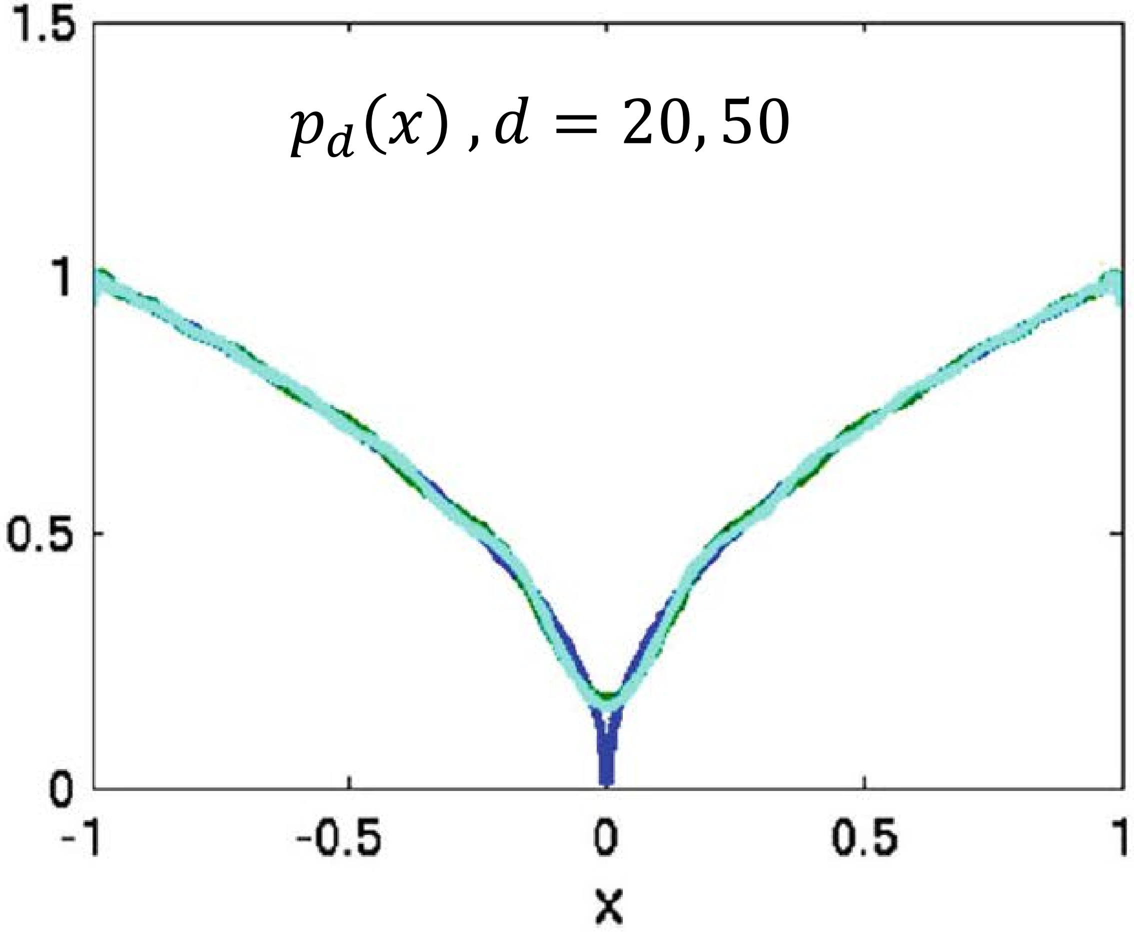
- y_α are moments of density $pr(x) = |x|$



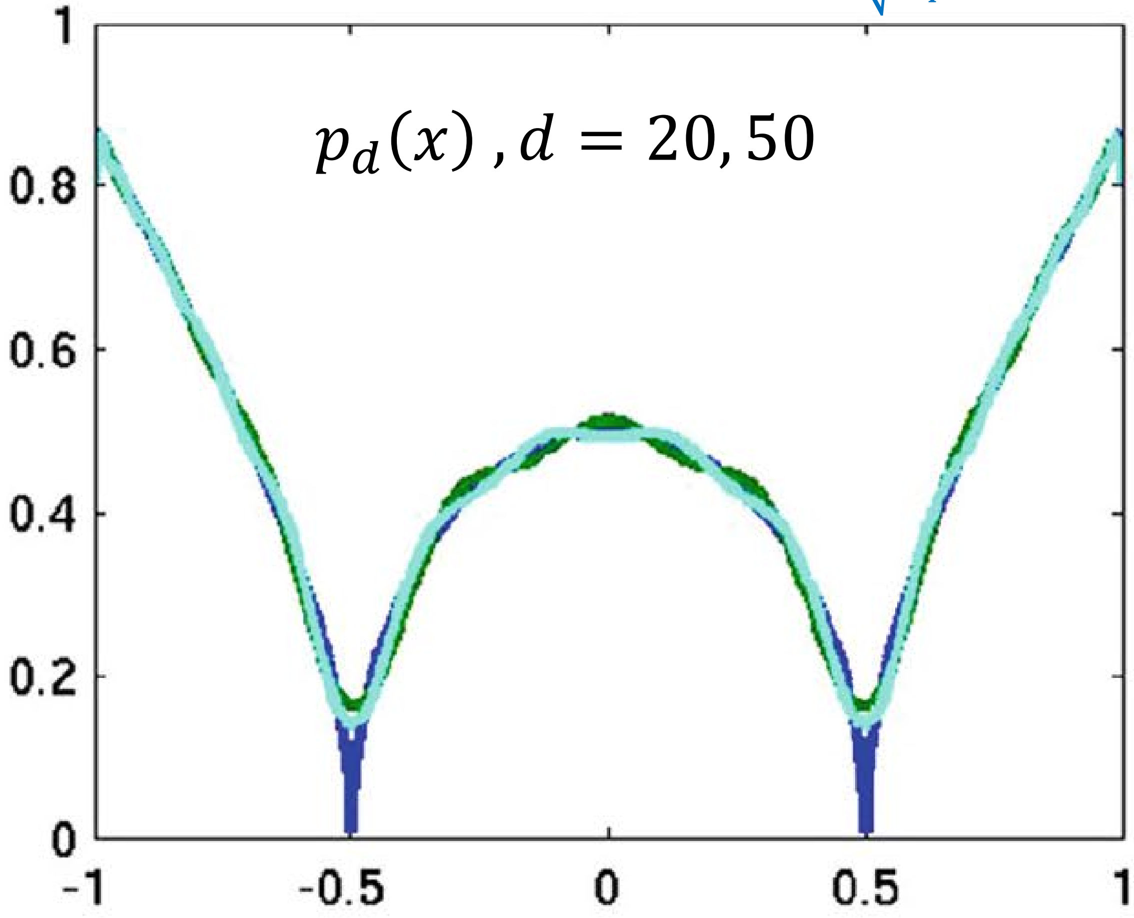
$\mathbf{B} = [-\mathbf{1}, \mathbf{1}]$

Example 3

- y_α are moments of density $pr(x) = \sqrt{|x|}$



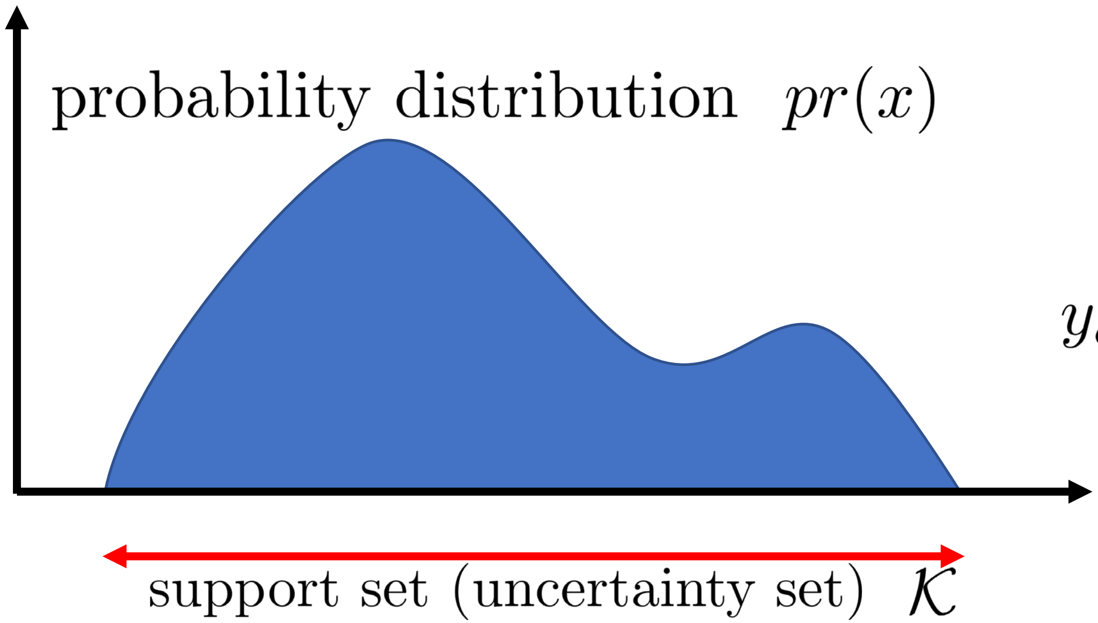
- y_α are moments of density $pr(x) = \sqrt{|\frac{1}{4} - x^2|}$



Probability Density Function from the moment information

Input: Finite sequence of the moments $y = [y_0, \ldots, y_d]$

Output: Probability density function (pdf) $pr(x)$



$$y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha pr(x)dx$$

Probability Density Function from the moment information

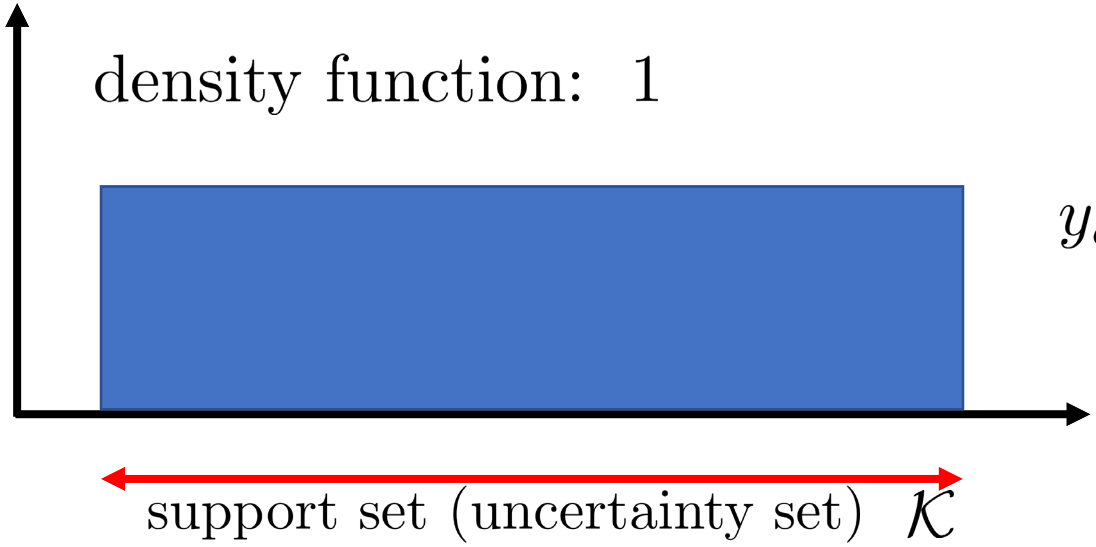
Input: Finite sequence of the moments $y = [y_0, \ldots, y_d]$

Output: Probability density function (pdf) $pr(x)$

Particular Case: moments belong to Lebesgue measure

Output: $\boldsymbol{p_d(x)}$ Polynomial approximation of Lebesgue measure

- We can recover the set $\boldsymbol{K}$ by looking at the level sets of $\boldsymbol{p_d}$, i.e., $\{\boldsymbol{x}: \boldsymbol{p_d(x)} \geq \frac{1}{2}\}$



$$y_\alpha = \mathrm{E}[x^\alpha] = \int x^\alpha dx$$

Example 4: Recovering Geometric Objects

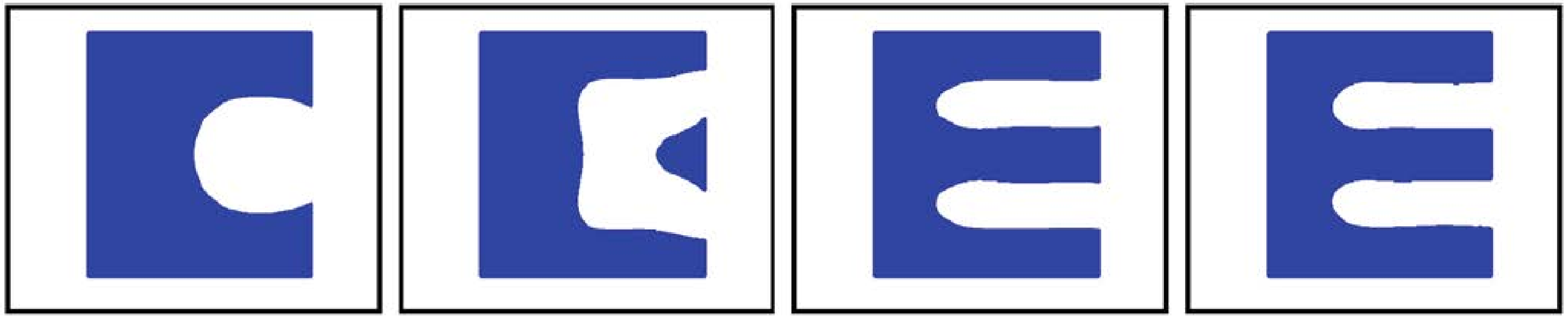
$$\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^2 : (x_1 \in [0,1] \wedge x_2 \in [0.8,1]) \vee (x_1 \in [0,0.3] \wedge x_2 \in [0.6,0.8]) \vee (x_1 \in [0,1] \wedge x_2 \in [0.4,0.6]) \vee (x_1 \in [0,0.3] \wedge x_2 \in [0.2,0.4]) \vee (x_1 \in [0,1] \wedge x_2 \in [0,0.2])\}$$

Input: moments of Lebesgue measure on the set $\mathbf{K}$

Output: polynomial approximation of the Lebesgue measure on the set $\mathbf{K}$ (density function 1)

- Obtained polynomial approximation $\boldsymbol{p_d(x)}$ is used to recover the set.

Recovered set $\{\boldsymbol{x}: \boldsymbol{p_d}(\boldsymbol{x}) \geq \frac{1}{2}\}$

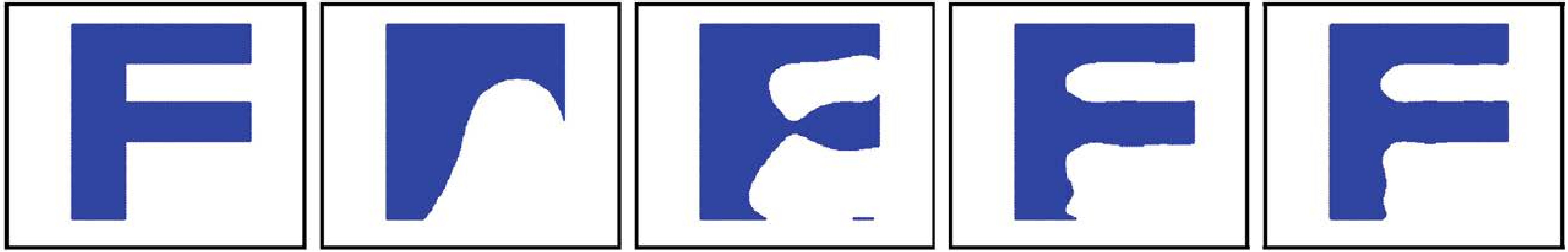


$\boldsymbol{d}$=3 $\boldsymbol{d}$=5 $\boldsymbol{d}$=8 $\boldsymbol{d}$=10

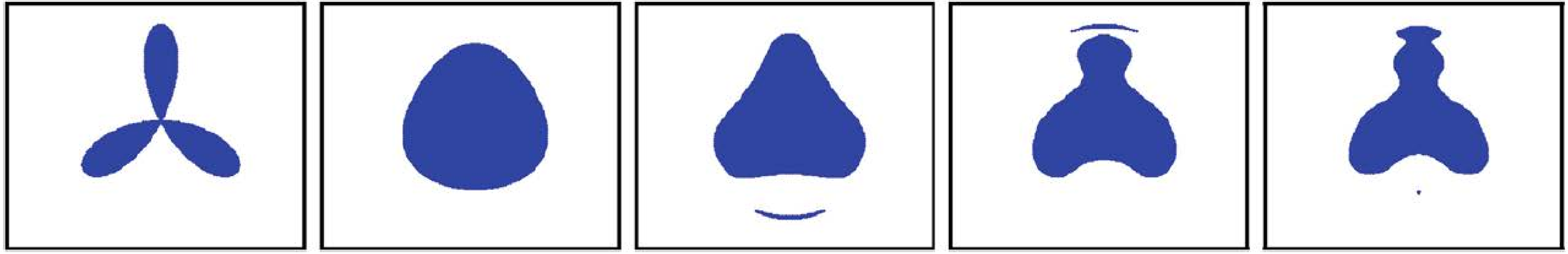
- Didier Henrion , Jean B. Lasserre , Martin Mevissen, "Mean Squared Error Minimization for Inverse Moment Problems" Applied Mathematics and Optimization archive Volume 70 Issue 1, August 2014 Pages 83-110
- Teague, M.R. "Image analysis via the general theory of moments". J. Opt. Soc. Am. **70**(8), 920–930 (1980)

Example 5: Recovering Geometric Objects

Input: moments of Lebesgue measure on the set $\mathbf{K}$ **Output:** Recovered set



set $\mathbf{K}$ $\boldsymbol{d}$=3 $\boldsymbol{d}$=5 $\boldsymbol{d}$=8 $\boldsymbol{d}$=10



set $\mathbf{K}$ $\boldsymbol{d}$=3 $\boldsymbol{d}$=5 $\boldsymbol{d}$=8 $\boldsymbol{d}$=10

Probabilistic Safety Verification:

1) Uncertainty (Moment) propagation through nonlinear uncertain dynamics

2) Risk estimation in presence of nonlinear safety constraints

3) Uncertainty set construction from the moment information

4) Probability density function construction from the moment information

Moment Based Uncertainty Propagation

- A. Jasour, "Convex Approximation of Chance Constrained Problems: Application in Systems and Control", School of Electrical Engineering and Computer Science, The Pennsylvania State University, 2016.

- A. Jasour, B. C. Williams, "Sequential Convex Chance Optimization for Flow-Tube based Control of Probabilistic Nonlinear Systems", IEEE Conference on Decision and Control, 2019.

Risk Estimation

- D. Henrion, J. B. Lasserre, and C. Savorgnan “Approximate Volume and Integration for Basic Semialgebraic Sets”, SIAM Rev., 51(4), 722–743, 2009.

- A. Jasour, A. Hofmann, B. C. Williams, "Moment-Sum-Of-Squares Approach for Fast Risk Estimation in Uncertain Environments", IEEE Conference on Decision and Control, 2018.

- J.B. Lasserre. ”Volume of sub-level sets of polynomials”, 18th European Control Conference (ECC), 2019.

- J.B. Lasserre, “Volume of Sublevel Sets of Homogeneous Polynomials”, SIAM Journal on Applied Algebra and Geometry, 2019.

Uncertainty Set Construction

- A. Jasour, C. Lagoa, ”Reconstruction of Support of a Measure From Its Moments”, 53 rd IEEE Conference on Decision and Control, Los Angeles, California, 2014.

Probability Density Function Construction

- D. Henrion, J. B. Lasserre, M. Mevissen “Mean Squared Error Minimization for Inverse Moment Problems”, Journal Applied Mathematics and Optimization archive Volume 70 Issue 1, Pages 83-110, 2014.

16.S498 Risk Aware and Robust Nonlinear Planning
Fall 2019