Lecture 8

Nonlinear Robust and Distributionally Robust Chance Constrained Optimization

SOS Based SDP Approach

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

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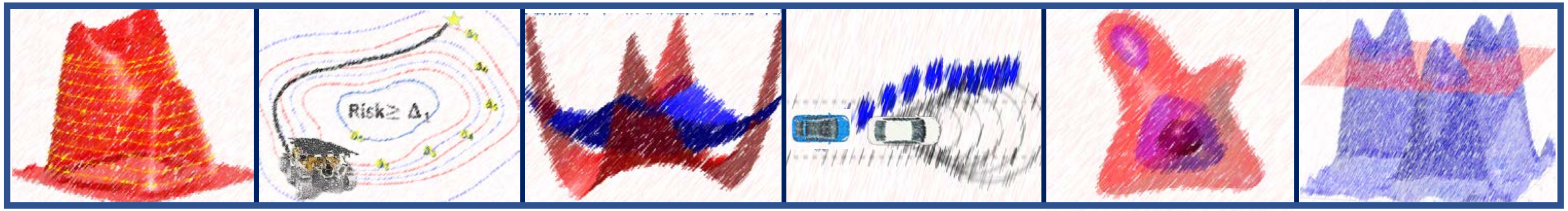


Topics

- Formulation of Robust Optimization
- Challenges
- Robust Sum-of-Squares
- Robust Sets for Robust Optimization
- Robust Games
- Distributionally Robust Chance Constrained Optimization

Robust Optimization Based Planning

$$\underset{\text{design parameters}}{\text{minimize}} \quad \text{Objective Function}(\text{design parameters})$$

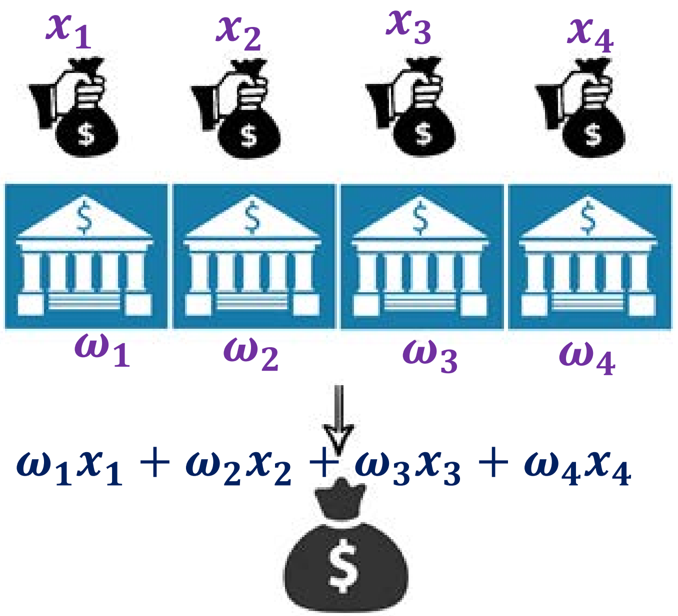
$$\text{subject to} \quad \text{satisfy constraints for all possible values of uncertainties}$$

Mathematical Formulation

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$

$$\text{subject to} \quad g_i(\underset{\substack{\downarrow \\ \text{Uncertainty}}}{x, \omega}) \geq 0, \; i = 1, ..., n_g, \quad \forall \omega \in \underset{\substack{\downarrow \\ \text{Uncertainty Set}}}{\Omega}$$

Example:



- Assets with uncertain rate of return:

$$\omega_i \in [l_i, u_i], i = 1, \ldots, 4$$

- x_i invested money in asset i

- Uncertain Constraints:

  Achive a return higher than "r^*“ $= \{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*\}$

Robust Optimization

$$\begin{aligned} \underset{x_1, x_2, x_3, x_4}{\text{minimize}} \quad & x_1 + x_2 + x_3 + x_4 \\ \text{subject to} \quad & \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^* \quad \forall \omega_i \in [l_i, u_i], \; i = 1, \ldots, 4 \end{aligned}$$

Example:

Robust Optimization

$$\begin{array}{ll}\underset{x_1,x_2,x_3,x_4}{\text{minimize}} & x_1 + x_2 + x_3 + x_4 \\ \text{subject to} & \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^* \quad \forall \omega_i \in [l_i, u_i],\ \ i = 1, ..., 4\end{array}$$

Chance Constrained Optimization

$$\begin{array}{ll}\underset{x_1,x_2,x_3,x_4}{\text{minimize}} & x_1 + x_2 + x_3 + x_4 \\ \text{subject to} & \text{Probability}(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*) \geq 1 - \Delta\end{array}$$

$$\omega_i \sim pr_{\omega_i}(\omega),\ \ i = 1, ..., 4$$

Example: Safe Control

- Uncertain Dynamical Model

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

states inputs Bounded Uncertainty $\in [l_k, u_k]$

- Unsafe Sets

$$\chi_{obs_i} = \{x : p_{obs_i}(x) < 0\},\ \ i = 1, \dots, \ell$$

- Safe Sets

$$\chi_{safe} = \{x : p_{obs_i}(x) \geq 0\ , i = 1, \dots, \ell\}$$

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➢ Source of uncertainty at time k:

i) uncertain states $x_k \in [l_{x_k}, u_{x_k}]$ and ii) disturbance $\omega_k \in [l_k, u_k]$

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Robust Optimization

$$\underset{u_k}{\text{minimize}} \quad ||u_k||_2^2$$

$$\text{subject to} \quad p_{obs_i}(x(k+1)) \geq 0|_{i=1}^{\ell} \qquad \forall \omega \in [l_k, u_k] \quad \forall x(k) \in [l_{x_k}, u_{x_k}]$$

$$u_k \in \mathcal{U} \qquad x_{k+1} = f(x_k, u_k, \omega_k)$$

Example: Safe Control

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Robust Optimization

$$\begin{aligned} \underset{u_k}{\text{minimize}}\quad & ||u_k||_2^2 \\ \text{subject to}\quad & p_{obs_i}\left(f\left(x_k, u_k, \omega_k\right)\right) \geq 0|_{i=1}^{\ell} \quad \forall \omega \in [l_k, u_k] \quad \forall x(k) \in [l_{x_k}, u_{x_k}] \\ & u_k \in \mathcal{U} \end{aligned}$$

Example: Safe Control

Robust Optimization

$$\begin{aligned} \underset{u_k}{\text{minimize}} \quad & ||u_k||_2^2 \\ \text{subject to} \quad & \boxed{p_{obs_i}\left(f\left(x_k, u_k, \omega_k\right)\right) \geq 0|_{i=1}^{\ell}} \quad \boxed{\forall \omega \in [l_k, u_k] \quad \forall x(k) \in [l_{x_k}, u_{x_k}]} \\ & u_k \in \mathcal{U} \end{aligned}$$

Chance Constrained Optimization

$$\begin{aligned} \underset{u_k}{\text{minimize}} \quad & ||u_k||_2^2 \\ \text{subject to} \quad & \boxed{\text{Prob}\left(p_{obs_i}\left(f\left(x_k, u_k, \omega_k\right)\right) \geq 0|_{i=1}^{\ell} \right) \geq 1 - \Delta} \quad \boxed{\omega(k) \sim pr_{\omega_k}(\omega) \quad x(k) \sim pr_{x_k}(x)} \\ & u_k \in \mathcal{U} \end{aligned}$$

Example: Safe Control

Examples on safe control of uncertain nonlinear dynamical systems in pages 23, 30, 64, 121

Challenges

Robust Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g_i(x, \omega) \geq 0, \; i = 1, ..., n_g, \quad \forall \omega \in \Omega \end{aligned}$$

Objective function and constraints are polynomial functions.

Existing Methods For Robust Optimization:

- Sampling based Approaches
  - R. Tempo, G. Calafiore, and F. Dabbene, Randomized Algorithms for Analysis and Control of Uncertain Systems, Communications and Control Engineering Series, Springer-Verlag, London, 2013.

- Particular Classes of Constraints and Uncertainties
  - A. Ben-Tal, L. El Ghaoui, A. Nemirovski. “Robust Optimization”, Princeton University Press, Princeton, 2009.
  - Uncertain Conic Optimization
  - Uncertain Quadratic Problems
  - Uncertain Semidefinite Problems
  - Ellipsoidal Uncertainty
  - Norm-Bounded Uncertainty
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In This Lecture: We use Sums-of-Squares approach to address a general class of Robust Optimizations.

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Robust Sum-of-Squares

Nonnegative Polynomials

$$p(x) \geq 0, \quad \forall x \in \chi$$

Robust Nonnegative Polynomials

$$p(x, \underbrace{\omega}_{\text{Uncertainty}}) \geq 0, \quad \forall x \in \chi \qquad \forall \omega \in \underbrace{\Omega}_{\text{Uncertainty Set}}$$

Nonnegative Polynomials

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Robust Nonnegative Polynomials

$$p(x, \underset{\text{Uncertainty}}{\underbrace{\omega}}) \geq 0, \quad \forall x \in \chi \quad \forall \omega \in \underset{\text{Uncertainty Set}}{\underbrace{\Omega}}$$

Application: **Stability Analysis of Uncertain Nonlinear Systems**

Given a dynamical system $\dot{x} = f(x, \omega) \qquad \forall \omega \in \Omega$

- **Stability in presence of uncertainties**
- To prove this, we need to find an **energy function** $V(x)$ with following properties

$V(x) = 0 \;\; on \;\; x = 0 \qquad V(x) > 0 \;\; on \;\; x \neq 0 \qquad -\dot{V}(x, \omega) = \frac{\partial V(x)}{\partial x}\dot{x} > 0 \;\; \forall x \;\; \forall \omega \in \Omega$ (Lyapunov function)

Robust Nonnegative Polynomial

$$p(x,\omega) \geq 0, \quad \forall x \in \chi \quad \forall \omega \in \Omega$$

Uncertainty

Uncertainty Set

SOS Relaxation:

Let $\chi = \{x \in \mathbb{R}^n : g_{x_i}(x) \geq 0, i = 1, \dots, n_x\}$ $\quad \Omega = \{\omega \in \mathbb{R}^m : g_{\omega_i}(\omega) \geq 0, i = 1, \dots, n_\omega\}$

$$p(x,\omega) = \sigma_0(x,\omega) + \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x_i}(x) + \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega_i}(\omega)$$

$\sigma_i(x,\omega) \in SOS_{2d_i}, \ i = 1, \dots, n_x$

$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i}, \ i = 1, \dots, n_\omega$

Robust Nonnegative Polynomial

$$p(\underset{\text{Uncertainty}}{\underset{\downarrow}{x,\omega}}) \geq 0, \quad \forall x \in \chi \qquad \forall \omega \in \underset{\text{Uncertainty Set}}{\underset{\downarrow}{\Omega}}$$

SOS Relaxation:

Let $\chi = \{x \in \mathbb{R}^n : g_{x\,i}(x) \geq 0, i = 1, \dots, n_x\}$ $\qquad \Omega = \{\omega \in \mathbb{R}^m : g_{\omega\,i}(\omega) \geq 0, i = 1, \dots, n_\omega\}$

$$p(x,\omega) = \sigma_0(x,\omega) + \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x\,i}(x) + \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega\,i}(\omega) \qquad \begin{aligned} &\sigma_i(x,\omega) \in SOS_{2d_i}, \ i = 1, \dots, n_x \\ &\bar{\sigma}_i(x,\omega) \in SOS_{2d_i}, \ i = 1, \dots, n_\omega \end{aligned}$$

Yalmip:

$\mathrm{SOS}(p(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x\,i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega\,i}(\omega))$

$\mathrm{SOS}(\sigma_i(x,\omega)) \quad i = 1, \dots, n_x$

$\mathrm{SOS}(\bar{\sigma}_i(x,\omega))\ , \quad i = 1, \dots, n_\omega$

Example: Stability Analysis of Uncertain Nonlinear Systems

Uncertain Nonlinear Dynamical System:

$$\dot{x}_1 = -\frac{3}{2}x_1^3 - \frac{1}{2}x_1^2 - x_2$$

$$\dot{x}_2 = 6x_1 - \omega x_2$$

- Uncertain Parameter: $\omega \in [3 \quad 5]$

$$\Downarrow$$

$$\Omega = \{\omega : (\omega - 3)(5 - \omega) \geq 0\}$$

Example: Stability Analysis of Uncertain Nonlinear Systems

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$\Downarrow$

$$\Omega = \{\omega : (\omega - 3)(5 - \omega) \geq 0\}$$

➢ We look for polynomial Lyapunov function $V(x) = \sum_{\alpha=0}^{2d} c_\alpha x^\alpha$

Stability Conditions:

$V(x) = 0 \quad on \quad x = 0$ $\rightarrow$ $c_0 = 0$

$\epsilon > 0$

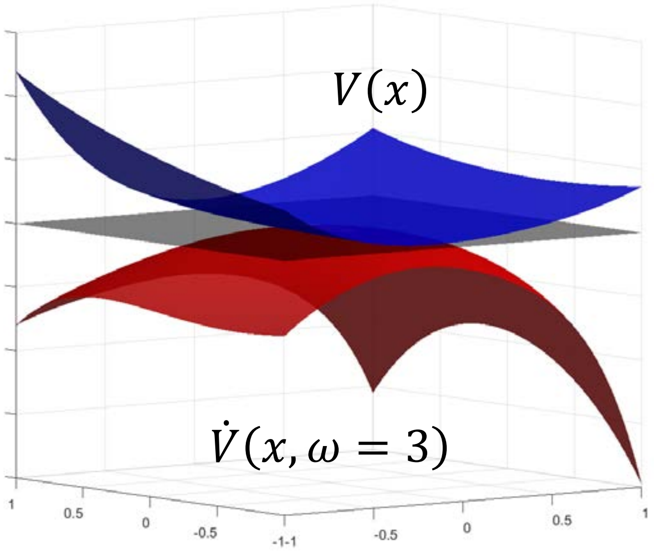
$V(x) > 0 \quad \forall x \neq 0$ $\rightarrow$ $V(x) \geq \epsilon \|x\|_2^2$ $\rightarrow$ $V(x) - \epsilon ||x||_2^2 \in SOS$

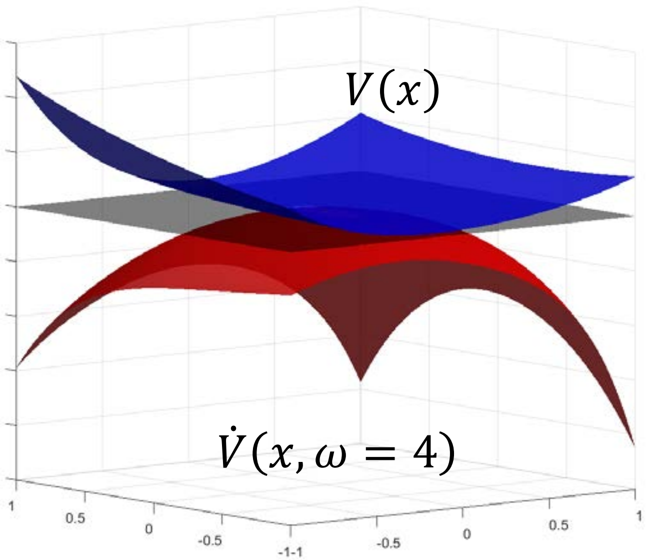
$-\dot{V}(x,\omega) > 0 \quad \forall x \neq 0 \quad \forall \omega \in \Omega$ $\rightarrow$ $-\dot{V}(x,\omega) \geq \epsilon \|x\|_2^2$ $\rightarrow$ $-\dot{V}(x) - \epsilon ||x||_2^2 - \sigma_1(x,\omega)(\omega - 3)(5 - \omega) \in SOS$

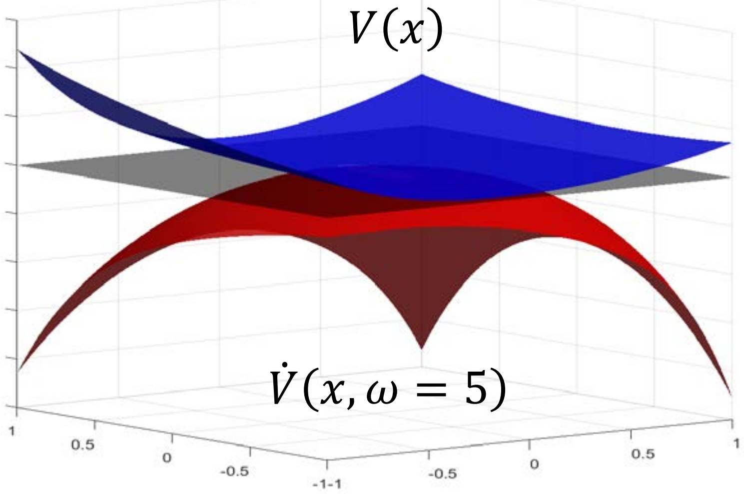
SOS Conditions:

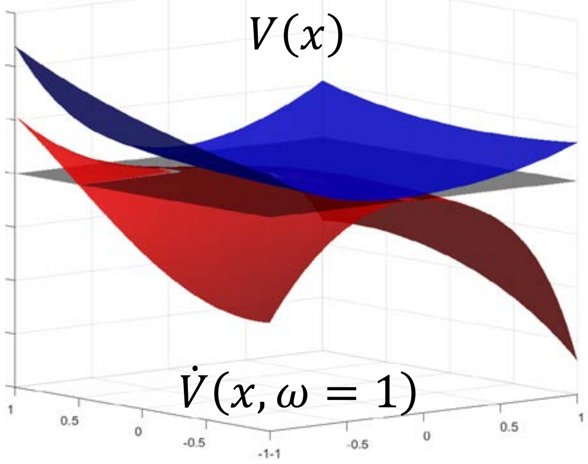
$$\dot{V}(x) = \frac{\partial V(x)}{\partial x_1}\dot{x}_1 + \frac{\partial V(x)}{\partial x_2}\dot{x}_2 = \frac{\partial V(x)}{\partial x_1}(-\frac{3}{2}x_1^3 - \frac{1}{2}x_1^2 - x_2) + \frac{\partial V(x)}{\partial x_2}(6x_1 - \omega x_2)$$

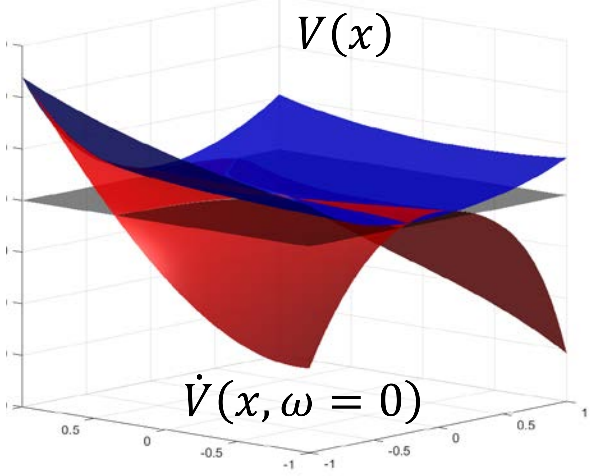
- Uncertain Nonlinear system is stable for Uncertain Parameter: $\omega \in [3 \quad 5]$











- For : $\omega \notin [3 \quad 5]$
  $V(x)$ does not satisfy the stability conditions.

https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_SOS/Example_1_Lyapunov.m

Based on : J. Lofberg "Modeling and solving uncertain optimization problems in YALMIP" 17th World Congress, The International Federation of Automatic Control, Seoul, Korea, July 6-11, 2008

https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_SOS/Example_2_Lyapunov.m

Example: Safety Verification

- Uncertain nonlinear dynamical system $\dot{x} = f(x, u, \omega)$
- Bounded Uncertainty $\omega \in \Omega$
- Unsafe Set $\mathcal{X}_{obs}$
- Initial state $x(0)$
- Policy $u(x)$

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➢ Given policy $u(x)$ is safe if there exist a function $V(x)$

$V(x(0)) = 0$ $\qquad$ $V(x) \geq 1 \quad \forall x \in \chi_{obs}$

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> Given policy $u(x)$ is safe if there exist a function $V(x)$ (Barrier function)
>
> $V(x(0)) = 0$ $\quad V(x) \geq 1 \ \ \forall x \in \mathcal{X}_{obs}$ $\quad \dot{V}(x, \omega) = \frac{\partial V(x)}{\partial x} f(x, u(x), \omega) \leq 0 \ \forall x, \quad \forall \omega \in \Omega$

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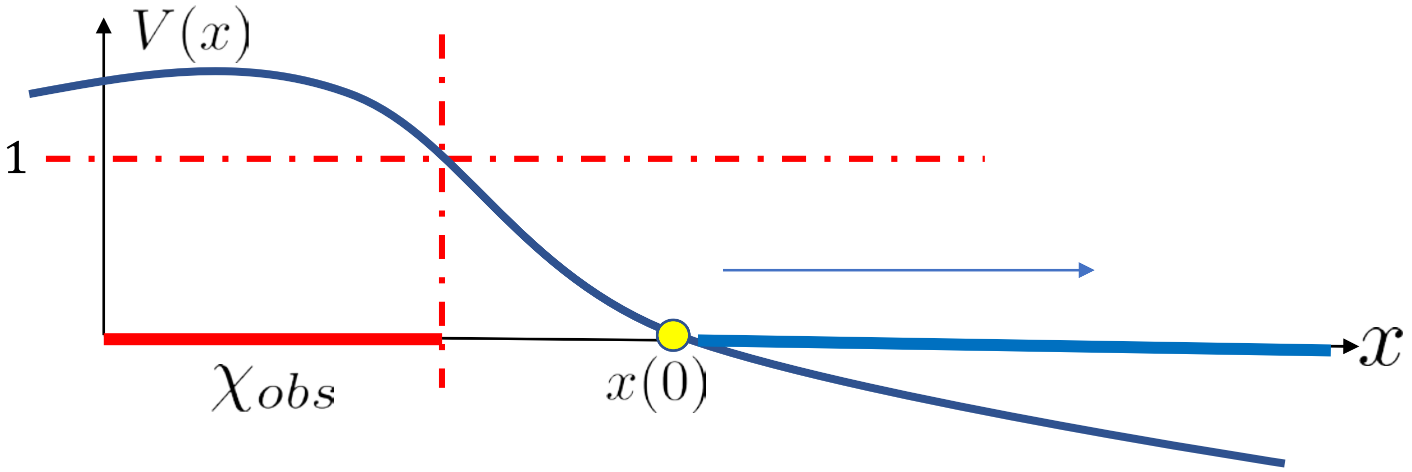


$u(x): \; \dot{V}(x,\omega) \leq 0 \; \forall x, \quad \forall \omega \in \Omega$

➢ x is constrained to evolve within the $\{x: V(x) \leq 0\}$

(0-level set of the function $V(x)$)

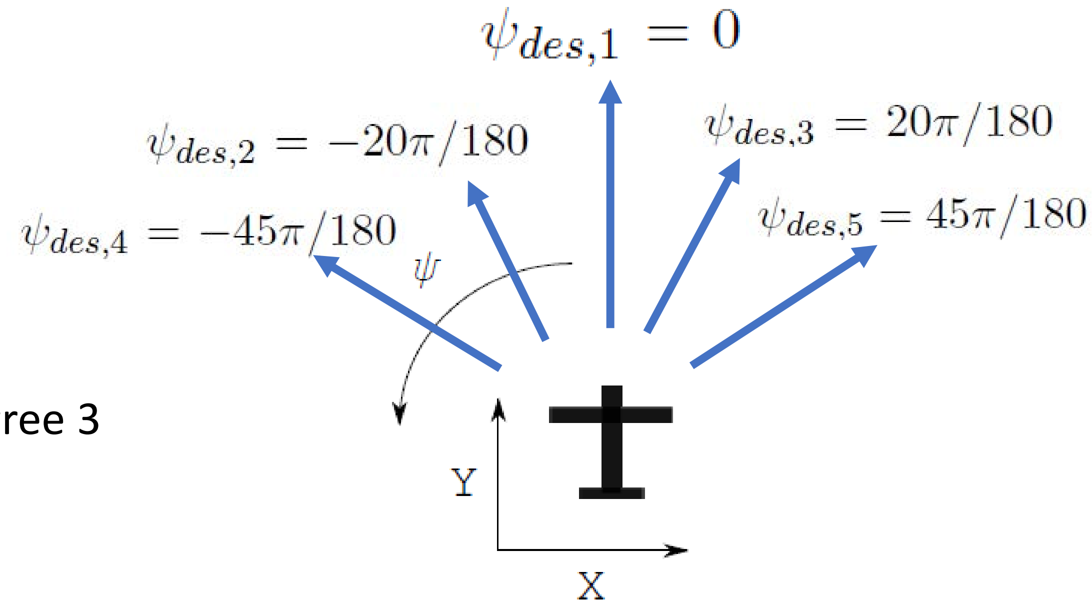
- Uncertain nonlinear dynamical system

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}, \qquad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -v \sin\psi + w \\ v\cos\psi \\ u \end{bmatrix}$$

- Wind Disturbance: $\omega \in [-0.05 \;\; 0.05]$

- Polynomial dynamics: Taylor expansion of nonlinear dynamics to degree 3

- Library of motion primitives



State-feedback control input for motion primitives:

$$u_i(\mathbf{x}) = -K(\psi - \psi_{des,i}), \;\; i = 1, \ldots, 5.$$

https://www.quantamagazine.org/a-classical-math-problem-gets-pulled-into-the-modern-world-20180523/

A. Ahmadi, A. Majumdary, “Some applications of polynomial optimization in operations research and real-time decision making”, Optimization Letters, Volume 10, Issue 4, pp 709–729, 2016.

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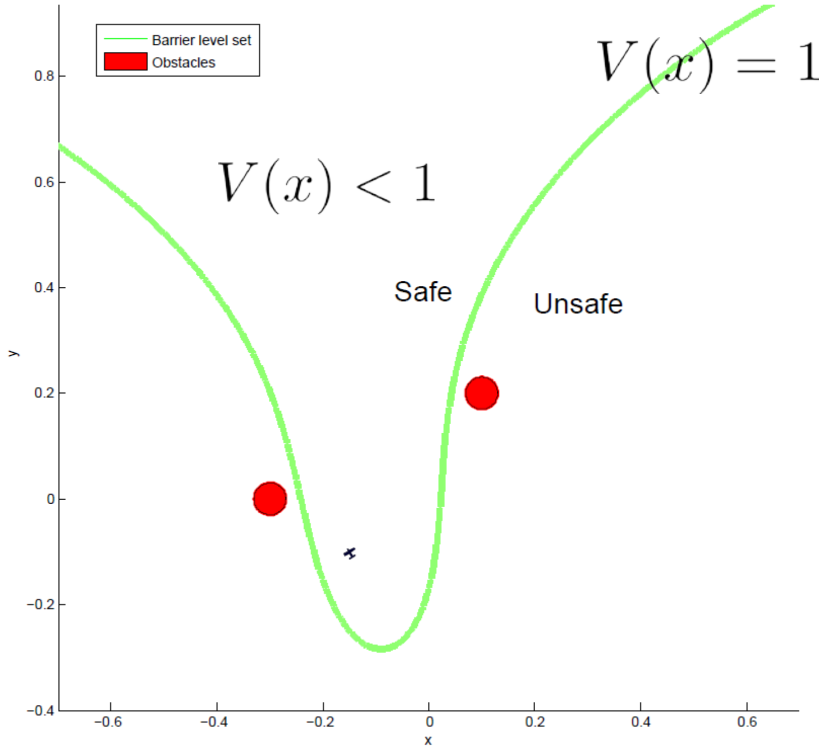

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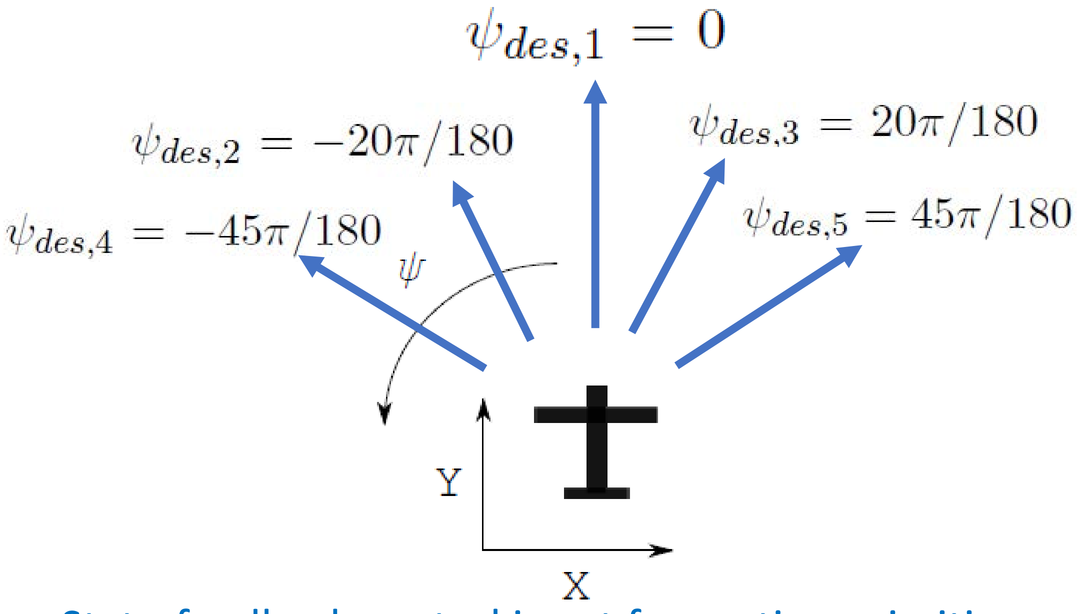

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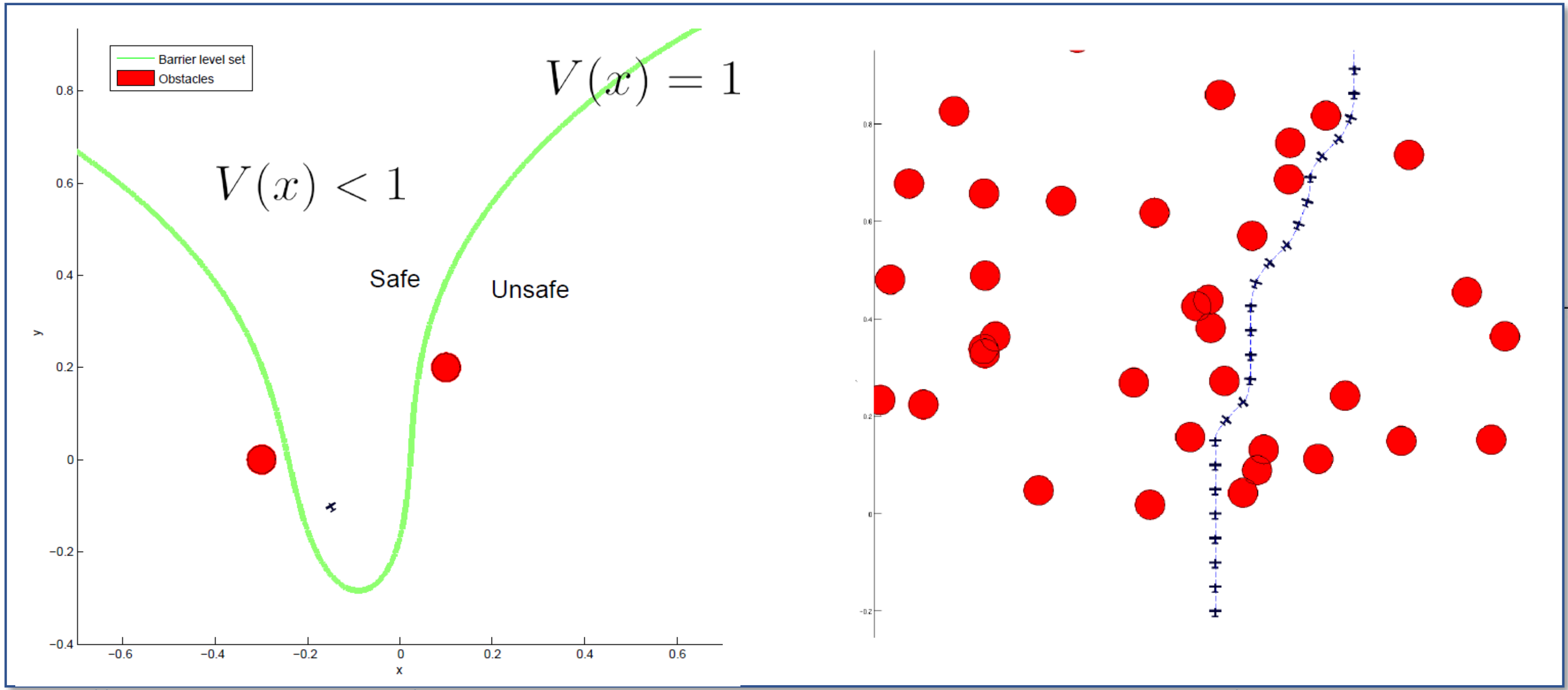

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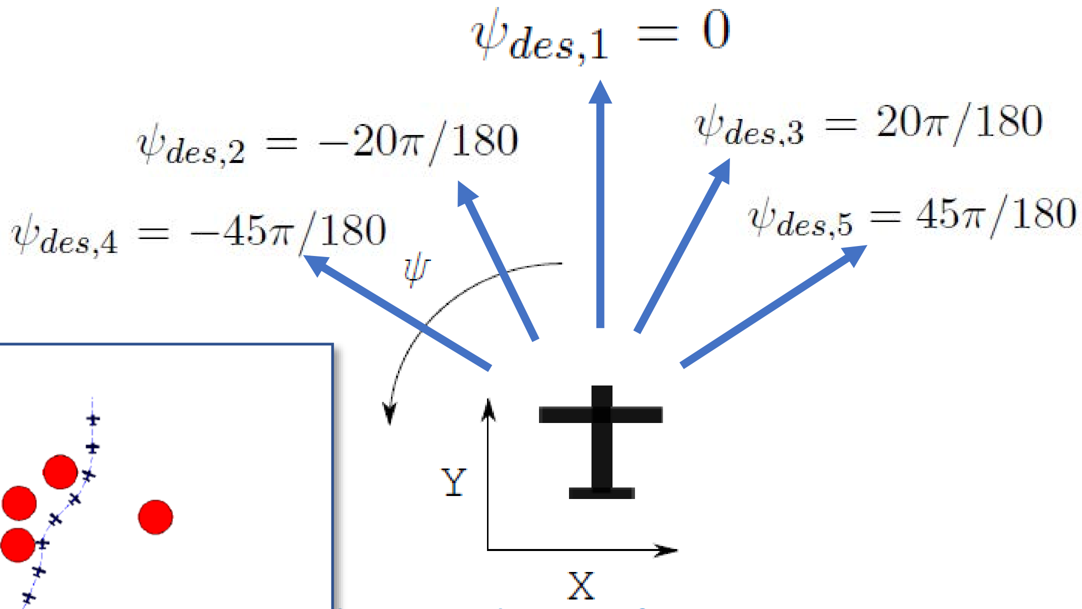

k control input for motion primitives:

$-K(\psi - \psi_{des,i}), \;\; i = 1, \ldots, 5.$

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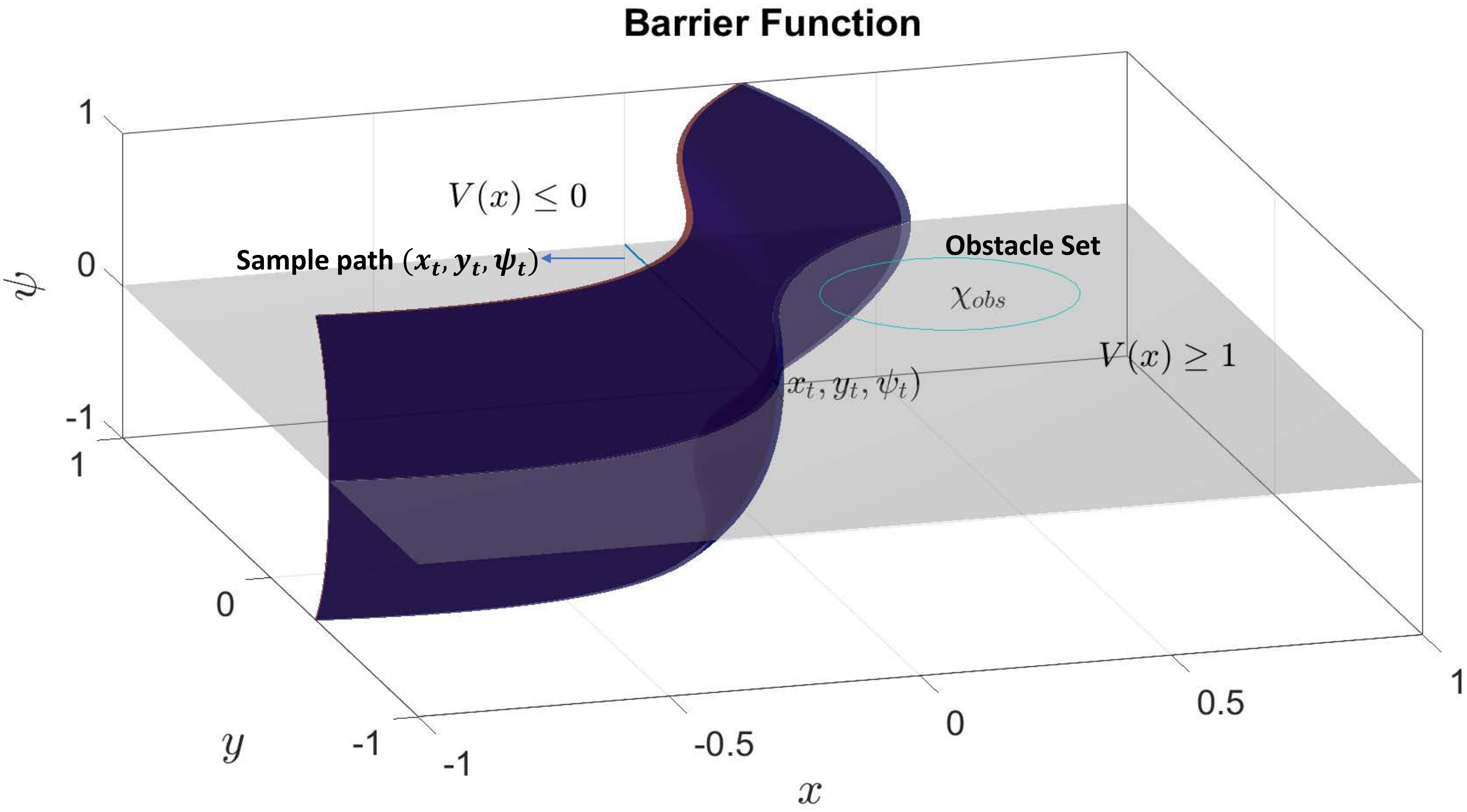

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Robust Optimization

$$\begin{aligned} \underset{x \in \chi}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g(x,\omega) \leq 0, \quad \forall \omega \in \Omega \end{aligned}$$

- **Design variables:** $x \in \chi \subset \mathbb{R}^n$
- **Uncertain parameters:** $\omega \in \Omega \subset \mathbb{R}^m$

Robust Optimization

$$\begin{aligned} \underset{x \in \chi}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g(x, \omega) \leq 0, \quad \forall \omega \in \Omega \end{aligned}$$

- **Design variables:** $x \in \chi \subset \mathbb{R}^n$
- **Uncertain parameters:** $\omega \in \Omega \subset \mathbb{R}^m$

➢ Design variables "x" should satisfy "safety/design constraints" for all possible values of uncertainty "ω".

$$g(x, \omega) \leq 0 \quad \forall \omega \in \Omega$$

Robust Optimization

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$$g(x, \omega) \leq 0 \quad \forall \omega \in \Omega$$

Robust design variable set:

Set of all design variable "x" that satisfies "safety/design constraints" for **all possible** values of uncertainty "ω".

$$\chi_R = \{x \in \chi : g(x, \omega) \leq 0, \ \forall \omega \in \Omega\}$$

Robust Optimization

$$\begin{aligned} &\underset{x \in \chi}{\text{minimize}} && p(x) \\ &\text{subject to} && g(x, \omega) \leq 0, \quad \forall \omega \in \Omega \end{aligned}$$

- **Design variables:** $x \in \chi \subset \mathbb{R}^n$
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Robust design variable set:
Set of all design variable "x" that satisfies "safety/design constraints" for **all possible** values of uncertainty "ω".

$$\chi_R = \{x \in \chi : g(x, \omega) \leq 0, \ \forall \omega \in \Omega\} \quad \Longrightarrow \quad \chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$$

Polynomial of order d

Robust Optimization

$$\begin{aligned} \underset{x\in\chi}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g(x,\omega) \le 0, \quad \forall \omega \in \Omega \end{aligned}$$

- **Design variables:** $x \in \chi \subset \mathbb{R}^n$
- **Uncertain parameters:** $\omega \in \Omega \subset \mathbb{R}^m$

➢ Design variables "x" should satisfy "safety/design constraints" for all possible values of uncertainty "ω".

$$g(x,\omega) \le 0 \;\; \forall \omega \in \Omega$$

Robust design variable set:

Set of all design variable "x" that satisfies "safety/design constraints" for **all possible** values of uncertainty "ω".

$$\chi_R = \{x \in \chi : g(x,\omega) \le 0, \; \forall \omega \in \Omega\} \quad \Longrightarrow \quad \chi_R^d = \{x \in \chi : \underbrace{p_d}_{\text{Polynomial of order } d}(x) \le 0\}$$

Deterministic Optimization

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & x \in \chi_R^d \end{aligned}$$

Robust Set:

$$\chi_R = \{x \in \chi : g(x, \omega) \leq 0, \ \forall \omega \in \Omega\}$$

Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\}$$

$$\Downarrow$$

$$\chi_R = \{x \in \chi : \max_{\omega \in \Omega} g(x,\omega) \leq 0\}$$

Worst Case Uncertainty: $$g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega)$$

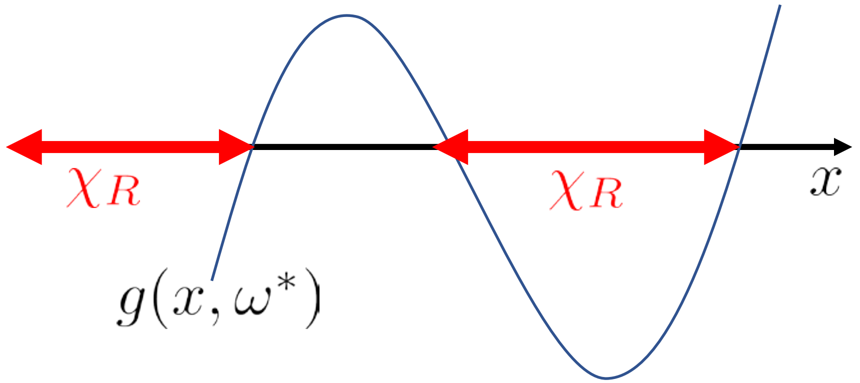
Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\}$$

$$\Downarrow$$

$$\chi_R = \{x \in \chi : \max_{\omega \in \Omega} g(x,\omega) \leq 0\}$$

Worst Case Uncertainty: $$g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega)$$



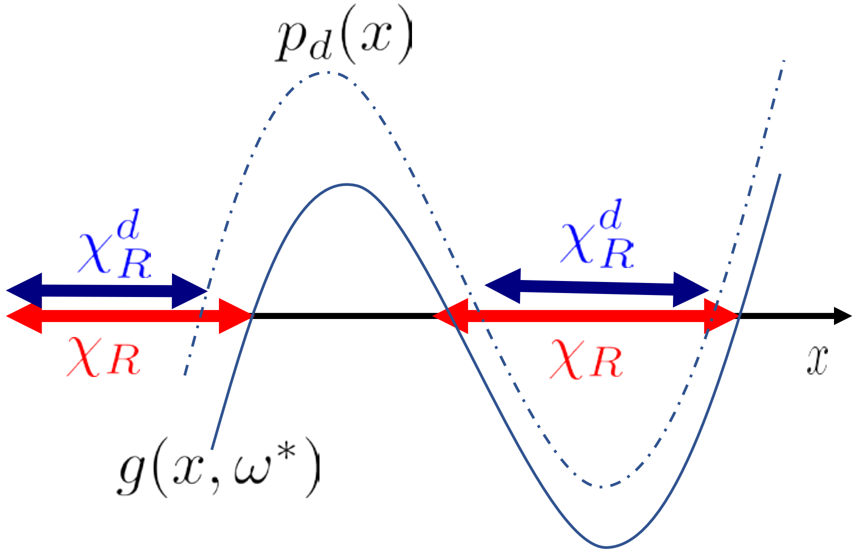
Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\}$$

$$\chi_R = \{x \in \chi : \max_{\omega \in \Omega} g(x,\omega) \leq 0\}$$

Worst Case Uncertainty: $g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega)$

Polynomial $p_d(x) \Rightarrow p_d(x) \geq g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega)$



Inner Approximation of Robust Set

$$\chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$$

Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\}$$

$$\Downarrow$$

$$\chi_R = \{x \in \chi : \max_{\omega \in \Omega} g(x,\omega) \leq 0\}$$

Inner Approximation of Robust Set $\chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$

$$p_d(x) \geq g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega)$$ (Worst case uncertainty)

Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\}$$

$$\chi_R = \{x \in \chi : \max_{\omega \in \Omega} g(x,\omega) \leq 0\}$$

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$$p_d(x) \geq g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega)$$ (Worst case uncertainty)

$$p_d(x) \geq g(x,\omega) \quad \forall x \in \chi,\ \forall \omega \in \Omega$$

Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \le 0,\ \forall \omega \in \Omega\}$$

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Inner Approximation of Robust Set $\chi_R^d = \{x \in \chi : p_d(x) \le 0\}$

$$p_d(x) \ge g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega)$$ (Worst case uncertainty)

$$p_d(x) \ge g(x,\omega) \quad \forall x \in \chi,\ \forall \omega \in \Omega$$

SOS Relaxation

$$p_d(x) - g(x,\omega) \ge 0 \quad \forall x \in \chi,\ \forall \omega \in \Omega$$

- Look for polynomial $p_d(x)$ such that polynomial $p_d(x) - g(x,\omega)$ be a nonnegative polynomials on the sets χ and Ω

SOS Program

Let $\chi = \{x \in \mathbb{R}^n : g_{x\,i}(x) \geq 0, i = 1, ..., n_x\}$ $\qquad \Omega = \{\omega \in \mathbb{R}^m : g_{\omega\,i}(\omega) \geq 0, i = 1, ..., n_\omega\}$

- Sets χ, Ω are Archimedean.

$$\underset{p_d(x)}{\text{minimize}} \quad \int_{\mathbf{B}} p_d(x)dx$$

$$\text{subject to} \quad p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x\,i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega\,i}(\omega) = \sigma_0(x,\omega)$$

$$\sigma_0(x,\omega) \in SOS_{2d}$$

$$\sigma_i(x,\omega) \in SOS_{2d_i}, \ i = 1, ..., n_x$$

$$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i}, \ i = 1, ..., n_\omega$$

$\chi \subset \mathbf{B}$: simple box

Assumption: After rescaling polynomials $\mathbf{B} = [-1, 1]^n$

- Section 3.3, Jean B. Lasserre, , “Tractable approximations of sets defined with quantifiers” Math. Program., Ser. B, 151:507–527, 2015.

SOS Program

Let $\chi = \{x \in \mathbb{R}^n : g_{x_i}(x) \geq 0, i = 1, ..., n_x\}$ $\quad \Omega = \{\omega \in \mathbb{R}^m : g_{\omega_i}(\omega) \geq 0, i = 1, ..., n_\omega\}$

- Sets χ, Ω are Archimedean.

$$\underset{p_d(x)}{\text{minimize}} \quad \int_{\mathbf{B}} p_d(x)dx$$

$$\text{subject to} \quad p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x_i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega_i}(\omega) = \sigma_0(x,\omega)$$

$$\sigma_0(x,\omega) \in SOS_{2d}$$

$$\sigma_i(x,\omega) \in SOS_{2d_i}, \ i = 1, ..., n_x$$

$$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i}, \ i = 1, ..., n_\omega$$

$$\Downarrow$$

$$p_d(x) - g(x,\omega) \geq 0 \quad \forall x \in \chi, \ \forall \omega \in \Omega$$

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SOS Program

Let $\chi = \{x \in \mathbb{R}^n : g_{x\,i}(x) \geq 0, i = 1, ..., n_x\}$ $\quad \Omega = \{\omega \in \mathbb{R}^m : g_{\omega\,i}(\omega) \geq 0, i = 1, ..., n_\omega\}$

- Sets χ, Ω are Archimedean.

$$\underset{p_d(x)}{\text{minimize}} \quad \int_{\mathbf{B}} p_d(x)dx$$

$$\text{subject to} \quad \boxed{\begin{aligned} & p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x\,i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega\,i}(\omega) = \sigma_0(x,\omega) \\ & \sigma_0(x,\omega) \in SOS_{2d} \\ & \sigma_i(x,\omega) \in SOS_{2d_i}, \ i = 1, ..., n_x \\ & \bar{\sigma}_i(x,\omega) \in SOS_{2d_i}, \ i = 1, ..., n_\omega \end{aligned}}$$

$$p_d(x) - g(x,\omega) \geq 0 \quad \forall x \in \chi, \ \forall \omega \in \Omega$$

Yalmip: $\mathrm{SOS}(p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x\,i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega\,i}(\omega))$

$\mathrm{SOS}(\sigma_i(x,\omega)) \quad i = 1, ..., n_x \quad \mathrm{SOS}(\bar{\sigma}_i(x,\omega)) \ , \quad i = 1, ..., n_\omega$

- Section 3.3, Jean B. Lasserre, , “Tractable approximations of sets defined with quantifiers” Math. Program., Ser. B, 151:507–527, 2015.

Robust Set

Inner Approximation of Robust Set

$$\chi_R^d = \{x \in \chi : p_d(x) \leq 0\} \quad \subset \chi_R = \{x \in \chi : g(x,\omega) \leq 0, \ \forall \omega \in \Omega\}$$

Theorem: Let polynomial $p_d(x)$ be the solution of SOS program $\left(p_d(x) \geq g(x,\omega) \ \ \forall x \in \chi, \ \forall \omega \in \Omega \right)$

$$d \to \infty \ : \ \chi_R^d \to \chi_R$$

Because: $\mathrm{Lim}_{d\to\infty} p_d(x) \to g(x,\omega^*) = \max_{\omega\in\Omega} g(x,\omega)$

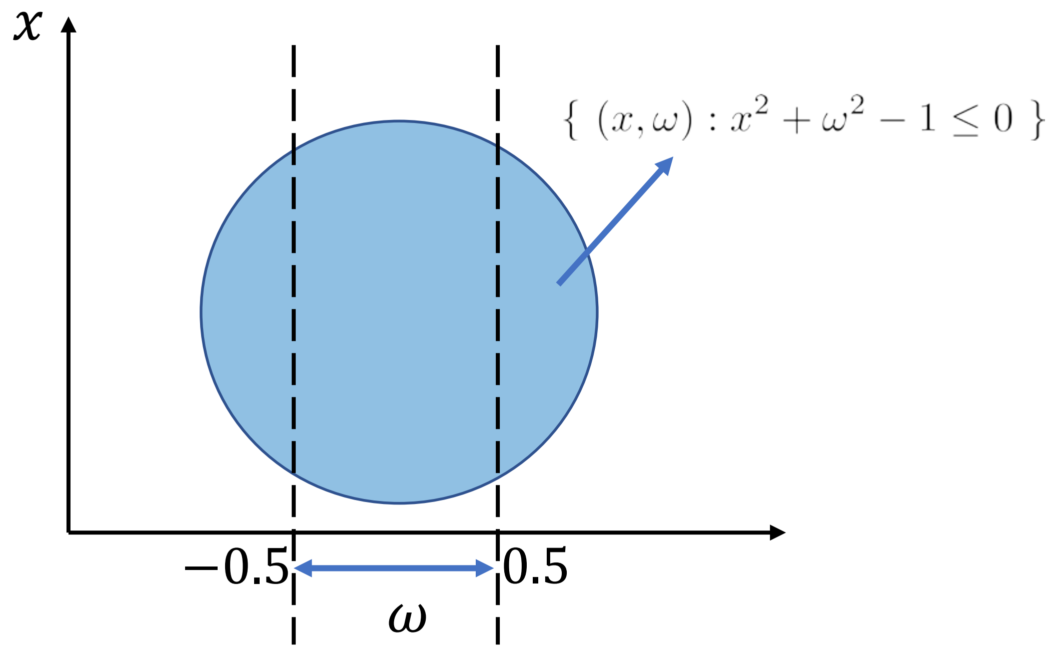
$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0, \ \forall \omega \in \Omega\}$$
$$= \{x \in \chi : \max_{\omega\in\Omega} g(x,\omega) \leq 0\}$$

- $\boldsymbol{g(x,\omega^*)}$ **is upper semi-continuous**, Lemma 1, Jean B. Lasserre, , “Tractable approximations of sets defined with quantifiers” Math. Program., Ser. B, 151:507–527, 2015.
- $\boldsymbol{\lim_{d\to\infty} \int_{\mathbf{B}} |p_d(x) - g(x,\omega^*)| dx = 0}$, Theorem 1 , Theorem 3 , Corollary 2, Jean B. Lasserre, , “Tractable approximations of sets defined with quantifiers” Math. Program., Ser. B, 151:507–527, 2015.

Example

Robust Safety Set

$$\chi_R = \{x \in \chi = [-2 \;\; 2] : \;\; x^2 + \omega^2 - 1 \leq 0, \;\; \forall \omega \in \Omega = [-0.5 \;\; 0.5] \;\}$$

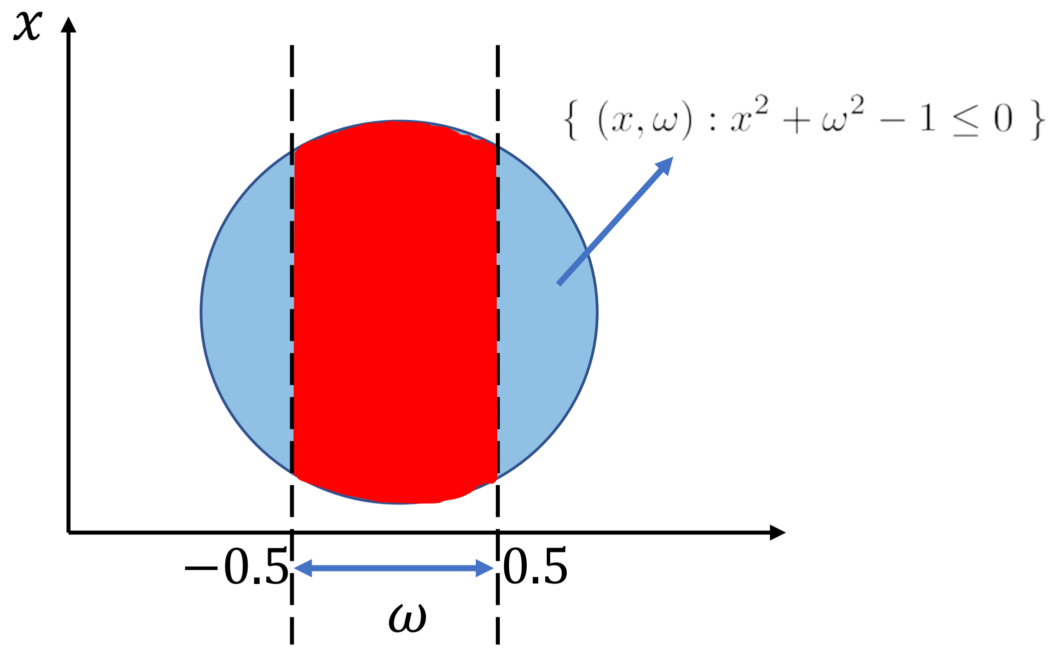


Example

Robust Safety Set

$$\chi_R = \{x \in \chi = [-2 \;\; 2] : \;\; x^2 + \omega^2 - 1 \leq 0, \;\forall \omega \in \Omega = [-0.5 \;\; 0.5] \;\}$$

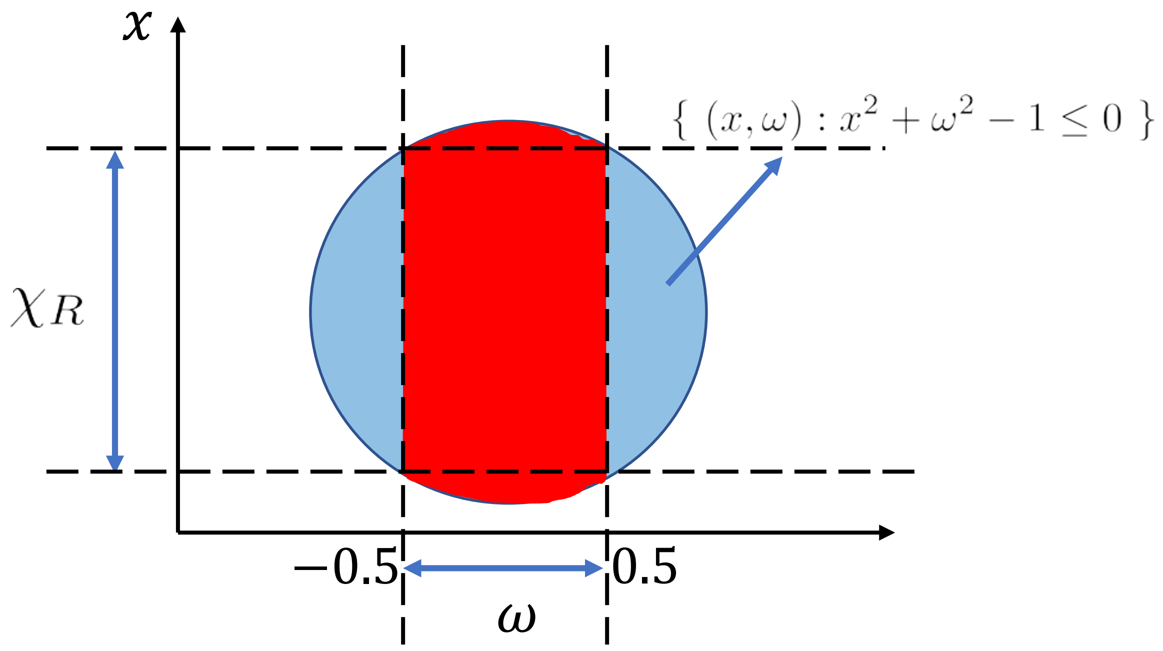
χ_R : Set of all x for which the point (x, ω) lies inside the red region $\forall \omega \in \Omega$



Example

Robust Safety Set

$$\chi_R = \{x \in \chi = [-2 \ \ 2] : \ \ x^2 + \omega^2 - 1 \leq 0, \ \forall \omega \in \Omega = [-0.5 \ \ 0.5] \ \}$$



Example

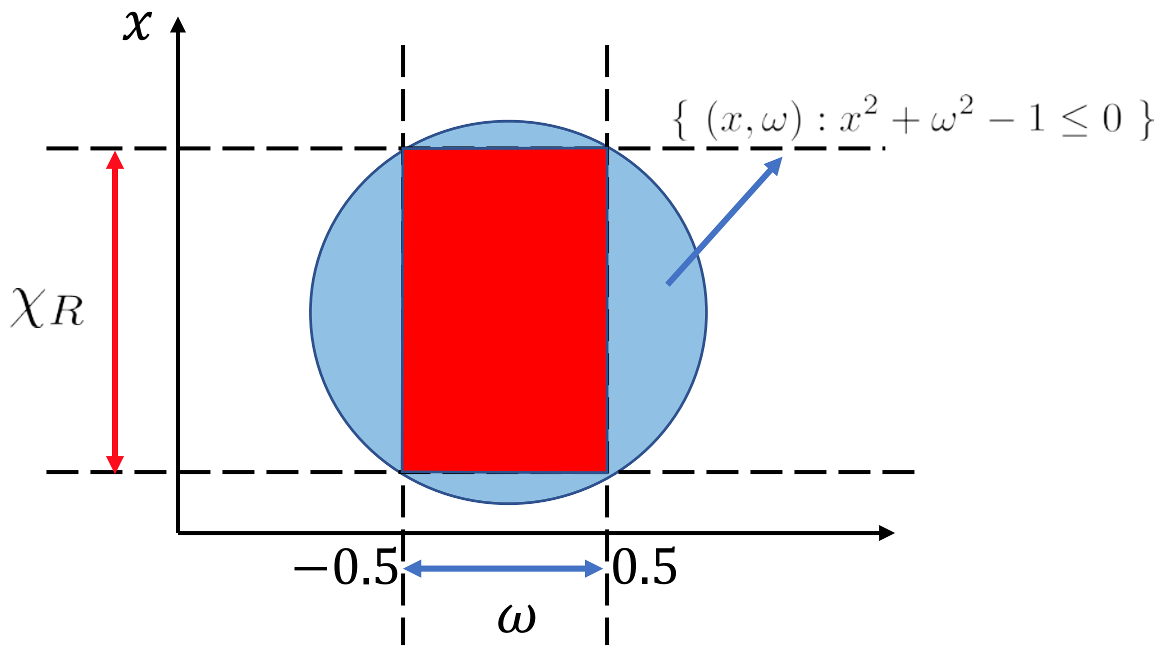
Robust Safety Set

$$\chi_R = \{x \in \chi = [-2 \;\; 2] : \;\; x^2 + \omega^2 - 1 \leq 0, \;\; \forall \omega \in \Omega = [-0.5 \;\; 0.5] \;\}$$

$$\chi_R = \{x : x^2 \leq 1 - \omega^2\} \quad -0.5 \leq \omega \leq 0.5$$

$$\chi_R = \{x : -\sqrt{1 - 0.5^2} \leq x \leq \sqrt{1 - 0.5^2}\}$$

$$\chi_R = \{x : -0.866 \leq x \leq 0.866\}$$



Example

Robust Safety Set

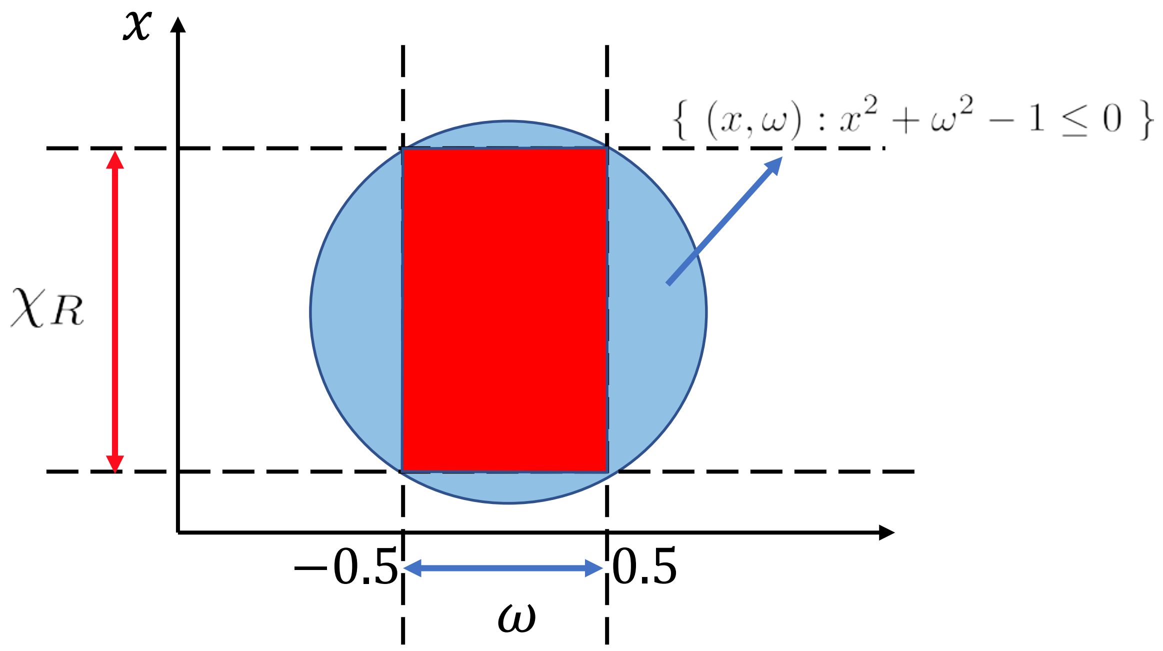
$$\chi_R = \{x \in \chi = [-2 \;\; 2] : \;\; x^2 + \omega^2 - 1 \leq 0, \;\forall \omega \in \Omega = [-0.5 \;\; 0.5] \;\}$$

$$g(x, \omega^*) = \max_{\omega \in \Omega} g(x, \omega)$$

$$= \max_{\omega \in \Omega = [-0.5 \;\; 0.5]} \;\; x^2 + \omega^2 - 1$$

$\omega^* = -0.5, 0.5$ (Worst case uncertainty)

$g(x, \omega^*) = x^2 + 0.5^2 - 1$ (Worst case maximum)



Example

Robust Safety Set

$$\chi_R = \{x \in \chi = [-2 \;\; 2] : \;\; x^2 + \omega^2 - 1 \leq 0, \;\; \forall \omega \in \Omega = [-0.5 \;\; 0.5] \;\}$$

$$g(x, \omega^*) = \max_{\omega \in \Omega} g(x, \omega)$$

$$= \max_{\omega \in \Omega = [-0.5 \;\; 0.5]} \quad x^2 + \omega^2 - 1$$

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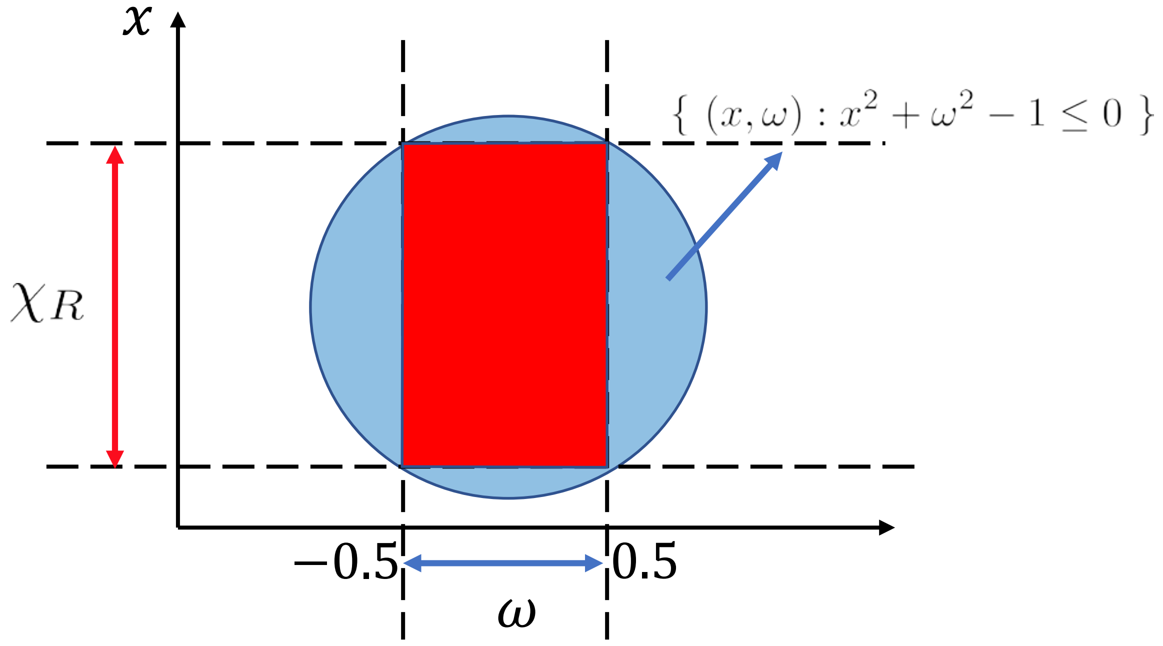


$$\chi_R^d = \{x \in \chi : p_d(x) \leq 0\} \qquad p_d(x) \geq g(x, \omega^*) = \max_{\omega \in \Omega} g(x, \omega)$$

Example

Robust Safety Set

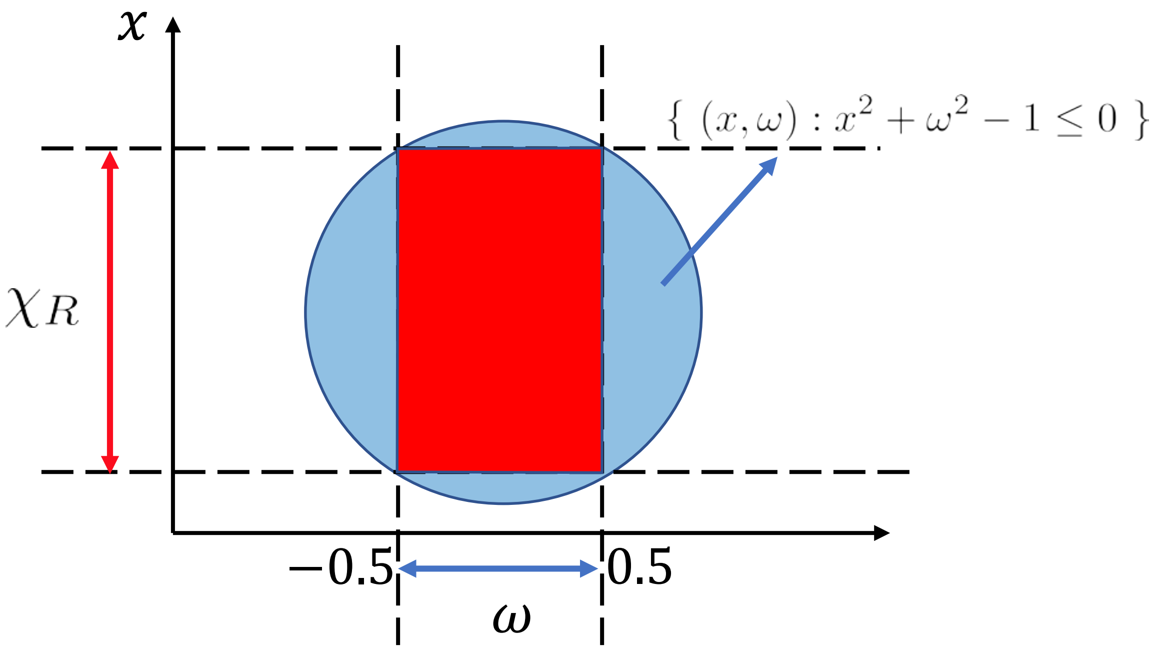
$$\chi_R = \{x \in \chi = [-2 \;\; 2] : \;\; x^2 + \omega^2 - 1 \leq 0, \;\; \forall \omega \in \Omega = [-0.5 \;\; 0.5] \;\}$$

$$g(x, \omega^*) = \max_{\omega \in \Omega} g(x, \omega)$$

$$= \max_{\omega \in \Omega = [-0.5 \;\; 0.5]} \quad x^2 + \omega^2 - 1$$

$\omega^* = -0.5, 0.5$ (Worst case uncertainty)

$g(x, \omega^*) = x^2 + 0.5^2 - 1$ (Worst case maximum)

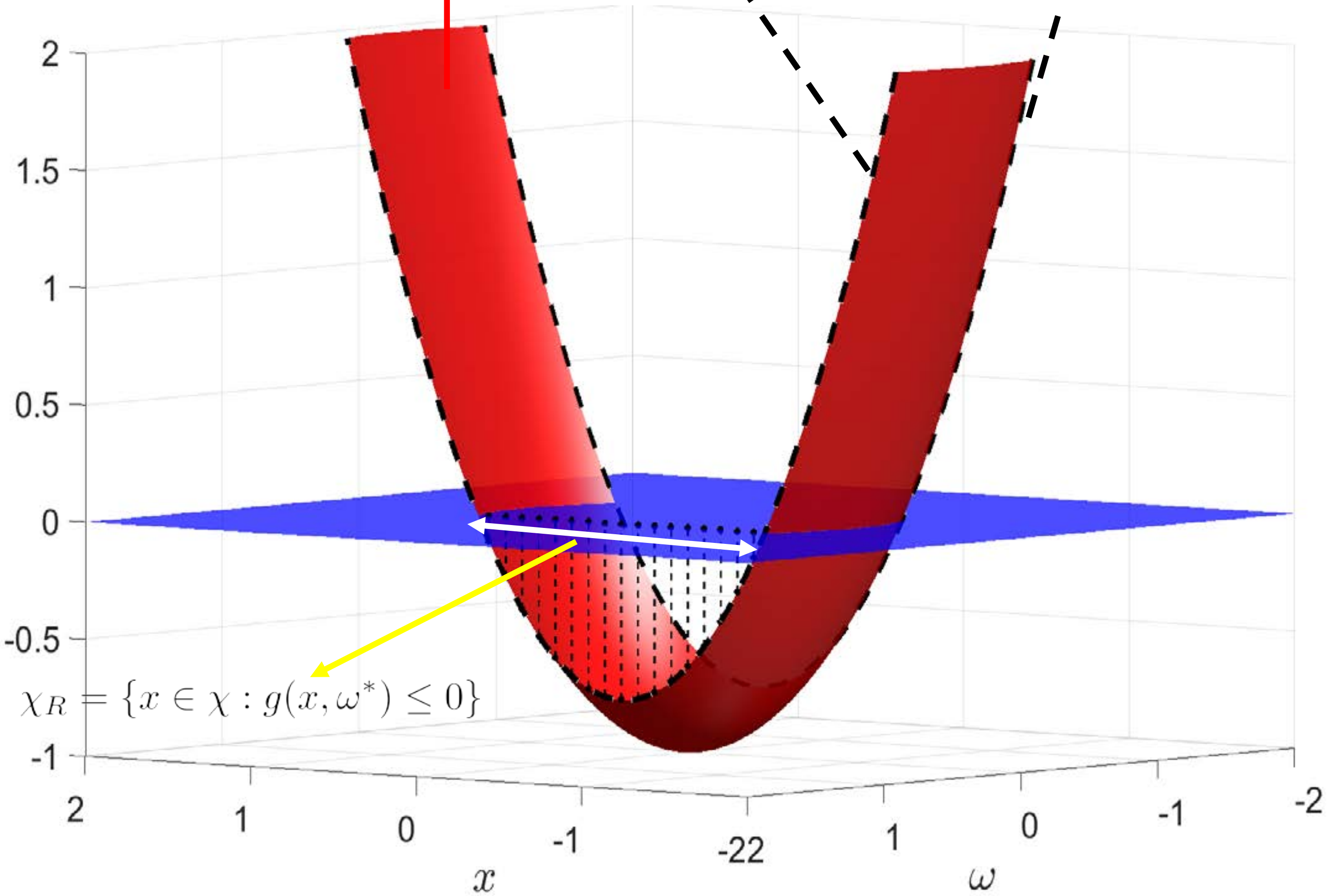


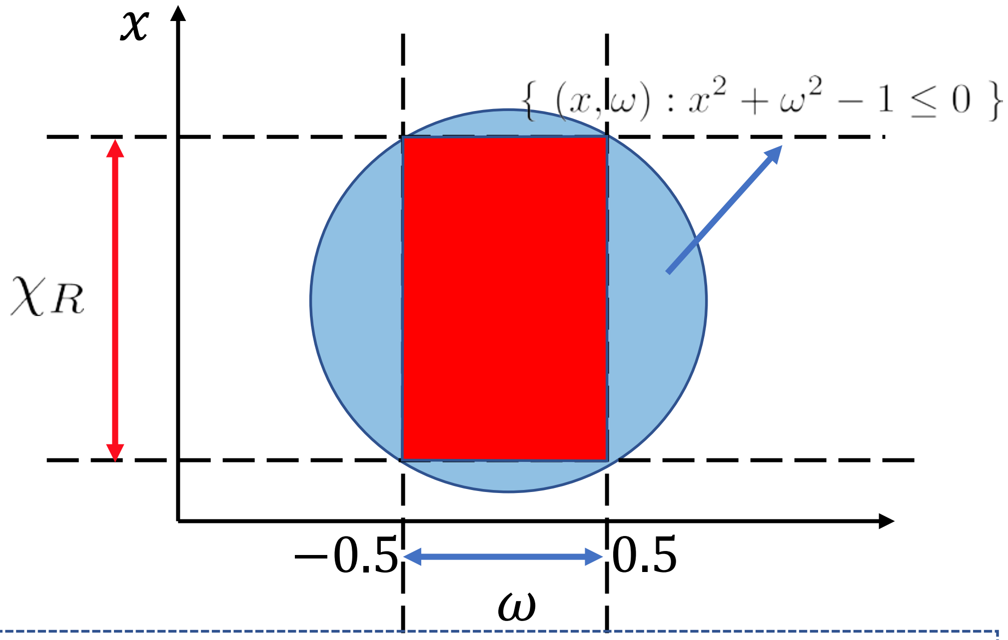
$\chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$ $\quad p_d(x) \geq g(x, \omega^*) = \max_{\omega \in \Omega} g(x, \omega)$ $\xrightarrow{\text{Replaced with}}$ $p_d(x) \geq g(x, \omega) \;\; \forall x \in \chi, \;\; \forall \omega \in \Omega$

$$g(x,\omega) = x^2 + \omega^2 - 1$$

$\omega \in \Omega = [-0.5\ \ 0.5]$

$x \in \chi = [-2\ \ 2]$





$$g(x,\omega^*) = \max_{\omega\in\Omega} g(x,\omega)$$

$$= \max_{\omega\in\Omega=[-0.5\ \ 0.5]} x^2 + \omega^2 - 1$$

$\omega^* = -0.5, 0.5$ (Worst case uncertainty)

$g(x,\omega^*) = x^2 + 0.5^2 - 1$ (Worst case maximum) https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_Set/Example_1_RobustSet.m

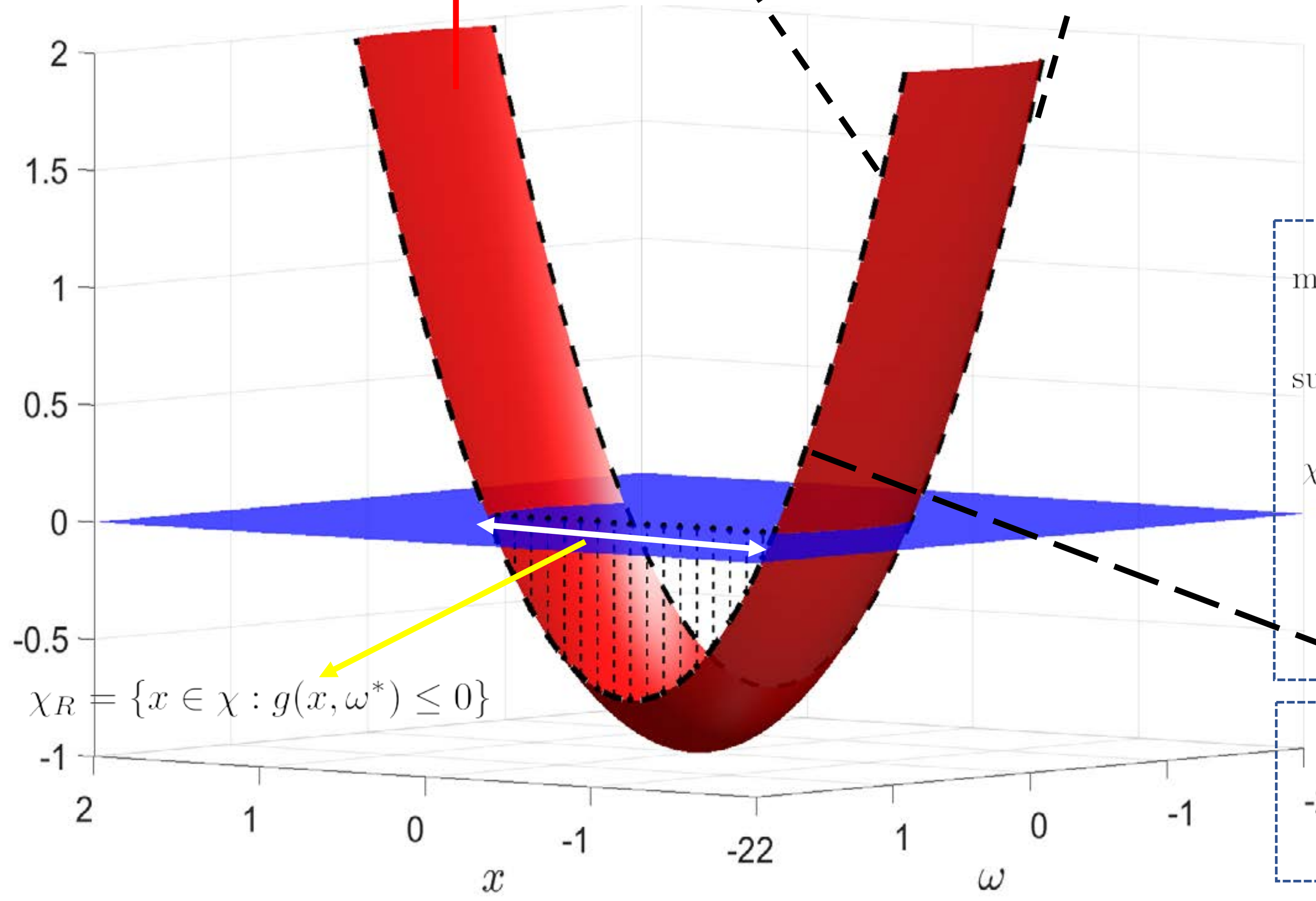
$$g(x,\omega) = x^2 + \omega^2 - 1$$

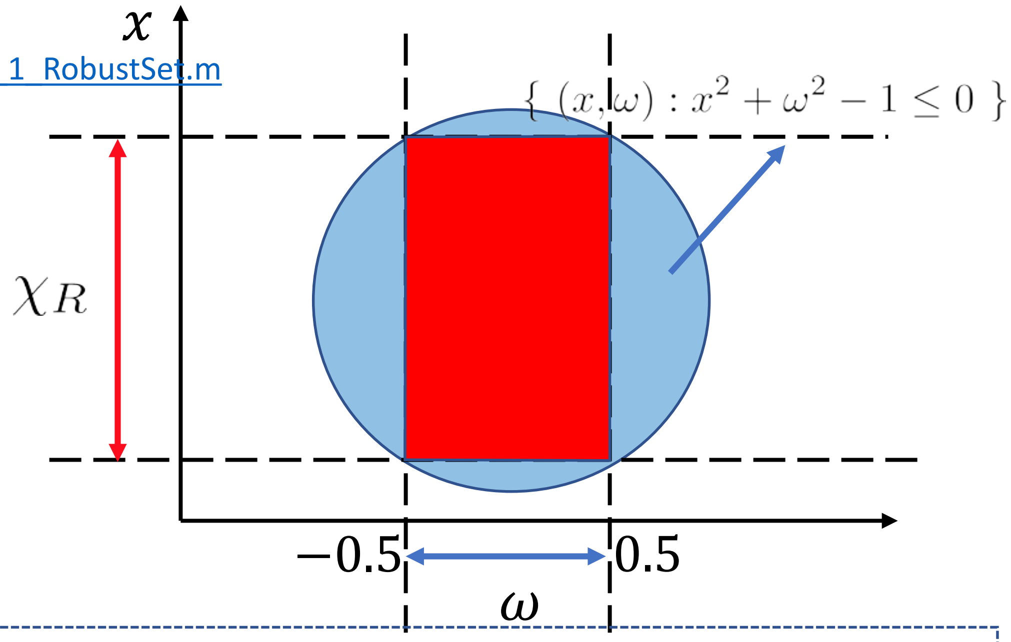
$\omega \in \Omega = [-0.5\ 0.5]$

$x \in \chi = [-2\ 2]$

$g(x,\omega^* = 0.5) = x^2 + 0.5^2 - 1$

$g(x,\omega^* = -0.5) = x^2 + (-0.5)^2 - 1$





$$\begin{aligned} \underset{p_d(x)}{\text{minimize}} \quad & \int_{\mathbf{B}} p_d(x)dx \\ \text{subject to} \quad & p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x_i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega_i}(\omega) = \sigma_0(x,\omega) \end{aligned}$$

$\chi = \mathbf{B} : [-2,\ 2]$

$\sigma_0(x,\omega) \in SOS_{2d}$

$\sigma_i(x,\omega) \in SOS_{2d_i},\ i = 1, ..., n_x$

$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i},\ i = 1, ..., n_\omega$

$$p_{d=1}(x) = -0.75 - x^2 = g(x,\omega^*)$$

$$\chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$$

Summary

Robust Optimization

$$\begin{aligned} \underset{x \in \chi}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & g(x,\omega) \leq 0, \quad \forall \omega \in \Omega \end{aligned}$$

- **Design variables:** $x \in \chi \subset \mathbb{R}^n$
- **Uncertain parameters:** $\omega \in \Omega \subset \mathbb{R}^m$

- SOS Program

$$\begin{aligned} \underset{p_d(x)}{\text{minimize}} \quad & \int_{\mathbf{B}} p_d(x)dx \\ \text{subject to} \quad & p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x_i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega_i}(\omega) = \sigma_0(x,\omega) \end{aligned}$$

$$\sigma_i(x,\omega) \in SOS_{2d_i}, \ i = 1,...,n_x$$
$$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i}, \ i = 1,...,n_\omega$$
$$\sigma_0(x,\omega) \in SOS_{2d}$$

- Based on worst case scenario $\quad p_d(x) \geq g(x,\omega^*) = \max_{\omega \in \Omega} g(x,\omega) \quad$ ω^*: worst case uncertainty

Deterministic Optimization

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & x \in \chi_R^d = \{x \in \chi : p_d(x) \leq 0\} \subset \chi_R = \{x \in \chi : g(x,\omega) \leq 0, \ \forall \omega \in \Omega\} \end{aligned}$$

Extension 1: Coupled Uncertainty

Robust Set : $$\chi_R = \{x \in \chi : g(x,\omega) \le 0,\ \forall \omega \in \Omega\}$$

Let $\Omega = \{\omega \in \mathbb{R}^m : g_{\omega_i}(\omega) \ge 0, i = 1, \ldots, n_\omega\}$

$$\underset{p_d(x)}{\text{minimize}} \quad \int_{\mathbf{B}} p_d(x)dx$$

$$\text{subject to} \quad p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x_i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega_i}(\omega) = \sigma_0(x,\omega)$$

$$\sigma_0(x,\omega) \in SOS_{2d}$$
$$\sigma_i(x,\omega) \in SOS_{2d_i},\ i = 1, \ldots, n_x$$
$$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i},\ i = 1, \ldots, n_\omega$$

Inner Approximation of Robust Set $$\chi_R^d = \{x \in \chi : p_d(x) \le 0\}$$

Extension 1: Coupled Uncertainty

Robust Set : $\chi_R = \{x \in \chi : g(x,\omega) \le 0,\ \forall \omega \in \Omega\}$

Let $\Omega = \{\omega \in \mathbb{R}^m : g_{\omega i}(\omega) \ge 0, i = 1, ..., n_\omega\}$

$$\underset{p_d(x)}{\text{minimize}} \quad \int_{\mathbf{B}} p_d(x)dx$$

$$\text{subject to} \quad p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{xi}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega i}(\omega) = \sigma_0(x,\omega)$$

$\sigma_0(x,\omega) \in SOS_{2d}$

$\sigma_i(x,\omega) \in SOS_{2d_i},\ i = 1, ..., n_x$

$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i},\ i = 1, ..., n_\omega$

Inner Approximation of Robust Set $\chi_R^d = \{x \in \chi : p_d(x) \le 0\}$

- Uncertainty set depends on design variables x

$$\omega \in \mathbf{\Omega}_x = \{\omega \in \mathbb{R}^m : \ g_{\omega_i}(x,\omega) \ge 0,\ j = 1, ..., n_\omega\}$$

Extension 1: Coupled Uncertainty

Robust Set : $\chi_R = \{x \in \chi : g(x,\omega) \le 0,\ \forall \omega \in \Omega\}$

Let $\Omega = \{\omega \in \mathbb{R}^m : g_{\omega i}(\omega) \ge 0, i = 1, \ldots, n_\omega\}$

$$\underset{p_d(x)}{\text{minimize}} \quad \int_{\mathbf{B}} p_d(x)dx$$

$$\text{subject to} \quad p_d(x) - g(x,\omega) - \sum_{i=1}^{n_x} \sigma_i(x,\omega) g_{x i}(x) - \sum_{i=1}^{n_\omega} \bar{\sigma}_i(x,\omega) g_{\omega i}(\omega) = \sigma_0(x,\omega)$$

$\sigma_0(x,\omega) \in SOS_{2d}$

$\sigma_i(x,\omega) \in SOS_{2d_i},\ i = 1, \ldots, n_x$

$\bar{\sigma}_i(x,\omega) \in SOS_{2d_i},\ i = 1, \ldots, n_\omega$

Inner Approximation of Robust Set $\chi_R^d = \{x \in \chi : p_d(x) \le 0\}$

- Uncertainty set depends on design variables x

$$\omega \in \mathbf{\Omega}_x = \{\omega \in \mathbb{R}^m : \ g_{\omega_i}(x,\omega) \ge 0,\ \ j = 1, \ldots, n_\omega\}$$

- We can apply the same methodology, to obtain the inner approximation.

$$\chi_R^d = \{x \in \chi : p_d(x) \le 0\}$$

Example: Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

Source of uncertainties: Initial states $\left(x_1(0), x_2(0), x_3(0)\right) \in \mathrm{X}_0$

Uncertain Parameter $\omega(k) \in \Omega_k = [\underline{\omega_k}, \overline{\omega_k}]$

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- Source of uncertainty at time k:

Coupled uncertainty $(x_1(k), x_2(k), x_3(k), \omega(k)) \in \Omega_x = \{(x_1, x_2, x_3, \omega) = 0.1^2 - x_1^2 - x_2^2 - x_3^2 - \omega^2 \geq 0\}$

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- We want to find a set of control inputs at time k that steer states $(x_1(k+1), x_2(k+1), x_3(k+1))$ to the neighborhood of the given way-point $(0,0,0.9)$, i.e. a ball around the way-point $1^2 - \left(\frac{x_1-0}{0.2}\right)^2 - \left(\frac{x_2-0}{0.2}\right)^2 - \left(\frac{x_3-0.9}{0.4}\right)^3 \geq 0$, for all possible values of uncertainty $(x_1(k), x_2(k), x_3(k), \omega(k)) \in \Omega_x = \{(x_1, x_2, x_3, \omega) = 0.1^2 - x_1^2 - x_2^2 - x_3^2 - \omega^2 \geq 0\}$.

Example: Control of Uncertain Nonlinear System

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Source of uncertainties: Initial states $\big(x_1(0), x_2(0), x_3(0)\big) \in \mathrm{X}_0$

Uncertain Parameter $\omega(k) \in \Omega_k = [\underline{\omega_k}\ ,\overline{\omega_k}]$

- Source of uncertainty at time k:

  Coupled uncertainty $(x_1(k), x_2(k), x_3(k), \omega(k)) \in \Omega_x = \{(x_1, x_2, x_3, \omega) = 0.1^2 - x_1^2 - x_2^2 - x_3^2 - \omega^2 \geq 0\}$

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$$U_R = \left\{ u(k) :\ 1 - \left(\frac{x_1(k+1)}{0.2}\right)^2 - \left(\frac{x_2(k+1)}{0.2}\right)^2 - \left(\frac{x_3(k+1)}{0.4}\right)^2 \geq 0,\ \forall x_k, \omega_k \in \Omega_x \right\}$$

Example: Control of Uncertain Nonlinear System

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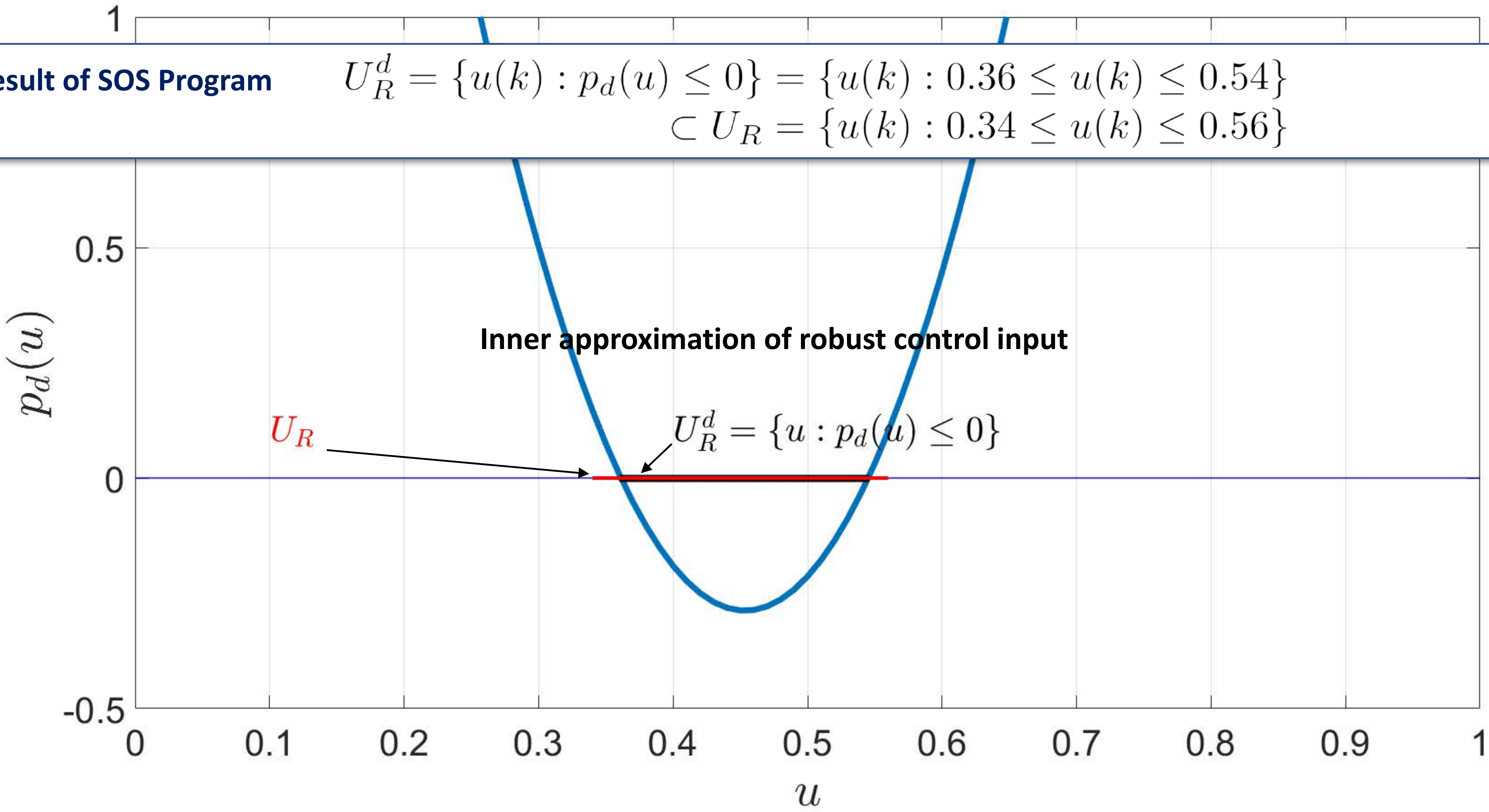
https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Robust_Set/Example_2_RobustSet_DynamicalSys.m

Example: Uncertain Nonlinear System

Result of SOS Program

$$U_R^d = \{u(k) : p_d(u) \leq 0\} = \{u(k) : 0.36 \leq u(k) \leq 0.54\}$$
$$\subset U_R = \{u(k) : 0.34 \leq u(k) \leq 0.56\}$$



- **Any control input that respects the obtained control bound, is robust in presence of uncertainties.**

Extension 2: Multiple Safety Constraints

$$\chi_R = \{x \in \chi : g_i(x,\omega) \le 0 \ i = 1, ..., n_g, \ \ \forall\omega \in \Omega\}$$

- Let $p_d^i(x)$ be a polynomial obtained by solving SOS program for single constraint $g_i(x,\omega)$

$$\chi_{R_i}^d = \{x \in \chi : p_d^i(x) \le 0\}$$

- Inner approximation of Robust set: $\cap_{i=1}^{n_g} \chi_{R_i}^d$

Extension 3: Failure Set

Robust Set:

Set of all design variable “x” that satisfies “safety/design constraints” for **all possible** values of uncertainty “ω”.

Robust Set: $$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\}$$

Failure Set:

Set of all design variable “x” that Does NOT satisfy “safety/design constraints” for **some** values of uncertainty “ω”.

Failure Set: $$\chi_F = \{x \in \chi : g(x,\omega) \geq 0,\ \exists \omega \in \Omega\}$$

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We can apply the same methodology to obtain the **outer approximation** of the failure set.

➢ $$\chi_F = \{x \in \chi : g(x,\omega) \geq 0,\ \exists \omega \in \Omega\} \ = \{x \in \chi : \max_{\omega \in \Omega} g(x,\omega) \geq 0\}$$

- Corollary 4, Jean B. Lasserre, , "Tractable approximations of sets defined with quantifiers" Math. Program., Ser. B, 151:507–527, 2015.

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- $\chi_F = \{x \in \chi : g(x,\omega) \geq 0, \ \exists \omega \in \Omega\} \ = \{x \in \chi : \max_{\omega \in \Omega} g(x,\omega) \geq 0\}$
- $\chi_F \subset \chi_F^d = \{x \in \chi : p_d(x) \geq 0\}$ where $p_d(x) \geq g(x,\omega) \ \ \forall x \in \chi, \ \forall \omega \in \Omega$

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- **SOS Program** $d \to \infty \ : \ \chi_F^d \to \chi_F$

- Corollary 4, Jean B. Lasserre, , "Tractable approximations of sets defined with quantifiers" Math. Program., Ser. B, 151:507–527, 2015.

Topics

- Formulation of Robust Optimization
- Challenges
- Robust Sum-of-Squares
- Robust Sets for Robust Optimization
- Robust Games
- Distributionally Robust Chance Constrained Optimization

Robust Games

- Based on worst case scenario
- Instead of looking for a set of all robust design values x, i.e., χ_R, we look for single optimal robust design value x^*
- 2-Player game formulation
- SOS Approach

Robust Games

Objective function: $p(x_1, x_2)$

Constraints: $g_i(x_1, x_2) \geq 0,\ i = 1, ..., n_g, \quad \forall y \in \Omega$

Two Sets of Variables: $x_1 \in \Omega \subset \mathbb{R}^m$ and $x_2 \in \mathbb{R}^n$

Two-Player Game: $\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \ \underset{x_2}{\text{minimize}} \ \{p(x_1, x_2) \ \text{ subject to } (x_1, x_2) \in \mathbf{K}\}$

$x_1 \in \chi_1 \qquad \mathbf{K} = \{(x_1, x_2) : g_i(x_1, x_2) \geq 0,\ i = 1, ..., n_g\}$

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$x_1 \in \chi_1 \qquad \mathbf{K} = \{(x_1, x_2) : g_i(x_1, x_2) \geq 0, \;\; i = 1, \ldots, n_g\}$

- In this game:
- Action set of player x_1: χ_1,
- Action set of player x_2 : $\{x_2 \colon g_i(x_1, x_2) \geq 0, i = 1, \ldots, n_g\}$ once player x_1 has chosen action.
- Player x_1 tries to maximize the objective function and Player x_2 tries to minimize the objective function.
- Player x_2 takes action after player x_1 chooses an action.

Example: Players x_1 and x_2 : ego and agent vehicles

Players x_1 and x_2 : design parameter and disturbance

Two-Player Game: $$\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \ \ \underset{x_2}{\text{minimize}} \ \{p(x_1, x_2) \ \ \text{subject to} \ (x_1, x_2) \in \mathbf{K}\}$$

$$x_1 \in \chi_1 \qquad \mathbf{K} = \{(x_1, x_2) : g_i(x_1, x_2) \geq 0, \ i = 1, ..., n_g\}$$

- From the point of view of player x_1, this is a robust optimization:
  - Player x_1 has to choose an action which is the best possible against the **worst action** by player x_2.

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- From the point of view of player x_1, this is a robust optimization:
  - Player x_1 has to choose an action which is the best possible against the **worst action** by player x_2.
  - If player x_1 knew the **optimal value function** of the player x_2

$$J(x_1) = \underset{x_2}{\text{minimize}} \; \{p(x_1, x_2) \;\; \text{subject to} \; (x_1, x_2) \in \mathbf{K}\}$$

$$x_2^*(x_1)$$

Then P_r reduces to optimization

$$\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \;\; J(x_1)$$

- In general there is no closed-form expression for optimal value function

$$J(x_1) = \underset{x_2}{\text{minimize}} \ \{p(x_1, x_2) \ \ \text{subject to} \ (x_1, x_2) \in \mathbf{K}\}$$ (parametric optimization)

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(parametric optimization)

- We find the polynomial approximation of (unknown) optimal value function $J(x_1)$

$$J_d(x_1) \leq J(x_1) = \underset{x_2}{\text{minimize}} \ \{p(x_1, x_2) \ \ \text{subject to} \ (x_1, x_2) \in \mathbf{K}\}$$

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- Similar to the methodology of the Robust set:

$$J_d(x_1) \leq p(x_1, x_2) \; \forall (x_1, x_2) \in \mathbf{K} \quad \Longrightarrow \quad J_d(x_1) \leq J(x_1) = \underset{x_1}{\text{minimize}} \; \{p(x_1, x_2) \;\; \text{subject to} \;\; (x_1, x_2) \in \mathbf{K}\}$$

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- Similar to the methodology of the Robust set:

$$J_d(x_1) \leq p(x_1, x_2) \ \forall (x_1, x_2) \in \mathbf{K} \quad \Rightarrow \quad J_d(x_1) \leq J(x_1) = \underset{x_1}{\text{minimize}} \ \{p(x_1, x_2) \ \ \text{subject to} \ (x_1, x_2) \in \mathbf{K}\}$$

- Look for polynomial $J_d(x_1)$ such that polynomial $p(x_1, x_2) - J_d(x_1)$ be a nonnegative polynomial on the set $\mathbf{K}$.

$$p(x_1, x_2) - J_d(x_1) \geq 0 \ \forall (x_1, x_2) \in \mathbf{K}$$

- SOS Condition

Two-Player Game: $$\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \ \ \underset{x_2}{\text{minimize}} \ \{p(x_1,x_2) \ \ \text{subject to} \ (x_1,x_2) \in \mathbf{K}\}$$

$$x_1 \in \chi_1 \qquad \mathbf{K} = \{(x_1,x_2) : g_i(x_1,x_2) \geq 0, \ \ i = 1,...,n_g\}$$

SOS Program

$$\underset{J_d(x_1)}{\text{maximize}} \quad \int_{\mathbf{B}} J_d(x_1)dx_1$$

$$\text{subject to} \quad p(x_1,x_2) - J_d(x_1) - \sum_{i=1}^{n_g} \sigma_i(x_1,x_2)g_i(x_1,x_2) = \sigma_0(x_1,x_2)$$

$$\sigma_0(x_1,x_2) \in SOS_{2d}$$

$\chi_1 \subset \mathbf{B}$: simple box

$$\sigma_i(x_1,x_2) \in SOS_{2d_i}, \ \ i = 1,...,n_g$$

Two-Player Game: $$\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \;\; \underset{x_2}{\text{minimize}} \; \{p(x_1, x_2) \;\; \text{subject to } (x_1, x_2) \in \mathbf{K}\}$$

$$x_1 \in \chi_1 \qquad \mathbf{K} = \{(x_1, x_2) : g_i(x_1, x_2) \geq 0, \;\; i = 1, ..., n_g\}$$

SOS Program

$$\underset{J_d(x_1)}{\text{maximize}} \quad \int_{\mathbf{B}} J_d(x_1) dx_1$$

$$\text{subject to} \quad p(x_1, x_2) - J_d(x_1) - \sum_{i=1}^{n_g} \sigma_i(x_1, x_2) g_i(x_1, x_2) = \sigma_0(x_1, x_2)$$

$$\sigma_0(x_1, x_2) \in SOS_{2d}$$

$$\sigma_i(x_1, x_2) \in SOS_{2d_i}, \;\; i = 1, ..., n_g$$

$\chi_1 \subset \mathbf{B}$: simple box

$$p(x_1, x_2) - J_d(x_1) \geq 0 \;\; \forall (x_1, x_2) \in \mathbf{K}$$

Yalmip: $\mathrm{SOS}(p(x_1, x_2) - J_d(x_1) - \sum_{i=1}^{n_g} \sigma_i(x_1, x_2) g_i(x_1, x_2))$

$\mathrm{SOS}(\sigma_i(x_1, x_2))$

Two-Player Game:

$$\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \ \underset{x_2}{\text{minimize}} \ \{p(x_1, x_2) \ \text{ subject to } (x_1, x_2) \in \mathbf{K}\}$$

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$$\sigma_0(x_1, x_2) \in SOS_{2d}$$

$$\sigma_i(x_1, x_2) \in SOS_{2d_i}, \ i = 1, ..., n_g$$

$\chi_1 \subset \mathbf{B}$: simple box

Theorem:

$$\mathrm{Lim}_{d \to \infty} J_d(x_1) \to J(x_1)$$

Strong Sense Convergence, Theorem 2.2: J. B. Lasserre, “Min-max and robust polynomial optimization”, Journal of Global Optimization, Volume 51, Issue 1, pp 1–10, 2011

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$$\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \ \underset{x_2}{\text{minimize}} \ \{p(x_1, x_2) \ \text{ subject to } (x_1, x_2) \in \mathbf{K}\}$$

$$x_1 \in \chi_1 \qquad \mathbf{K} = \{(x_1, x_2) : g_i(x_1, x_2) \geq 0, \ i = 1, \ldots, n_g\}$$

SOS Program

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$$\text{subject to} \quad p(x_1, x_2) - J_d(x_1) - \sum_{i=1}^{n_g} \sigma_i(x_1, x_2) g_i(x_1, x_2) = \sigma_0(x_1, x_2)$$

$$\sigma_0(x_1, x_2) \in SOS_{2d}$$

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$\chi_1 \subset \mathbf{B}$: simple box

Theorem:

$$\mathrm{Lim}_{d \to \infty} J_d(x_1) \to J(x_1)$$

Theorem:

$$\mathrm{P}_r^d = \underset{x_1 \in \chi_1}{\text{maximize}} \ J_d(x_1) \xrightarrow{\hat{\mathrm{P}}_r^d = \underset{j = d_0, \ldots, d}{\text{maximize}} \ \mathrm{P}_r^j} \mathrm{Lim}_{d \to \infty} \hat{\mathrm{P}}_r^d \to \mathrm{P}_r$$

- If J is continuous and x_1^* is the unique global maximizer of P_r then $x_1^{*d} \to x_1^*$ *as* $d \to \infty$

Theorem 2.4: J. B. Lasserre, “Min-max and robust polynomial optimization”, Journal of Global Optimization, Volume 51, Issue 1, pp 1–10, 2011

Summary

Two-Player Game: $$\mathrm{P}_r = \underset{x_1 \in \chi_1}{\text{maximize}} \;\; \underset{x_2}{\text{minimize}} \; \{p(x_1, x_2) \;\; \text{subject to} \; (x_1, x_2) \in \mathbf{K}\}$$

$$x_1 \in \chi_1 \qquad \mathbf{K} = \{(x_1, x_2) : g_i(x_1, x_2) \geq 0, \;\; i = 1, \ldots, n_g\}$$

- SOS Program

$$\underset{J_d(x_1)}{\text{maximize}} \quad \int_{\mathbf{B}} J_d(x_1) dx_1$$

$$\text{subject to} \quad p(x_1, x_2) - J_d(x_1) - \sum_{i=1}^{n_g} \sigma_i(x_1, x_2) g_i(x_1, x_2) = \sigma_0(x_1, x_2)$$

$$\chi_1 \subset \mathbf{B} : \text{simple box} \qquad \sigma_0(x_1, x_2) \in SOS_{2d} \qquad \sigma_i(x_1, x_2) \in SOS_{2d_i}, \;\; i = 1, \ldots, n_g$$

- Based on worst case scenario $J_d(x_1) \leq J(x_1) = \underset{x_2}{\text{minimize}} \; \{p(x_1, x_2) \;\; \text{subject to} \; (x_1, x_2) \in \mathbf{K}\}$

$$\mathrm{P}_r^d = \underset{x_1 \in \chi_1}{\text{maximize}} \;\; J_d(x_1)$$

Topics

- Formulation of Robust Optimization
- Challenges
- Robust Sum-of-Squares
- Robust Sets for Robust Optimization
- Robust Games
- Distributionally Robust Chance Constrained Optimization

4. Distributionally Robust Chance Constrained Optimization

- Combination of Chance Constrained and Robust Optimizations

- Robust Set

$$\chi_R = \{x \in \chi : g(x, \omega) \leq 0, \ \forall \omega \in \Omega\}$$

- Chance Constrained Set

$$\chi_{CC} = \{x \in \chi : Prob(Success) \geq 1 - \Delta\} \qquad \omega \sim pr(\omega)$$

- Distributionally Robust Chance Constrained Set

$$\chi_{DR} = \{x \in \chi : Prob(Success) \geq 1 - \Delta, \ \ \forall pr_a(\omega), a \in \mathcal{A}\}$$

$$\omega \sim \text{Family of distributions} \ \ pr_a(\omega)$$

4. Distributionally Robust Chance Constrained Optimization

- Probability distribution with uncertain parameters $a \in \mathcal{A}$

e.g., Gaussian probability distribution with uncertain mean and deviation

$$\underbrace{\frac{1}{\sqrt{2\pi}\, a_2} e^{\frac{-(x-a_1)^2}{2a_2^2}}, \qquad a_1 \in [l_1, u_2], a_2 \in [l_2, u_2]}_{\text{Family of probability distributions}}$$

4. Distributionally Robust Chance Constrained Optimization

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$$\begin{array}{ll} \underset{\text{design parameters}}{\text{minimize}} & \text{Objective Function}(\text{design parameters}) \\ \text{subject to} & \text{uncertainty} \sim \text{Probability distribution}(a), \;\; a \in \mathcal{A} \\ & \underbrace{\text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \geq 1 - \Delta, \quad \forall a \in \mathcal{A}}_{\text{Chance constraints should be satisfied for the family of the probability distributions of the uncertainties}} \end{array}$$

4. Distributionally Robust Chance Constrained Optimization

$$\begin{aligned}
\underset{\text{design parameters}}{\text{minimize}} \quad & \text{Objective Function}(\text{design parameters}) \\
\text{subject to} \quad & \text{uncertainty} \sim \text{Probability distribution}(a),\ a \in \mathcal{A} \\
& \underbrace{\text{Probability}(\text{Success}(\text{ design parameters, probabilistic uncertainty})) \geq 1 - \Delta, \quad \forall a \in \mathcal{A}}_{\text{Chance constraints should be satisfied for the family of the probability distributions of the uncertainties}}
\end{aligned}$$

Mathematical Formulation

$$\begin{aligned}
\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\
\text{subject to} \quad & \omega \sim \text{pr}_a(\omega),\ a \in \mathcal{A} \\
& \text{Probability}_{\text{pr}_a(\omega)}(\ g_i(x,\omega) \geq 0,\ i = 1, \ldots, n_g\ ) \geq 1 - \Delta, \quad \forall a \in \mathcal{A}
\end{aligned}$$

Chance Constrained Set

Set of all design variable "x" that satisfies probabilistic "safety/design constraints" with respect to **probability distribution of uncertainty** "ω".

$$\chi_{CC} = \{x \in \chi : \text{Probability}(g_i(x,\omega) \geq 0, i = 1, ..., n_g) \geq 1 - \Delta\}$$

Chance Constrained Set

Set of all design variable "x" that satisfies probabilistic "safety/design constraints" with respect to **probability distribution of uncertainty** "ω".

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Robust Set

Set of all design variable "x" that satisfies "safety/design constraints" for all **possible values of uncertainty** "ω".

$$\chi_R = \{x \in \chi : g_i(x,\omega) \leq 0 i = 1, ..., n_g, \ \forall \omega \in \Omega\}$$

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$$\chi_R = \{x \in \chi : g_i(x,\omega) \leq 0 i = 1, \dots, n_g, \ \forall\omega \in \Omega\}$$

Distributionally Robust Set

Set of all design variable “x” that satisfies “safety/design constraints” for all **possible probability distribution of uncertainty** “ω”.

$$\chi_{DR} = \{x \in \chi : \text{Probability}(\ g_i(x,\omega) \geq 0,\ i = 1, \dots, n_g\ ) \geq 1 \quad -\Delta, \forall \text{pr}_a(\omega),\ a \in \mathcal{A}\}$$

Review of Inner Approximation of Chance Constrained Set (Lecture 7)

Review of Inner Approximation of Chance Constrained Set (Lecture 7)

Chance Constrained Sets

$$\chi_{CC} = \{x \in \chi : \text{Probability}(g_i(x,\omega) \geq 0, i = 1, ..., n_g) \geq 1 - \Delta\}$$

SOS Programing for Chance Constrained Set

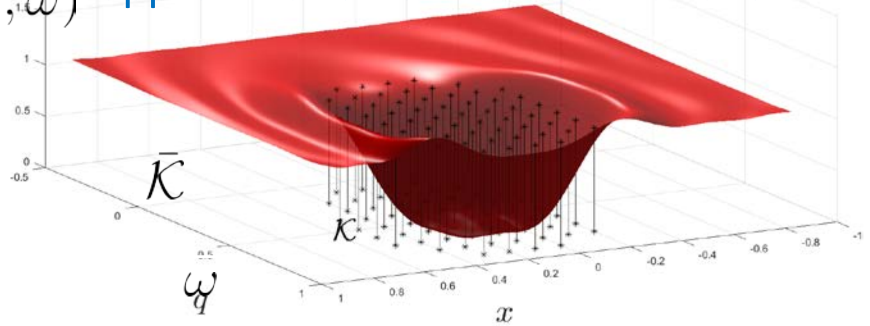
$$\begin{aligned} \bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \bar{\mathcal{W}}(x,\omega)pr(\omega)d\omega dx \\ \text{subject to} \quad & \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}} \\ & \bar{\mathcal{W}}(x,\omega) \geq 0 \end{aligned}$$

(complement set)

$$\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \ g_i(x,\omega) \geq 0, \ i = 1, ..., n_g \ \}$$

Inner approximation $\quad \bar{\chi}_{CC} = \left\{x \in \mathbb{R}^n : \bar{h}(x) \leq \Delta\right\} \quad \bar{h}(x) = \int \bar{\mathcal{W}}(x,\omega)pr(\omega) \ d\omega$

$\bar{\mathcal{W}}(x,\omega)$: approximation of Indicator function



Review of Inner Approximation of Chance Constrained Set (Lecture 7)

Chance Constrained Sets

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- Moments of the probability distributions of uncertainties are known $\quad y_{\omega\,\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr(\omega)d\omega$

objective function: $\int \bar{\mathcal{W}}(x,\omega)pr(\omega)d\omega dx \xrightarrow[y_{\omega\,\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr(\omega)d\omega \quad y_{x\,\alpha} = \mathrm{E}[x^\alpha] = \int x^\alpha dx]{\bar{\mathcal{W}}(x,\omega) = \sum c_{\alpha\beta}x^\alpha\omega^\beta} \sum c_{\alpha\beta}y_{x\,\alpha}y_{\omega\,\beta}$

$\bar{h}(x) = \int \bar{\mathcal{W}}(x,\omega)\ pr(\omega)d\omega \longrightarrow \bar{h}(x) = \sum c_{\alpha\beta}y_{\omega\,\beta}x^\beta$

Review of Inner Approximation of Chance Constrained Set (Lecture 7)

Chance Constrained Sets

$$\chi_{CC} = \{x \in \chi : \text{Probability}(g_i(x,\omega) \geq 0, i = 1, ..., n_g) \geq 1 - \Delta\}$$

SOS Programing for Chance Constrained Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \bar{\mathcal{W}}(x,\omega) pr(\omega) d\omega dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}} \text{ (complement set)}$$

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- Moments of the probability distributions of uncertainties are known $\; y_{\omega\,\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr(\omega) d\omega$

objective function: $\int \bar{\mathcal{W}}(x,\omega) pr(\omega) d\omega dx \xrightarrow{\bar{\mathcal{W}}(x,\omega) = \sum c_{\alpha\beta} x^\alpha \omega^\beta} \sum c_{\alpha\beta} y_{x\,\alpha} y_{\omega\,\beta} \rightarrow$ Known

$y_{\omega\,\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr(\omega) d\omega \qquad y_{x\,\alpha} = \mathrm{E}[x^\alpha] = \int x^\alpha dx$

$\bar{h}(x) = \int \bar{\mathcal{W}}(x,\omega) \, pr(\omega) d\omega \longrightarrow \bar{h}(x) = \sum c_{\alpha\beta} y_{\omega\,\beta} x^\beta$

Distributionally Robust Chance Constrained Set

- If we apply the same methodology to Distributionally Robust problems, we need moments of probability distributions $pr_a(\omega)$.

- Moments of the probability distributions are **unknown,** (depend on the parameters of the probability distributions).

$$y_{\omega_\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr_a(\omega) d\omega$$

Example:

Uncertain Mean and deviation

$$pr_a(\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(\omega - m)^2}{2\sigma^2}} \xrightarrow{\quad a = (m, \sigma) \quad} (m, \sigma) \in \mathcal{A} = [\underline{m}, \bar{m}] \times [\underline{\sigma} > 0, \bar{\sigma}] \subset \mathbb{R}^2$$

Parametric Moments: $y_\alpha = \mathrm{E}[\omega^\alpha] = \int \omega^\alpha \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(\omega - m)^2}{2\sigma^2}} d\omega$

$$\begin{aligned}
y_0 &= 1, \\
y_1 &= p_1(a) = m, \\
y_2 &= p_2(a) = m^2 + \sigma^2, \\
y_3 &= p_3(a) = m^3 + 3m\sigma^2, \\
y_4 &= p_4(a) = m^4 + 6m^2\sigma^2 + 3\sigma^4, \\
y_5 &= p_5(a) = m^5 + 10m^3\sigma^2 + 15m\sigma^4, \\
y_6 &= p_6(a) = m^6 + 15m^4\sigma^2 + 45m^2\sigma^4 + 15\sigma^6
\end{aligned}$$

- Moments are polynomial functions in terms of uncertain parameter $a = (m, \sigma)$

- Same methodology applied to Distributionally Robust Problems:

Distributionally Robust Chance Constrained Set

$$\chi_{DR} = \{x \in \chi : \text{Probability}(\ g_i(x,\omega) \geq 0,\ i = 1, ..., n_g\ ) \geq 1\ -\Delta, \forall \text{pr}_a(\omega),\ a \in \mathcal{A}\}$$

SOS Programing

$$\begin{aligned} \bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \bar{\mathcal{W}}(x,\omega) pr_a(\omega) d\omega dx \\ \text{subject to} \quad & \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}} \quad \text{(complement set)} \\ & \bar{\mathcal{W}}(x,\omega) \geq 0 \end{aligned}$$

$$\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ g_i(x,\omega) \geq 0,\ i = 1, ..., n_g\ \}$$

Inner approximation: $\bar{\chi}_{DR} = \left\{x \in \mathbb{R}^n : \bar{h}(x) \leq \Delta\right\}$ $\quad h(x) = \int \bar{\mathcal{W}}(x,\omega) pr_a(\omega) d\omega$

- Moments of the probability distribution are **unknown**. $\quad y_{\omega^\beta} = \text{E}[\omega^\beta] = \int \omega^\beta pr_a(\omega) d\omega$

Distributionally Robust Chance Constrained Set

$$\chi_{DR} = \{x \in \mathbb{R}^n : \text{Probability}(\ g_i(x,\omega) \geq 0,\ \ i = 1, ..., n_g\ ) \geq 1 - \Delta, \forall \text{pr}_a(\omega),\ \ a \in \mathcal{A}\}$$

SOS Programing

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$$

(complement set)

$$\bar{\mathcal{W}}(x,\omega) \geq 0$$

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polynomial moments

- Moments of the probability distributions are unknown. $y_{\omega_\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr(\omega)d\omega = p_\beta(a)$

objective function: $\int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega dx \xrightarrow[y_{x_\alpha} = \mathrm{E}[x^\alpha] = \int x^\alpha dx \qquad y_{\omega_\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr_a(\omega)d\omega = p_\beta(a)]{\bar{\mathcal{W}}(x,\omega) = \sum c_{\alpha\beta}x^\alpha\omega^\beta} \sum c_{\alpha\beta} y_{x_\alpha} p_\beta(a)$

$$\int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega \longrightarrow \bar{h}(x,a) = \sum c_{\alpha\beta} p_\beta(a) x^\beta$$

$$\bar{\chi}_{DR} = \{x \in \mathbb{R}^n : h(x,a) \leq \Delta\}$$

Distributionally Robust Chance Constrained Set

$$\chi_{DR} = \{x \in \mathbb{R}^n : \text{Probability}(\ g_i(x,\omega) \geq 0,\ \ i = 1, ..., n_g\ ) \geq 1 - \Delta, \forall \text{pr}_a(\omega),\ \ a \in \mathcal{A}\}$$

SOS Programing

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \int \bar{\mathcal{W}}(x,\omega) pr_a(\omega) d\omega dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$$ (complement set)

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- Moments of the probability distributions are unknown. $y_{\omega_\beta} = \mathrm{E}[\omega^\beta] = \int \omega^\beta pr(\omega) d\omega = p_\beta(a)$

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$$\int \bar{\mathcal{W}}(x,\omega) pr_a(\omega) d\omega \longrightarrow \bar{h}(x,a) = \sum c_{\alpha\beta} p_\beta(a) x^\beta$$

- To construct inner approximation of distributionally robust set, we look for an upper bound $h_d(x) \geq h(x,a)$

$$\bar{\chi}_{DR} = \{x \in \mathbb{R}^n : \bar{h}_d(x) \leq \Delta\}$$

Same methodology:

Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\} \xrightarrow{p_d(x) \geq g(x,\omega)\ \ \forall x \in \chi,\ \forall \omega \in \Omega} \chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$$

Distributionally Robust Chance Constrained Sets:

$$\chi_{DR} = \{x \in \chi : \bar{h}(x,a) \leq \Delta\}$$

$$\bar{h}(x,a) = \int \bar{\mathcal{W}}(x,\omega) pr_a(\omega) d\omega$$

$$\xrightarrow{\bar{h}_d(x) \geq \bar{h}(x,a)\ \ \forall x \in \chi,\ \forall a \in \mathcal{A}} \bar{\chi}_{DR}^d = \{x \in \chi : \bar{h}_d(x) \leq \Delta\}$$

Same methodology:

Robust Set:

$$\chi_R = \{x \in \chi : g(x,\omega) \leq 0,\ \forall \omega \in \Omega\} \xrightarrow{p_d(x) \geq g(x,\omega)\ \ \forall x \in \chi,\ \forall \omega \in \Omega} \chi_R^d = \{x \in \chi : p_d(x) \leq 0\}$$

Distributionally Robust Chance Constrained Sets:

$$\chi_{DR} = \{x \in \chi : \bar{h}(x,a) \leq \Delta\} \xrightarrow{\bar{h}_d(x) \geq \bar{h}(x,a)\ \ \forall x \in \chi,\ \forall a \in \mathcal{A}} \bar{\chi}_{DR}^d = \{x \in \chi : \bar{h}_d(x) \leq \Delta\}$$

$$\bar{h}(x,a) = \int \bar{\mathcal{W}}(x,\omega) pr_a(\omega) d\omega$$

- To be able to use SOS program, $h(x,a)$ should be a **polynomial** in terms of x and a
- Assumption: parametric moments of $pr_a(\omega)$ are **polynomials**.

SOS Programing for Chance Constrained Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \bar{\mathcal{W}}(x,\omega)pr(\omega)d\omega dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$$ (complement set)

$$\bar{\mathcal{W}}(x,\omega) \geq 0$$

$$\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \ g_i(x,\omega) \geq 0, \ i = 1, ..., n_g \ \}$$

Inner approximation $\bar{\chi}_{CC} = \left\{x \in \chi : \bar{h}(x) \leq \Delta\right\}$ $h(x) = \int \bar{\mathcal{W}}(x,\omega) \ pr(\omega)d\omega$

SOS Programing for Chance Constrained Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \bar{\mathcal{W}}(x,\omega)pr(\omega)d\omega dx$$

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$$\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \; g_i(x,\omega) \geq 0, \; i = 1,...,n_g \}$$

Inner approximation $\quad \bar{\chi}_{CC} = \left\{x \in \chi : \bar{h}(x) \leq \Delta\right\} \quad h(x) = \int \bar{\mathcal{W}}(x,\omega)\, pr(\omega)d\omega$

SOS Programing for Distributionally Robust Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega],\bar{h}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int \bar{h}_d(x)dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$$

$$\bar{\mathcal{W}}(x,\omega) \geq 0$$ (complement set)

$$\bar{h}_d(x) \geq \bar{h}(x,a) \quad \forall x \in \chi, \; \forall a \in \mathcal{A}$$

$\bar{h}_d(x) \geq \bar{h}(x,a) = \int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega$ (In $\bar{\mathcal{W}}(x,\omega)$ Replace ω^α with polynomial moments $p_\alpha(a)$)

SOS Programing for Chance Constrained Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \bar{\mathcal{W}}(x,\omega)pr(\omega)d\omega dx$$

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(complement set)

$$\bar{\mathcal{W}}(x,\omega) \geq 0$$

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SOS Programing for Distributionally Robust Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega],\bar{h}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int \bar{h}_d(x)dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$$

$$\bar{\mathcal{W}}(x,\omega) \geq 0$$

(complement set)

$$\bar{h}_d(x) \geq \bar{h}(x,a) \quad \forall x \in \chi, \; \forall a \in \mathcal{A}$$

$\bar{h}_d(x) \geq \bar{h}(x,a) = \int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega$ (In $\bar{\mathcal{W}}(x,\omega)$ Replace ω^α with polynomial moments $p_\alpha(a)$)

$$\text{objective function} : \int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega dx = \int \left(\int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)\right) dx = \int \bar{h}(x,a)dx \longrightarrow \int \bar{h}_d(x,a)dx$$

SOS Programing for Chance Constrained Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad \int \bar{\mathcal{W}}(x,\omega)pr(\omega)d\omega dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega)-1\geq 0 \quad \forall (x,\omega)\in\bar{\mathcal{K}}$$

(complement set)

$$\bar{\mathcal{W}}(x,\omega)\geq 0$$

$$\mathcal{K}=\{(x,\omega)\in\mathbb{R}^n\times\mathbb{R}^m:\ g_i(x,\omega)\geq 0,\ i=1,...,n_g\ \}$$

Inner approximation $\quad \bar{\chi}_{CC}=\left\{x\in\chi:\bar{h}(x)\leq\Delta\right\} \qquad h(x)=\int\bar{\mathcal{W}}(x,\omega)\ pr(\omega)d\omega$

SOS Programing for Distributionally Robust Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega],\bar{h}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int \bar{h}_d(x)dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega)-1\geq 0 \quad \forall (x,\omega)\in\bar{\mathcal{K}}$$

$$\bar{\mathcal{W}}(x,\omega)\geq 0$$

(complement set)

$$\bar{h}_d(x)\geq\bar{h}(x,a) \quad \forall x\in\chi,\ \forall a\in\mathcal{A}$$

$\bar{h}_d(x)\geq\bar{h}(x,a)=\int\bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega$ (In $\bar{\mathcal{W}}(x,\omega)$ Replace ω^α with polynomial moments $p_\alpha(a)$)

$$\text{objective function}:\int\bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega dx=\int\left(\int\bar{\mathcal{W}}(x,\omega)pr_a(\omega)\right)dx=\int\bar{h}(x,a)dx \longrightarrow \int\bar{h}_d(x,a)dx$$

Inner approximation $\quad \bar{\chi}_{DR}=\left\{x\in\mathbb{R}^n:\bar{h}_d(x)\leq\Delta\right\}$

SOS Programing for Distributionally Robust Set

$$\bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega],\bar{h}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad \int \bar{h}_d(x)dx$$

$$\text{subject to} \quad \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}}$$

(complement set)

$$\bar{\mathcal{W}}(x,\omega) \geq 0$$

$$\bar{h}_d(x) \geq \bar{h}(x,a) \quad \forall x \in \chi, \ \forall a \in \mathcal{A}$$

$\bar{h}_d(x) \geq \bar{h}(x,a) = \int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega$ (In $\bar{\mathcal{W}}(x,\omega)$ Replace ω^α with polynomial moments $p_\alpha(a)$)

objective function : $\int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)d\omega dx = \int \left(\int \bar{\mathcal{W}}(x,\omega)pr_a(\omega)\right)dx = \int \bar{h}(x,a)dx \longrightarrow \int \bar{h}_d(x,a)dx$

Inner approximation $\quad \bar{\chi}^d_{DR} = \{x \in \mathbb{R}^n : \bar{h}_d(x) \leq \Delta\}$

Theorem: $\quad d \to \infty \ : \ \bar{\chi}^d_{DR} \to \chi_{DR}$

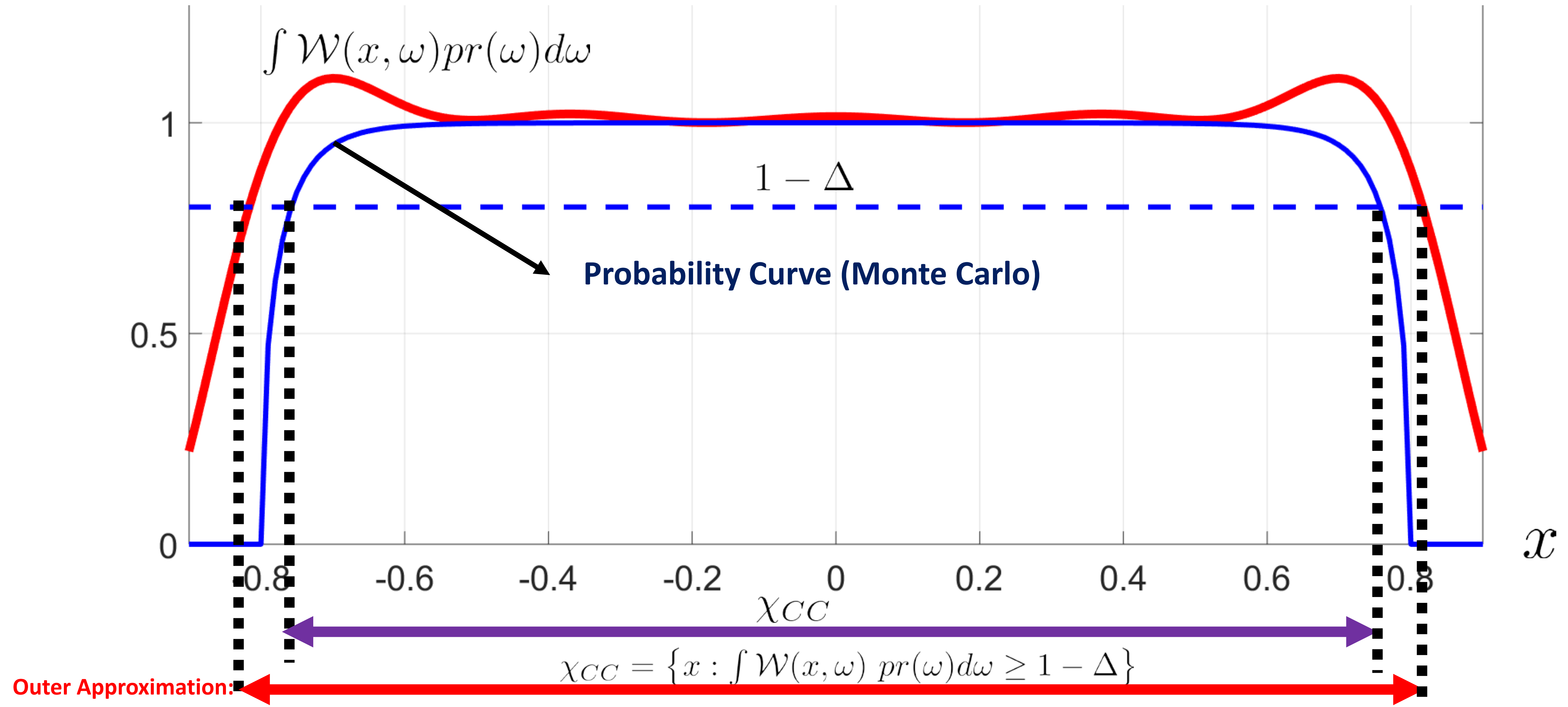
Theorem 4.3, Jean B. Lasserre,Tillmann Weisser,”Distributionally robust polynomial chance-constraints under mixture ambiguity sets”, Mathematical Programming pp 1–45, 2019

Example: Outer Approximation of Chance Constrained Set

$$\chi_{CC} = \{x \in \chi = [-1 \quad 1] : \quad \text{Prob}\left(0.8^2 - x^2 - \omega^2 \geq 0\right) \geq 1 - \Delta, \;\; \omega \sim N(0, 0.2) \;\}$$

Example: Outer Approximation of Chance Constrained Set

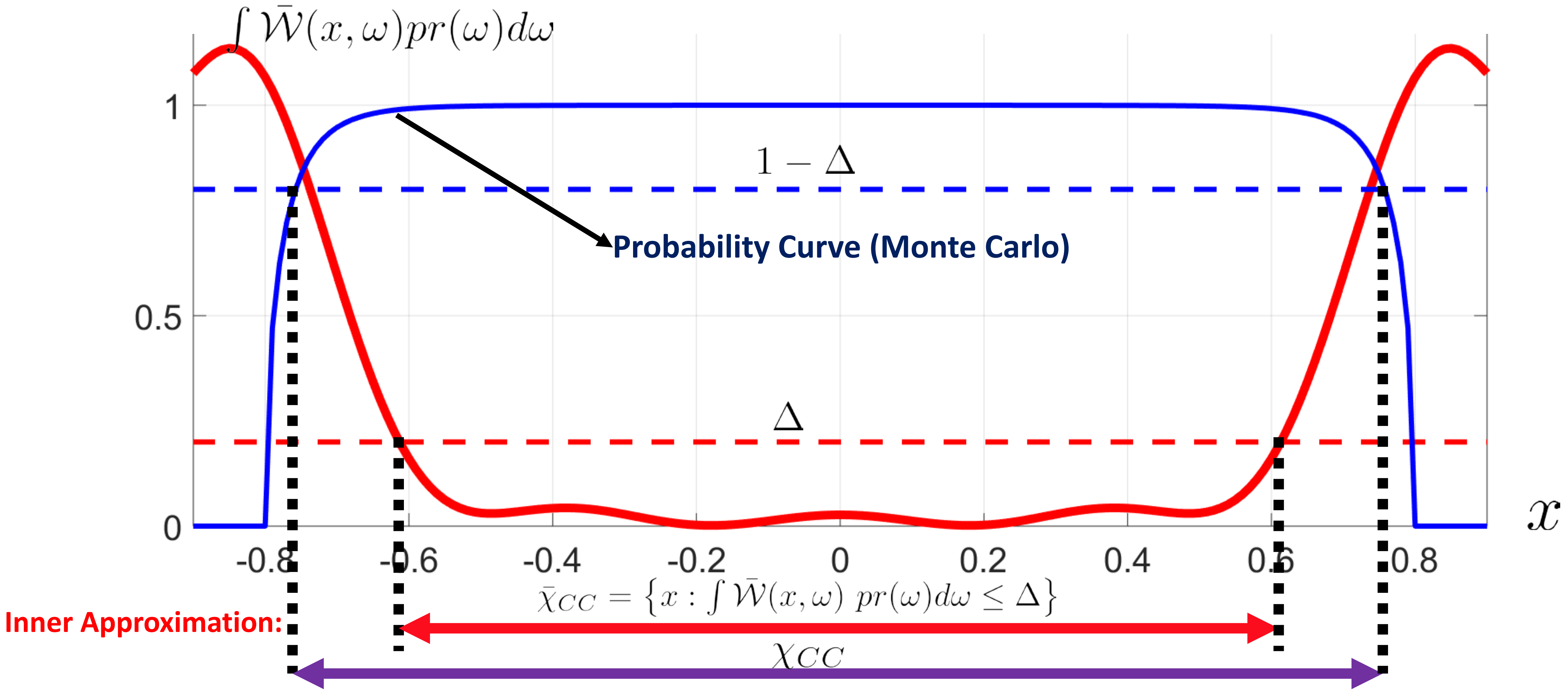
$$\chi_{CC} = \{x \in \chi = [-1 \;\; 1] : \;\; \text{Prob}\left(0.8^2 - x^2 - \omega^2 \geq 0\right) \geq 1-\Delta, \;\; \omega \sim N(0, 0.2) \;\}$$



https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Distributionally_Robust_Chance%20Constrained_Set/Example_1/Example_ChanceConSet_Outer.m

Example: Inner Approximation of Chance Constrained Set

$$\chi_{CC} = \{x \in \chi = [-1 \quad 1] : \quad \text{Prob}\left(0.8^2 - x^2 - \omega^2 \geq 0\right) \geq 1 - \Delta, \ \omega \sim N(0, 0.2) \ \}$$



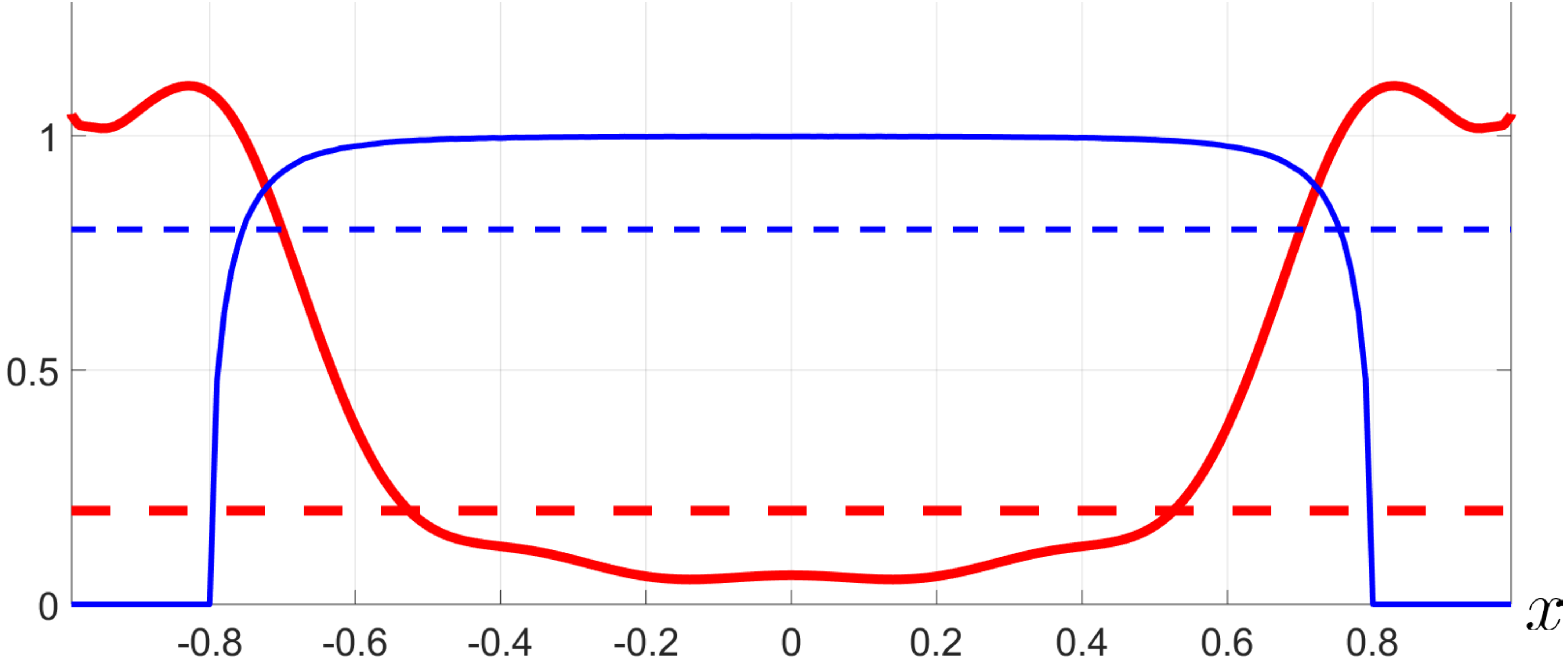
https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Distributionally_Robust_Chance%20Constrained_Set/Example_1/Example_ChanceConSet_Inner.m

Example Inner Approximation of Distributionally Robust Chance Constrained Set

$$\chi_{DR} = \{x \in \chi = [-1 \quad 1] : \quad \text{Prob}\left(0.8^2 - x^2 - \omega^2 \geq 0\right) \geq 1 - \Delta, \ \omega \sim N(m, \sigma), \forall m \in [-0.1, 0.1], \forall \sigma \in [0.3, 0.5] \ \}$$

Example Inner Approximation of Distributionally Robust Chance Constrained Set

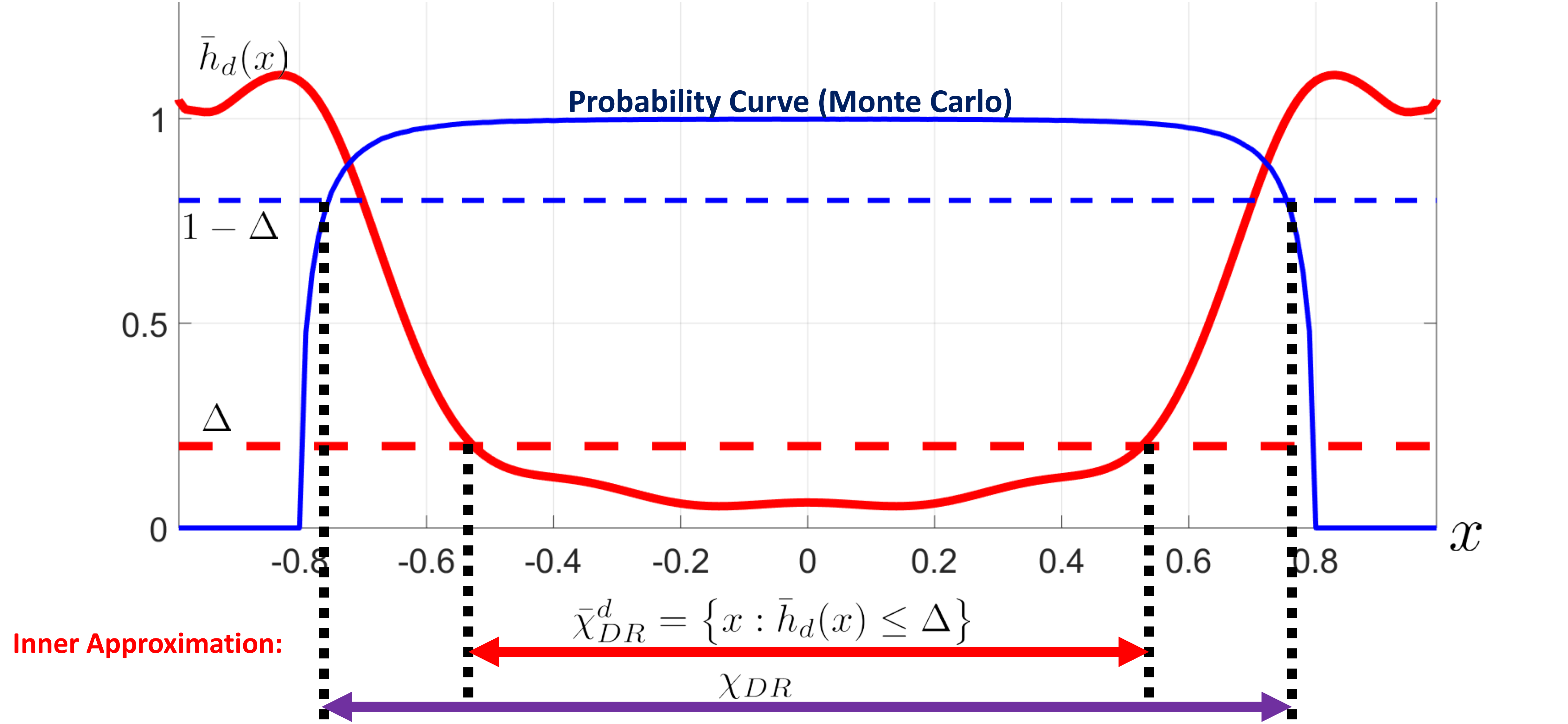
$$\chi_{DR} = \{x \in \chi = [-1 \quad 1] : \quad \text{Prob}\left(0.8^2 - x^2 - \omega^2 \geq 0\right) \geq 1 - \Delta, \ \omega \sim N(m, \sigma), \forall m \in [-0.1, 0.1], \forall \sigma \in [0.3, 0.5] \ \}$$



https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Distributionally_Robust_Chance%20Constrained_Set/Example_1/Example_DistributionallyRobustCCSet_Inner.m

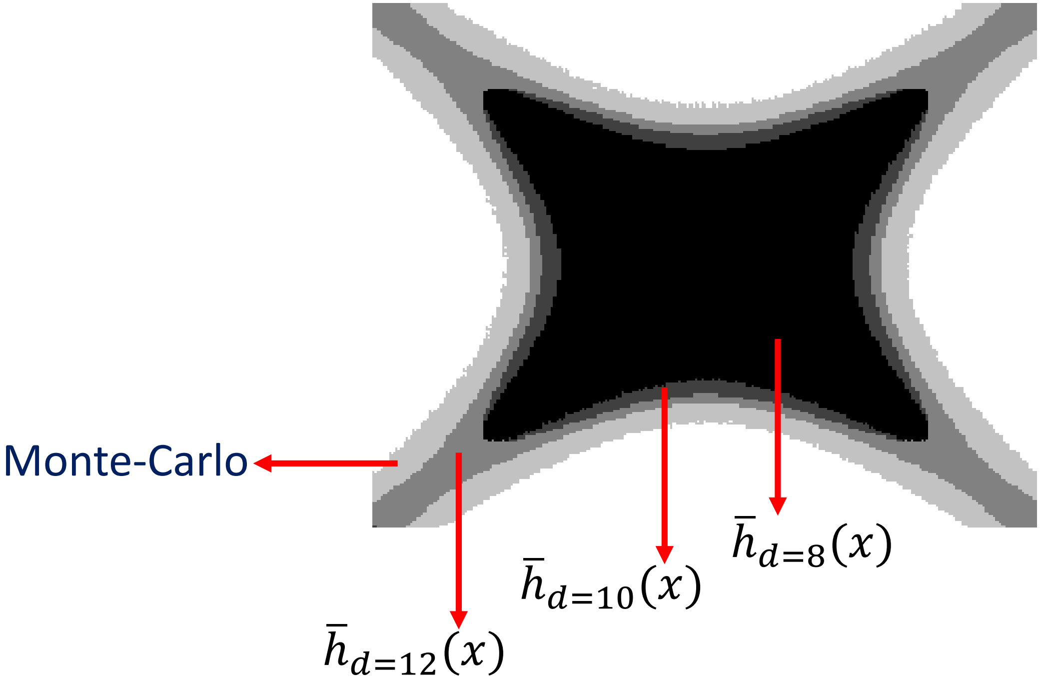
Example **Inner Approximation of Distributionally Robust Chance Constrained Set**

$$\chi_{DR} = \{x \in \chi = [-1 \;\; 1] : \quad \text{Prob}\left(0.8^2 - x^2 - \omega^2 \geq 0\right) \geq 1 - \Delta, \;\; \omega \sim N(m, \sigma), \forall m \in [-0.1, 0.1], \forall \sigma \in [0.3, 0.5] \;\}$$



Example:

$$\chi_{DR} = \{x \in \chi = [-1 \ \ 1] : \ \ \text{Prob}(2\omega x_2^2 - 2\omega x_1^2 - 1 \le 0) \ge 1 - \Delta, \ \omega \sim N(m,\sigma), \forall m \in [-0.1, 0.1], \forall \sigma \in [0.8, 1] \ \}$$



Jean B. Lasserre,Tillmann Weisser,"Distributionally robust polynomial chance-constraints under mixture ambiguity sets", Mathematical Programming, pp 1–45, 2019

Example: Safe Control of Uncertain Nonlinear System

Uncertain Nonlinear System:

$$x_1(k+1) = \omega(k)x_2(k)$$
$$x_2(k+1) = x_1(k)x_3(k)$$
$$x_3(k+1) = 1.2x_1(k) - 0.5x_2(k) + 2u(k)$$

Source of uncertainties: Initial states $\left(x_1(0), x_2(0), x_3(0)\right)$ and disturbance $\omega(k)$

Unsafe Set: $X_{obs} = \{\ (x_1, x_2, x_3): 1^2 - \left(\frac{x_1 - 0.1}{0.2}\right)^2 - \left(\frac{x_2 - 0.1}{0.2}\right)^2 - \left(\frac{x_3 - 0.4}{0.3}\right)^3 \geq 0\}$

Example: Safe Control of Uncertain Nonlinear System

Case 1:

Chance Constrained and Distributionally Robust Chance constrained set for control input at time k :

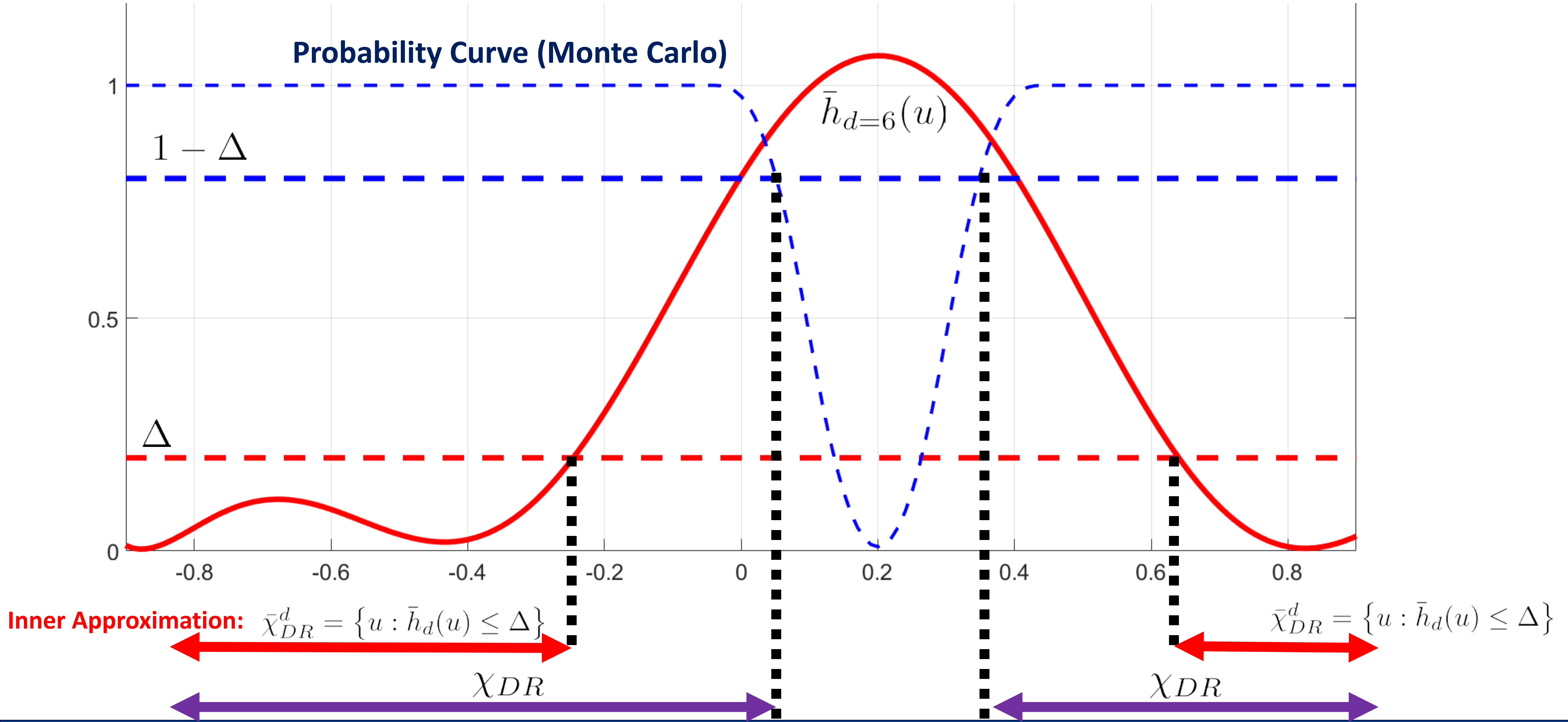
$$\forall u_k : \ Prob\big(x(k+1) \in X_{safe}\big) \geq 1 - \Delta$$

- $x(k) \sim Uniform[-0.1, 0.1]^3$
- For all probability distributions
  $\omega(k) \sim N(m, \sigma), \qquad m \in [-0.1 \quad 0.1], \sigma \in [0.1 \quad 0.3]$

https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Distributionally_Robust_Chance%20Constrained_Set/Example_2_DynamicalSys/Example_DistributionallyRobustCCSet_Inner.m

Example: Safe Control of Uncertain Nonlinear System

Case 1: $\chi_{DR} = \{\, \forall u_k \in [-1,1\,]\colon Prob\big(x(k+1) \in X_{safe}\big) \geq 1-\Delta\,\}$



Example: Safe Control of Uncertain Nonlinear System

Case 1:
Chance Constrained and Distributionally Robust Chance constrained set for control input at time k :

$$\forall u_k : \ Prob\big(x(k+1) \in X_{safe}\big) \geq 1 - \Delta$$

- $x(k) \sim Uniform[-0.1, 0.1]^3$
- For all probability distributions
  $\omega(k) \sim N(m, \sigma), \qquad m \in [-0.1 \ \ 0.1], \sigma \in [0.1 \ \ 0.3]$

https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Distributionally_Robust_Chance%20Constrained_Set/Example_2_DynamicalSys/Example_DistributionallyRobustCCSet_Inner.m

Case 2:
Coupled Distributionally Robust Chance constrained set for control input at time k :

$$\forall u_k : \ Prob\big(x(k+1) \in X_{safe}\big) \geq 1 - \Delta$$

- For all probability distributions
  $x_1(k) \sim N(m_1, \sigma_1),$
  $x_2(k) \sim N(m_2, \sigma_2),$
  $x_3(k) \sim N(m_3, \sigma_3),$
  $\omega(k) \sim N(m_4, \sigma_4),$

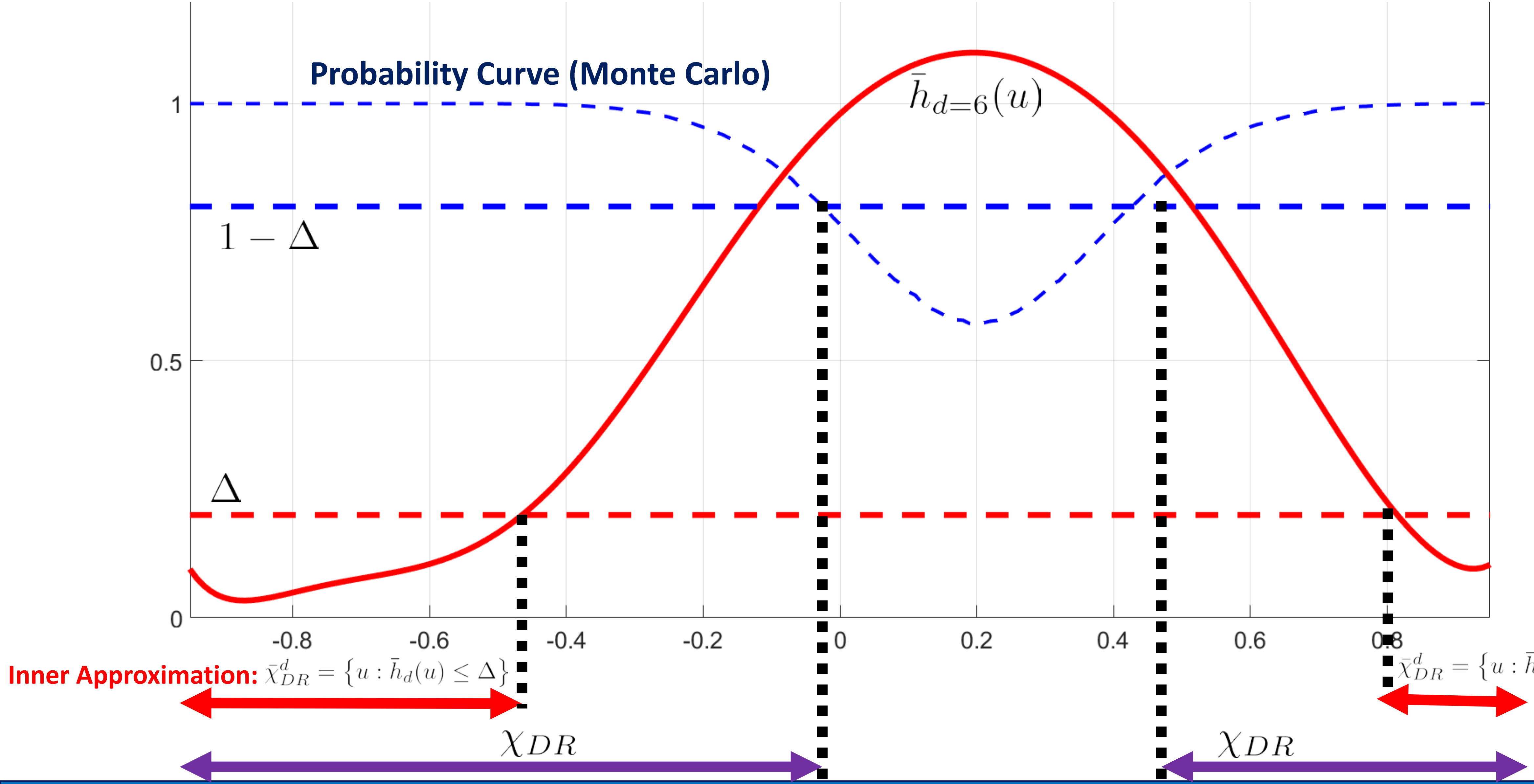
https://github.com/jasour/rarnop19/blob/master/Lecture8%20_RobustOptimization/Distributionally_Robust_Chance%20Constrained_Set/Example_2_DynamicalSys/Example_Coupled_DistributionallyRobustCCSet_Inner.m

Mean set: $\{\, (m_1, m_2, m_3, m_4): \ 0.1 - m_1^2 - m_2^2 - m_3^2 - m_4^2 \geq 0\}$

Sigma set: $\{\, (\sigma_1, \sigma_2, \sigma_3, \sigma_4): \ 0.1 - (\sigma_1 - 0.2)^2 - (\sigma_2 - 0.2)^2 - (\sigma_3 - 0.2)^2 - (\sigma_4 - 0.2)^2 \geq 0\}$

Example: Safe Control of Uncertain Nonlinear System

Case 2: $\chi_{DR} = \{\, \forall u_k \in [-1,1\,]\colon Prob\big(x(k+1) \in X_{safe}\big) \geq 1 - \Delta \,\}$



Summary

Distributionally Robust Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \omega \sim \text{pr}_a(\omega),\ a \in \mathcal{A} \\ & \text{Probability}_{\text{pr}_a(\omega)}\big(\ g_i(x,\omega) \geq 0,\ i = 1,\ldots,n_g\ \big) \geq 1 - \Delta,\quad \forall a \in \mathcal{A} \end{aligned}$$

- SOS Program

$$\begin{aligned} \bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega)\in\mathbb{R}_d[x,\omega],\bar{h}_d(x)\in\mathbb{R}_d[x]}{\text{minimize}} \quad & \int \bar{h}_d(x)dx \\ \text{subject to} \quad & \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}} \\ & \bar{\mathcal{W}}(x,\omega) \geq 0 \\ & \bar{h}_d(x) \geq \bar{h}(x,a) \quad \forall x \in \chi,\ \forall a \in \mathcal{A} \end{aligned}$$

- Based on worst case scenario $\quad \bar{h}_d(x) \geq \bar{h}(x,a) = \int \bar{\mathcal{W}}(x,\omega) pr_a(\omega) d\omega \quad$ (Worst case probability distribution)

Deterministic Optimization

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & x \in \bar{\chi}_{DR} = \{x \in \mathbb{R}^n : \bar{h}_d(x) \leq \Delta\} \subset \chi_{DR} = \{x \in \chi : \text{Probability}(\ g_i(x,\omega) \geq 0,\ i = 1,\ldots,n_g\ ) \geq 1-\Delta, \forall \text{pr}_a(\omega),\ a \in \mathcal{A}\} \end{aligned}$$

Topics

- Formulation of Robust Optimization
- Challenges
- Robust Sum-of-Squares
- Robust Sets for Robust Optimization
- Robust Games
- Distributionally Robust Chance Constrained Optimization

Robust Optimization

- A. Ben-Tal, L. El Ghaoui, A. Nemirovski. “Robust Optimization”, Princeton University Press, Princeton, 2009.

Robust Sets

- Jean B. Lasserre, , “Tractable approximations of sets defined with quantifiers” Math. Program., Ser. B, 151:507–527, 2015.

Robust Game

- J. B. Lasserre, “Min-max and robust polynomial optimization”, Journal of Global Optimization, Volume 51, Issue 1, pp 1–10, 2011

Distributionally Robust Chance Constrained Optimization

- Jean B. Lasserre, Tillmann Weisser,”Distributionally robust polynomial chance-constraints under mixture ambiguity sets”, Mathematical Programming, pp 1–45, 2019

Book

- Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.

- Jean B. Lasserre,”An Introduction to Polynomial and Semi-Algebraic Optimization”, Cambridge University Press, 2015