Lecture 12

Mean and Covariance Based Control and Safety Verification of Probabilistic Dynamical Systems

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

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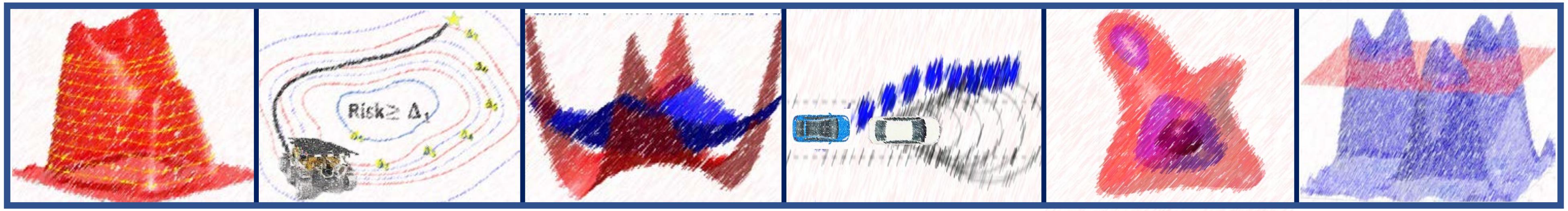

- In this lecture,

  instead of using **higher order moments** to represent probabilistic uncertainties(Lectures 10,11),
  we just use information of **first and second moments,** i.e., mean and covariance.

- In the presence of exact information of first and second moments i.e., mean and variance, of uncertainties:

  - Gaussian representation of uncertainty
    - In discrete time, we model uncertainty with Gaussian Random Variable $\mathcal{N}$(mean, Variance)
    - In continuous time, we model uncertainty with Gaussian Stochastic Process

  - Distributionally Robust representation of uncertainty
    - We model uncertainty with family of probability distributions whose first and second moments matches the given moments

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    - In continuous time, we model uncertainty with Gaussian Stochastic Process
  - Distributionally Robust representation of uncertainty
    - We model uncertainty with family of probability distributions whose first and second moments matches the given moments

- In the presence of uncertain information of first and second moments :

  - Moment ambiguity sets:

    Uncertainty set for first order moments (mean vector): $m \in [\underline{m}, \overline{m}]$

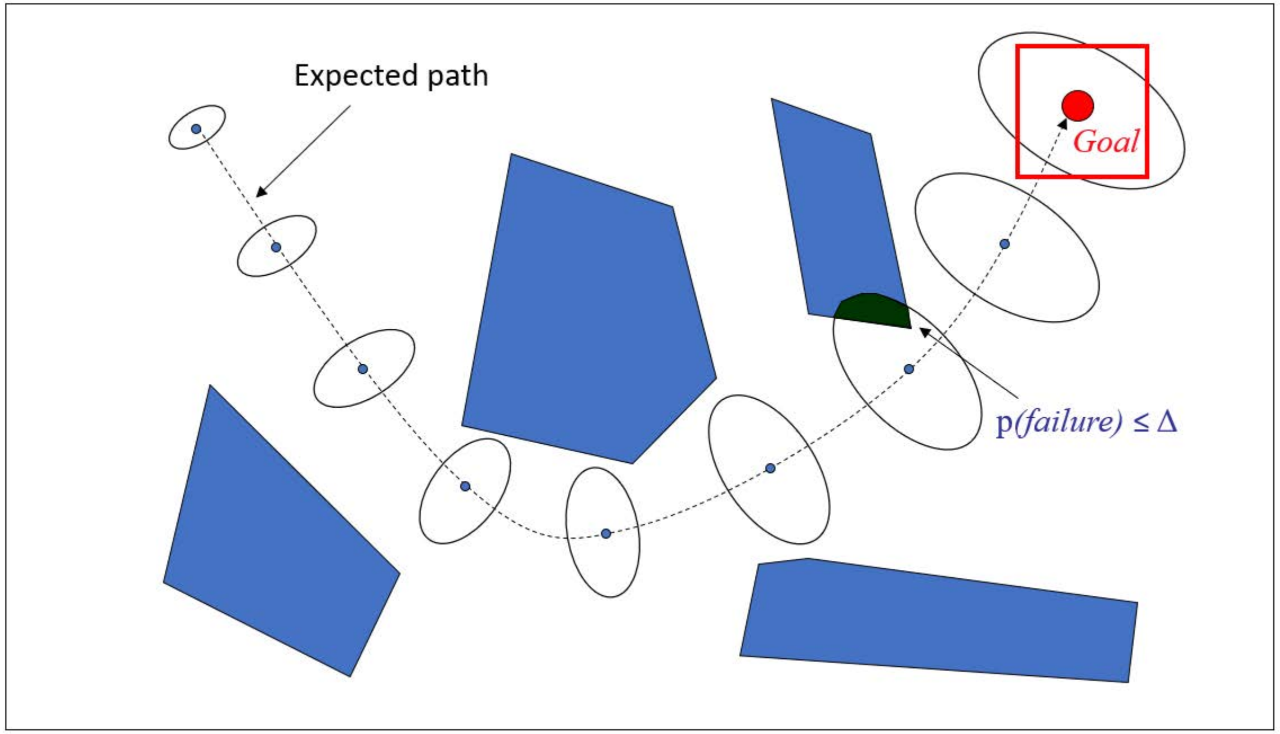
    Uncertainty set for second order moments (Covariance Matrix): $\Sigma \in [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \overline{\delta} I]$ for some given $\underline{\delta}, \overline{\delta} > 0$
  - Distributionally Robust representation of uncertainty
    - We model uncertainty with family of probability distributions whose first and second moments are in the given sets.

Topics:

- Chane Constrained Control

  i) Trajectory optimization,   ii) RRT*,   iii) PRM,   iv) Motion Primitive

  v) Continuous-Time Safety Guarantees



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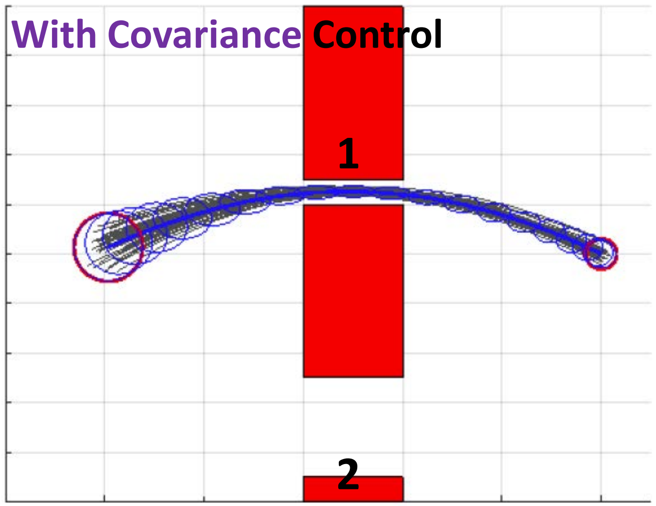

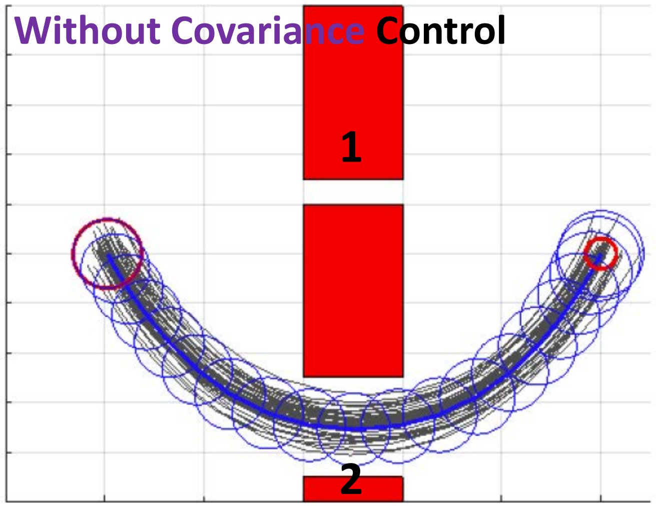

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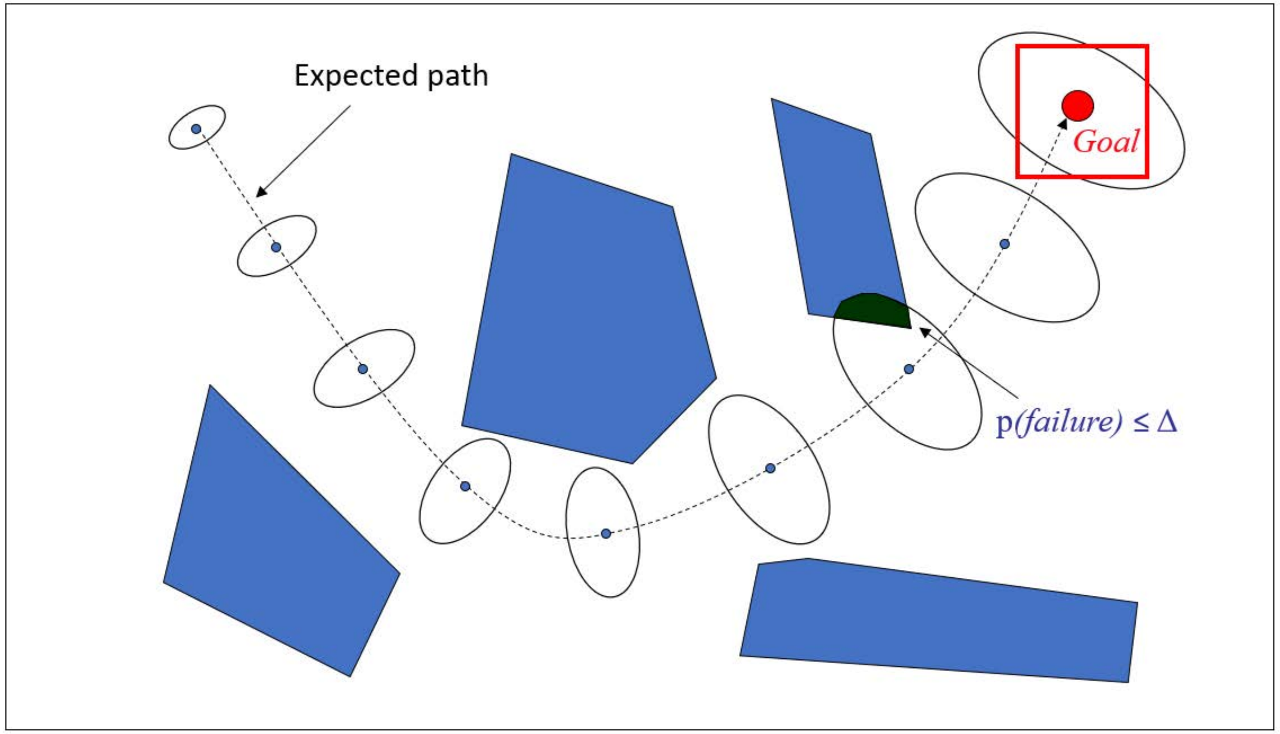

Chance Constrained Trajectory Optimization

Gaussian Linear System

$$x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

- Uncertainties with Normal distribution $\mathcal{N}$(mean, Variance) $x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma_{x_0}^2)$, $\omega_k \sim \mathcal{N}(\overline{\omega}, \Sigma_{\omega_k}^2)$

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➢ Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \ldots, T-1$

$$\min_{[u_0, \ldots, u_{T-1}]} E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

Probabilistic Safety Constraints

$$E[x_T] = x_G$$

Chance Constrained Trajectory Optimization

Gaussian Linear System

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Probabilistic Safety Constraints

$$E[x_T] = x_G$$

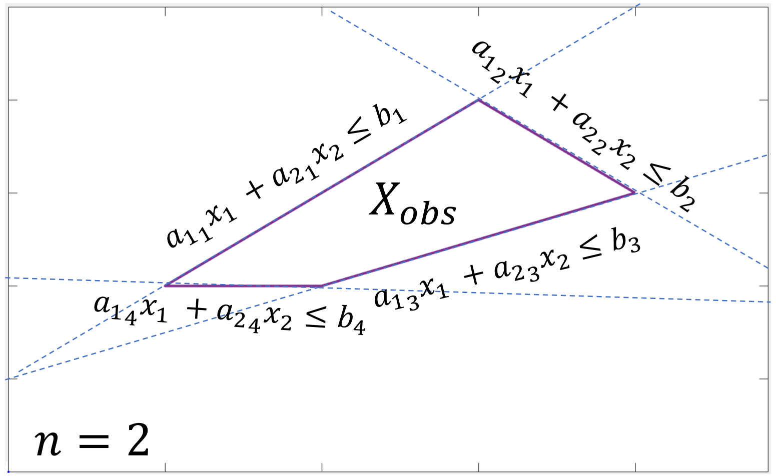
➢ To solve the chance constrained control problem:

Probabilistic Safety Constraints **Replace** ⇨ **Deterministic Linear Constraints In terms of (Mean & Variance) of x_k**

Probabilistic Safety Constraints

- **Obstacle set:** Conjunction of linear constraints:

$$X_{obs} = \{ \ (x_1, \dots, x_n) : \cap_{i=1}^{\ell} (a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i) \ \}$$ (convex linear set)



➢ **Chance Constraint:** probability of avoiding the obstacle

- $prob(x_k \notin X_{obs}) \geq 1 - \Delta_k \quad k = 1, \dots, T-1$

$\equiv$

- $prob(x_k \in X_{obs}) \leq \Delta_k \quad k = 1, \dots, T-1$

Probabilistic Safety Constraints

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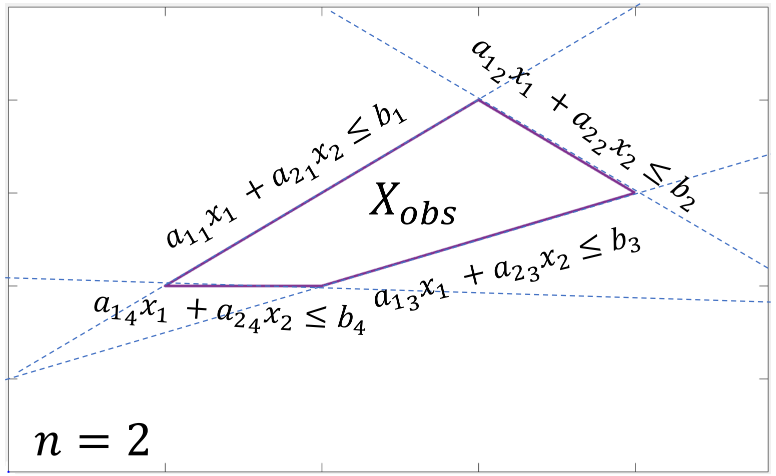

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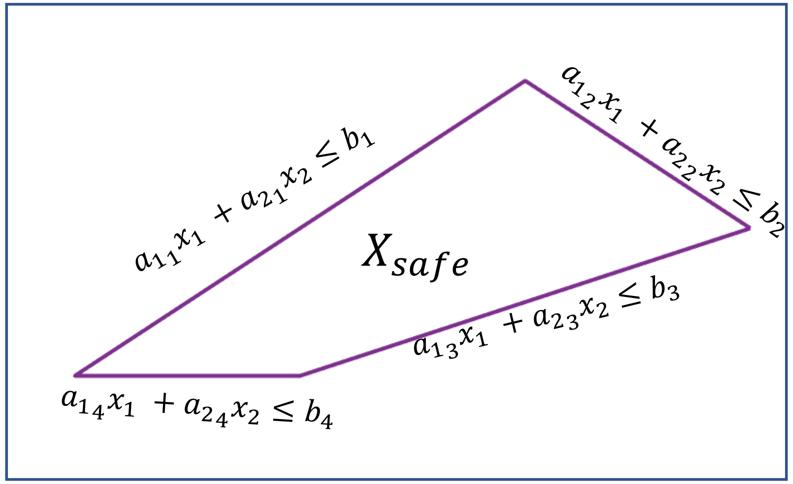
- **Safe Set:** Conjunction of linear constraints:

$$X_{safe} = \{ \ (x_1, \dots, x_n): \cap_{i=1}^{\ell} (a_{1_i} x_1 + \cdots + a_{n_i} x_n \le b_i) \ \}$$ (convex linear set)



➢ **Chance Constraint:** probability of remaining o the safe region

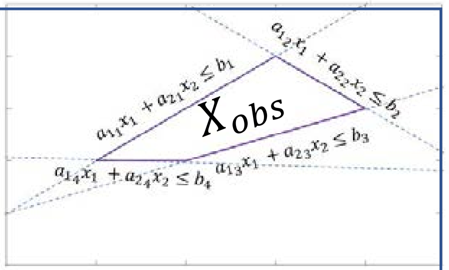
- $prob(x_k \notin X_{safe}) \le \Delta_k \quad k = 1, \dots, T-1$

$\equiv$

- $prob(x_k \in X_{safe}) \ge 1 - \Delta_k \quad k = 1, \dots, T-1$

Probabilistic Safety Constraints

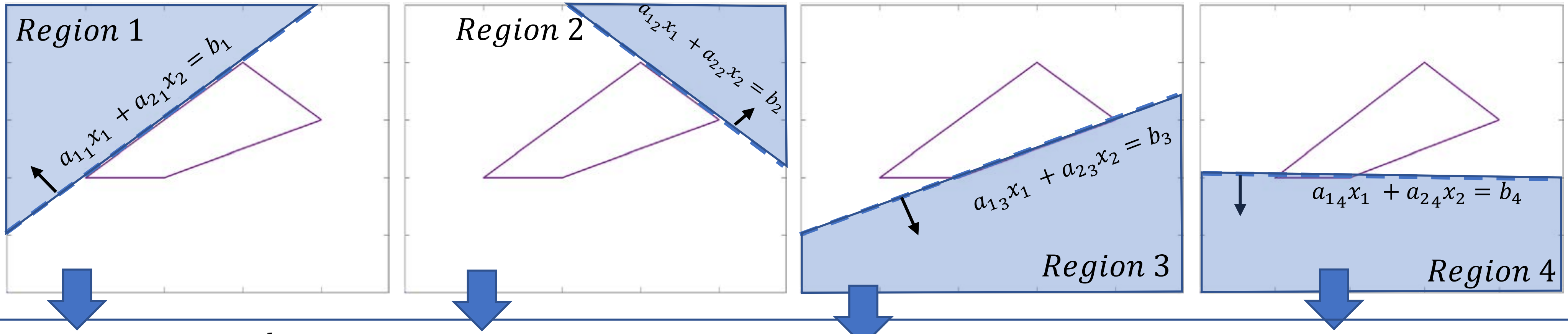
$$X_{obs} = \{\, (x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} \, (a_{1_i} x_1 + \dots + a_{n_i} x_n < b_i)\}$$



- Probabilistic safety constraints at time step k:

$$prob(x_k \notin X_{obs}) \geq 1 - \Delta_k$$

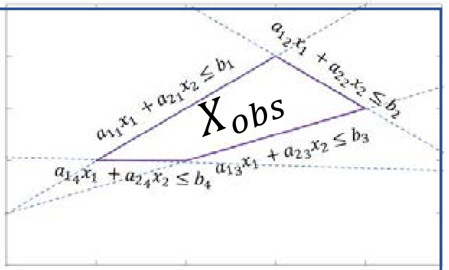
Safety Constraints



Deterministic Case: $a_{1_1}x_{1_k} + a_{2_1}x_2 \geq b_1$ **OR** $a_{1_2}x_1 + a_{2_2}x_2 \geq b_2$ **OR** $a_{1_3}x_1 + a_{2_3}x_2 \geq b_3$ **OR** $a_{1_4}x_1 + a_{2_4}x_2 \geq b_4$

Probabilistic Safety Constraints

$$X_{obs} = \{\, (x_1, \ldots, x_n) \colon \cap_{i=1}^{\ell} \, (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i)\}$$



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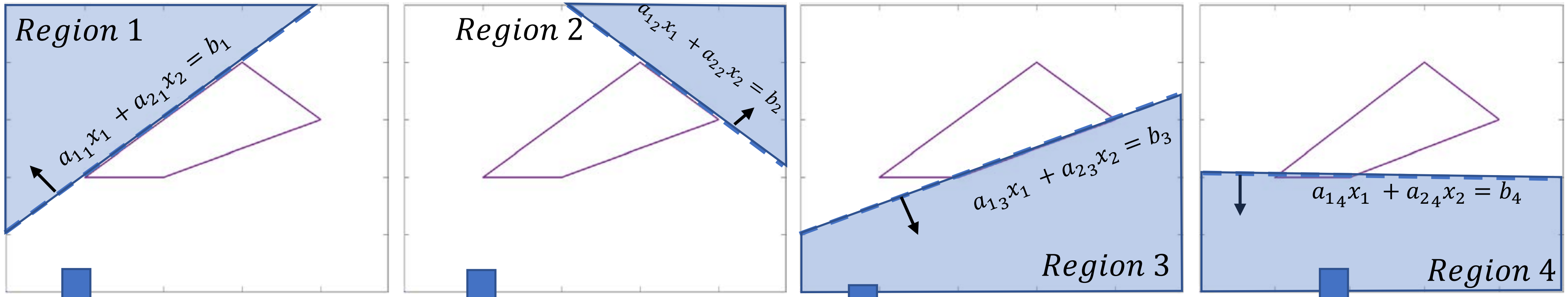

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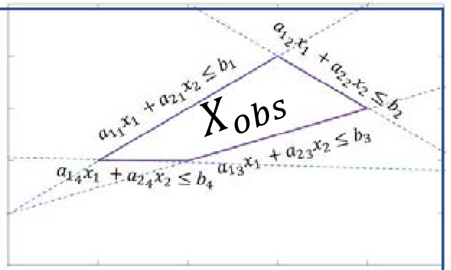
Chance Constraints:

$prob\{a_{1_1}x_1 + a_{2_1}x_2 \ge b_1\} \ge 1 - \Delta_k$ **OR** $prob\{a_{1_2}x_1 + a_{2_2}x_2 \ge b_2\} \ge 1 - \Delta_k$ **OR** $prob\{a_{1_3}x_1 + a_{2_3}x_2 \ge b_3\} \ge 1 - \Delta_k$ **OR** $prob\{a_{1_4}x_1 + a_{2_4}x_2 \ge b_4\} \ge 1 - \Delta_k$

Probabilistic Safety Constraints

$$X_{obs} = \{ (x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} (a_{1_i}x_1 + \dots + a_{n_i}x_n < b_i) \}$$



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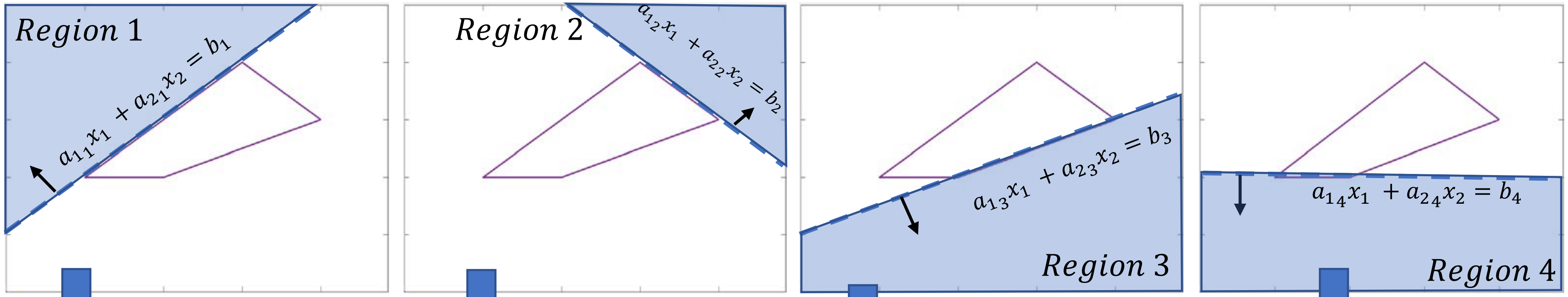
Safety Constraints



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- Joint chance constraints:

$$prob\left(x_k \notin \cap_{i=1}^{\ell} (a_{1_i}x_1 + \dots + a_{n_i}x_n < b_i)\right) \ge 1 - \Delta_k$$

- Disjunction of chance constraints:

$$\cup_{i=1}^{\ell} \ prob\left(a_{1_i}x_1 + \dots + a_{n_i}x_n \ge b_i\right) \ge 1 - \Delta_k$$

Chance Constrained Trajectory Optimization

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0, \dots, u_{T-1}]} E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob\big(x_k \notin \underbrace{\cap_{i=1}^{\ell} (a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i)}_{X_{obs}}\big) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$

$$E[x_T] = x_G$$

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$$\cup_{i=1}^{\ell} \; prob\big(a_{1_i} x_1 + \cdots + a_{n_i} x_n \geq b_i\big) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$

$$E[x_T] = x_G$$

- $prob\big(a_{1_i} x_1 + \cdots + a_{n_i} x_n \geq b_i\big) \xrightarrow{\textbf{Replace}}$ Deterministic Linear Constraints

- To replace $prob(a_{1_i}x_1 + \cdots + a_{n_i}x_n \geq b_i)$ with deterministic constraints:

  - Uncertainty propagation to obtain mean and variance of the x_k

  - Represent $prob(a_{1_i}x_1 + \cdots + a_{n_i}x_n \geq b_i)$ in terms of the mean and variance of x_k

Linear systems and Gaussian Uncertainty Propagation:

$$x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k \qquad x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma^2_{x_0}) \qquad \omega_k \sim \mathcal{N}(\overline{\omega}, \Sigma^2_{\omega_k})$$

- x_1 is a sum of normal distributions $\omega_0 \sim \mathcal{N}(\overline{\omega}_0, \Sigma^2_{\omega_0})$ and $x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma^2_{x_0})$ and deterministic input u_k.
  Hence $x_1 \sim \mathcal{N}(\bar{x}_1, \Sigma^2_{x_1})$

- At each time step k, states are normal $x_k \sim \mathcal{N}(\bar{x}_k, \Sigma^2_{x_k})$

➢ In uncertainty propagation, we just need to obtain mean and variance of the states.

Linear systems and Gaussian Uncertainty Propagation:

$$x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k \qquad x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma_{x_0}^2) \qquad \omega_k \sim \mathcal{N}(\bar{\omega}, \Sigma_{\omega_k}^2)$$

- **Mean:** $\bar{x}_{k+1} = \mathrm{E}[x_{k+1}] = \mathrm{E}[A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k]$

- By the assumption that u_k is deterministic i.e., $\mathrm{E}[u_k] = u_k$

$$\bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \bar{\omega}_k$$

- Linear in control input
- By recursion we can describe $\bar{x}_k$ as a linear function of known $\bar{x}_0, \bar{\omega}_k|_{k=0}^{k-1}$ and unknown $u_k|_{k=0}^{k-1}$

Linear systems and Gaussian Uncertainty Propagation:

$$x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k \qquad x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma^2_{x_0}) \qquad \omega_k \sim \mathcal{N}(\overline{\omega}, \Sigma^2_{\omega_k})$$

- **Variance:** $\Sigma^2_{x_{k+1}} = \mathrm{E}[(x_{k+1} - \mathrm{E}[x_{k+1}])^2]$

$\mathrm{E}[x_k - \bar{x}_k] = 0 \qquad \mathrm{E}[\omega_k - \overline{\omega}_k] = 0$

$B_{u_k} u_k$ term in x_{k+1} and $\bar{x}_{k+1}$ cancels out

- By the assumption that u_k is deterministic.

$$\Sigma^2_{x_{k+1}} = A_k \Sigma^2_{x_k} A_k^T + B_{\omega_k} \Sigma^2_{\omega_k} B_{\omega_k}^T$$

- Independent of the **control input**.
- Hence, given **initial state variance** and **disturbance variance**, we can obtain the covariance of states at future time steps beforehand.

- To replace $prob(a_{1_i}x_1 + \cdots + a_{n_i}x_n \geq b_i)$ with deterministic constraints:

- Uncertainty propagation to obtain mean and variance of the x_k

Mean is a linear in function of control input

$$\bar{x}_{k+1} = A_k\bar{x}_k + B_{u_k}u_k + B_{\omega_k}\bar{\omega}_k$$

Precompute the variance of the states over $k = 1, \ldots T-1$

$$\Sigma^2_{x_{k+1}} = A_k\Sigma^2_{x_k}A_k^T + B_{\omega_k}\Sigma^2_{\omega_k}B^T_{\omega_k}$$

- At each time step k, we represent $prob(a_{1_i}x_1 + \cdots + a_{n_i}x_n \geq b_i)$ in terms of the mean $\bar{x}_k$ and variance $\Sigma^2_{x_k}$.

  We leverage on the properties of Gaussian random variables.

Gaussian Linear Chance Constraints

Case 1:

- **Univariate** chance constraint

Probability($x \leq b$) $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$ $\bar{x}$:mean
Σ_x^2 :variance

Case 2:

- **Multivariate** chance constraint

Probability($a_1 x_1 + \cdots + a_n x_n \leq b$) $\leq \Delta$ $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$ $\bar{x}$:mean vector
Σ_x^2 :Covariance Matrix

Univariate Gaussian Linear Chance Constraints

- Consider the chance constraint Probability($x \leq b$) $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$
- We rewrite the chance constraint in terms of standard Normal distribution $\mathcal{N}(0,1)$

- Normal random variable: $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$ Pdf: $\frac{1}{\sqrt{2\pi\Sigma_x^2}} e^{-\frac{(x-\bar{x})^2}{2\Sigma_x^2}}$ $\bar{x}$:mean Σ_x^2 :variance Σ_x :deviation

Univariate Gaussian Linear Chance Constraints

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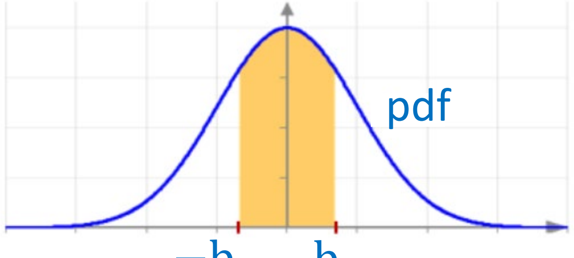
- Standard Normal random variable:

$x \sim \mathcal{N}(0,1)$

- pdf: $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
- CDF: $\phi(b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{x^2}{2}} dx$: Probability($x \leq b$)
- error function: $erf(b) = \frac{2}{\sqrt{\pi}} \int_{0}^{b} e^{-\frac{x^2}{2}} dx$: Probability($-$b $\leq x \leq b$)



- CDF and erf :

$$\phi(b) = \frac{1}{2}[1 + \mathrm{erf}(\frac{b}{\sqrt{2}})]$$ =Probability($x \leq b$)

Univariate Gaussian Linear Chance Constraints

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- Standard Normal random variable:

$x \sim \mathcal{N}(0,1)$

- pdf: $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
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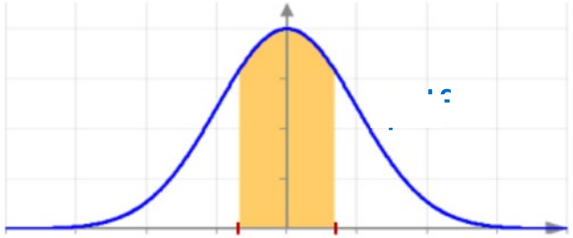


- CDF and erf :

$$\phi(b) = \frac{1}{2}[1 + \mathrm{erf}(\frac{b}{\sqrt{2}})]$$ =Probability($x \le b$)

- We can represent any normal distribution in terms of a **standard** Normal distribution $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2) \longrightarrow z = \frac{x - \bar{x}}{\Sigma_x} \sim \mathcal{N}(0,1)$

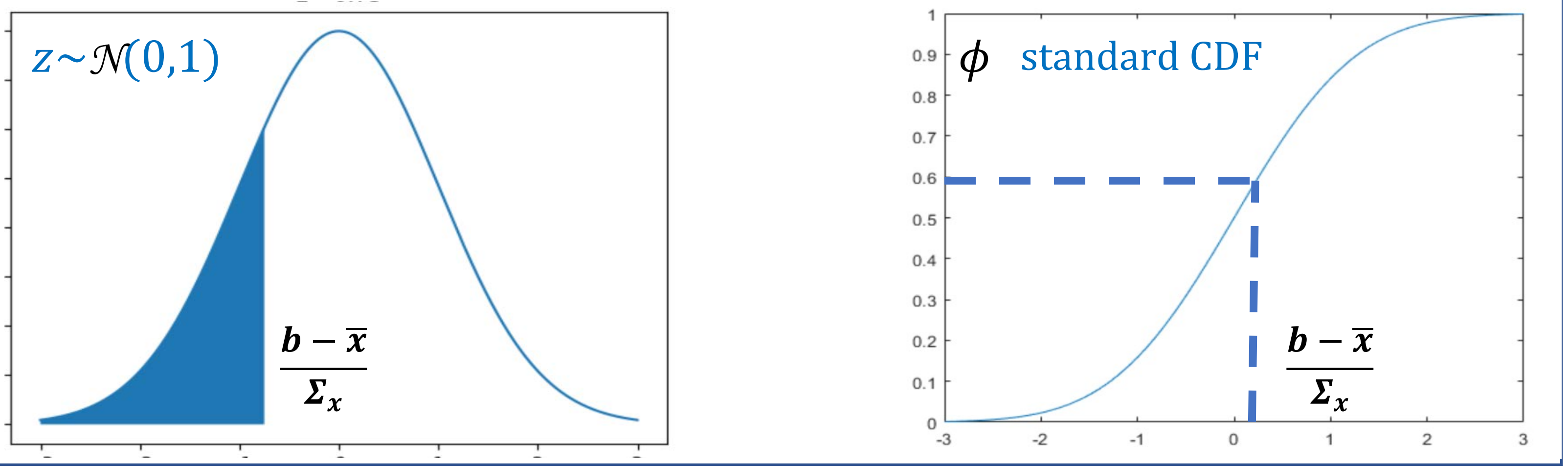
deviation

- $x = \bar{x} + \Sigma_x z$ $z \sim \mathcal{N}(0,1)$

Univariate Gaussian Linear Chance Constraints

- Probability($x \leq b$) = Probability($\bar{x} + \Sigma_x z \leq b$) = Probability($z \leq \frac{b-\bar{x}}{\Sigma_x}$) = $\phi\left(\frac{b-\bar{x}}{\Sigma_x}\right) = \frac{1}{2}[1 + \mathrm{erf}(\frac{b-\bar{x}}{\sqrt{2}\Sigma_x})]$ error function

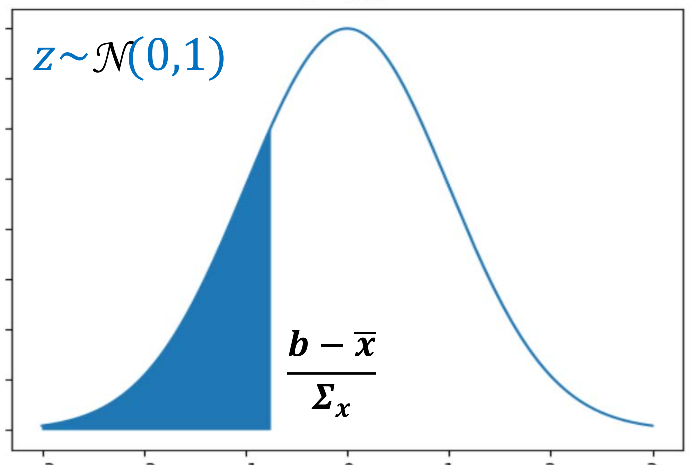
  $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$ $z \sim \mathcal{N}(0,1)$

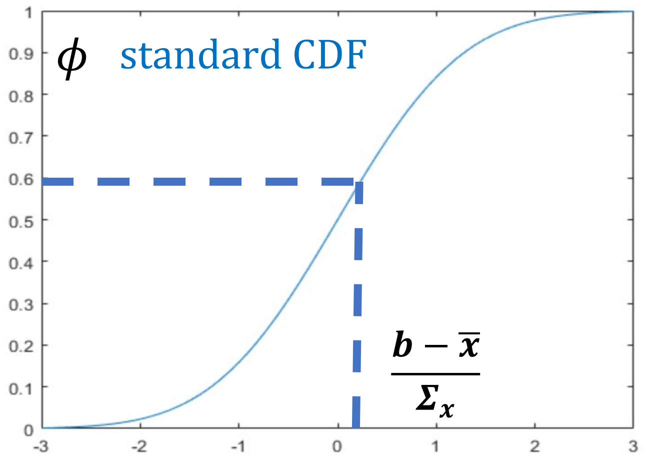


Univariate Gaussian Linear Chance Constraints

- Probability($x \le b$) = Probability($\bar{x} + \Sigma_x z \le b$) = Probability($z \le \frac{b-\bar{x}}{\Sigma_x}$) = $\phi\left(\frac{b-\bar{x}}{\Sigma_x}\right) = \frac{1}{2}[1 + \text{erf}(\frac{b-\bar{x}}{\sqrt{2}\Sigma_x})]$ error function

  $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$





- Probability($x \le b$) $\le \Delta$ Chance constraint

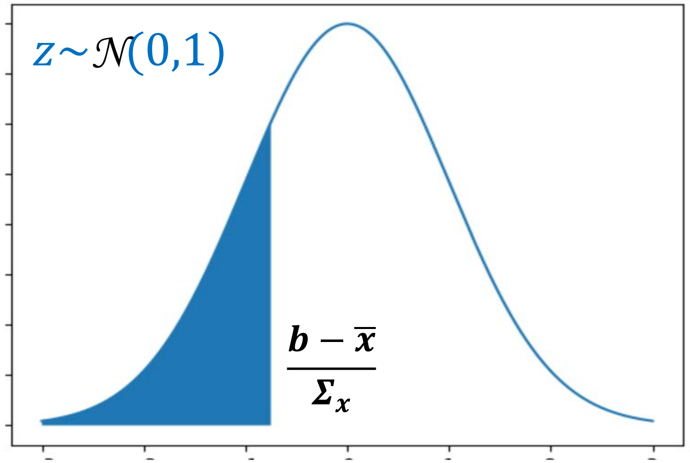
- $\phi\left(\frac{b-\bar{x}}{\Sigma_x}\right) \le \Delta \longrightarrow \frac{b-\bar{x}}{\Sigma_x} \le \phi^{-1}(\Delta) \longrightarrow$ $\boxed{\bar{x} \ge \text{b} - \Sigma_x \phi^{-1}(\Delta) \quad \text{Chance constraint in terms mean and variance}}$

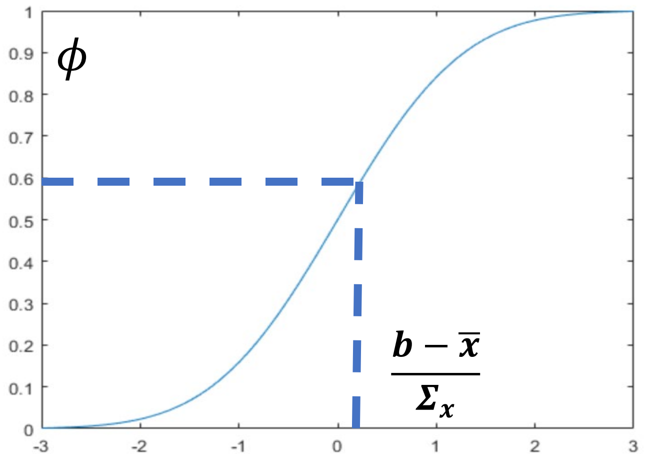
- To satisfy chance constraint, mean and variance of x should satisfy $\bar{x} \ge \text{b} - \Sigma_x \phi^{-1}(\Delta)$

Univariate Gaussian Linear Chance Constraints

- Probability($x \leq b$) = Probability($\bar{x} + \Sigma_x z \leq b$) = Probability($z \leq \frac{b-\bar{x}}{\Sigma_x}$) = $\phi\left(\frac{b-\bar{x}}{\Sigma_x}\right) = \frac{1}{2}[1 + \mathrm{erf}(\frac{b-\bar{x}}{\sqrt{2}\Sigma_x})]$

$x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$





- Probability($x \leq b$) $\leq \Delta$ Chance constraint

- $\frac{1}{2}[1 + \mathrm{erf}(\frac{b-\bar{x}}{\sqrt{2}\Sigma_x})] \leq \Delta \rightarrow \mathrm{erf}\left(\frac{b-\bar{x}}{\sqrt{2}\Sigma_x}\right) \leq 2\Delta - 1 \rightarrow \frac{b-\bar{x}}{\sqrt{2}\Sigma_x} \leq \mathrm{erf}^{-1}(2\Delta - 1) \longrightarrow \bar{x} \geq b - \sqrt{2}\Sigma_x \mathrm{erf}^{-1}(2\Delta - 1)$

Chance constraint in terms mean and variance: $\xrightarrow{\mathrm{erf}^{-1}(-a) = -\mathrm{erf}^{-1}(a)}$ $\bar{x} \geq b + \sqrt{2}\Sigma_x \mathrm{erf}^{-1}(1 - 2\Delta)$

- To satisfy chance constraint, mean and variance of x should satisfy $\bar{x} \geq b + \sqrt{2}\Sigma_x \mathrm{erf}^{-1}(1 - 2\Delta)$

Univariate Gaussian Linear Chance Constraints

Chance constraint

- Probability($x \leq b$) $\leq \Delta$

  $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

linear constraint

- $\bar{x} \geq \mathrm{b} - \Sigma_x \phi^{-1}(\Delta)$ (or $\bar{x} \geq b + \sqrt{2}\Sigma_x \mathrm{erf}^{-1}(1 - 2\Delta)$)

- For a given risk level Δ, constraints

$$\text{“} \bar{x} \geq \mathrm{b} - \Sigma_x \phi^{-1}(\Delta)\text{”} \qquad \text{“} \bar{x} \geq b + \sqrt{2}\Sigma_x \mathrm{erf}^{-1}(1 - 2\Delta)\text{”}$$

  are linear in $(\bar{x}, \Sigma_x)$.

Multivariate Gaussian Linear Chance Constraints

- Gaussian random vector: $[x_1, \dots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$ $\quad \bar{x}$ Mean Vector $\quad \Sigma_x^2$ Covariance Matrix

- **Chance Constraint:** Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$

Multivariate Gaussian Linear Chance Constraints

- Gaussian random vector: $[x_1, \dots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

  $\bar{x}$ Mean Vector

  Σ_x^2 Covariance Matrix

- **Chance Constraint:** Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$

- New random variable
  - $y = a_1 x_1 + \cdots + a_n x_n$
  - Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$ = Probability$(y \leq b)$

y is a linear sum of Gaussian random variables $y \sim \mathcal{N}(\bar{y}, \Sigma_y^2)$

Mean $\bar{y} = a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n$

Variance $\Sigma_y^2 = [a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]$

Multivariate Gaussian Linear Chance Constraints

- Gaussian random vector: $[x_1, \dots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

$\bar{x}$ Mean Vector

Σ_x^2 Covariance Matrix

- **Chance Constraint:** Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$

- New random variable
  - $y = a_1 x_1 + \cdots + a_n x_n$
  - Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$ = Probability$(y \leq b)$

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Mean $\bar{y} = a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n$

Variance $\Sigma_y^2 = [a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]$

- **Multivariate Chance Constraint:**

Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$

$[x_1, \dots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

$\Rightarrow$

- **Univariate Chance Constraint:**

Probability$(y \leq b)$

$y \sim \mathcal{N}(\bar{y}, \Sigma_y^2)$

Multivariate Gaussian Linear Chance Constraints

- **Multivariate Chance Constraint:**

Probability$(a_1x_1 + \cdots + a_nx_n \leq b)$

$[x_1, \ldots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- **Univariate Chance Constraint:**

Probability$(y \leq b)$

$y \sim \mathcal{N}(\bar{y}, \Sigma_y^2)$

Mean $\bar{y} = a_1\bar{x}_1 + \cdots + a_n\bar{x}_n$

Variance $\Sigma_y^2 = [a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]$

$$\text{Probability}(y \leq b) \xrightarrow{\; y = \bar{y} + \Sigma_y z \qquad z \sim \mathcal{N}(0,1) \;} \text{Probability}\left(z \leq \frac{b-\bar{y}}{\Sigma_y}\right) = \phi\left(\frac{b-\bar{y}}{\Sigma_y}\right) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{b-\bar{y}}{\sqrt{2}\Sigma_y}\right)\right]$$

Multivariate Gaussian Linear Chance Constraints

- **Multivariate Chance Constraint:**

Probability$(a_1x_1 + \cdots + a_nx_n \leq b)$

$[x_1, \dots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- **Univariate Chance Constraint:**

Probability$(y \leq b)$

$y \sim \mathcal{N}(\bar{y}, \Sigma_y^2)$

Mean $\bar{y} = a_1\bar{x}_1 + \cdots + a_n\bar{x}_n$

Variance $\Sigma_y^2 = [a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]$

Probability$(y \leq b)$ $\xrightarrow{y = \bar{y} + \Sigma_y z \qquad z \sim \mathcal{N}(0,1)}$ Probability$(z \leq \frac{b-\bar{y}}{\Sigma_y})$= $\phi\left(\frac{b-\bar{y}}{\Sigma_y}\right) = \frac{1}{2}[1 + \mathrm{erf}(\frac{b-\bar{y}}{\sqrt{2}\Sigma_y})]$

➢ Chance constraint Probability$(a_1x_1 + \cdots + a_nx_n \leq b) \leq \Delta$

Multivariate Gaussian Linear Chance Constraints

- **Multivariate Chance Constraint:**

  Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$

  $[x_1, \ldots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- **Univariate Chance Constraint:**

  Probability$(y \leq b)$

  $y \sim \mathcal{N}(\bar{y}, \Sigma_y^2)$

  Mean $\quad \bar{y} = a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n$

  Variance $\quad \Sigma_y^2 = [a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]$

Probability$(y \leq b)$ $\xrightarrow{\; y = \bar{y} + \Sigma_y z \quad z \sim \mathcal{N}(0,1) \;}$ Probability$(z \leq \frac{b - \bar{y}}{\Sigma_y})$= $\phi\left(\frac{b - \bar{y}}{\Sigma_y}\right) = \frac{1}{2}[1 + \text{erf}(\frac{b - \bar{y}}{\sqrt{2}\Sigma_y})]$

➢ Chance constraint Probability$(a_1 x_1 + \cdots + a_n x_n \leq b) \leq \Delta$

- Probability$(y \leq b) \leq \Delta$ $\longrightarrow$ $\bar{y} \geq \mathrm{b} - \Sigma_y \phi^{-1}(\Delta)$ (Linear in deviation Σ_y)

Multivariate Gaussian Linear Chance Constraints

- **Multivariate Chance Constraint:**

  Probability($a_1 x_1 + \cdots + a_n x_n \le b$)

  $[x_1, \ldots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- **Univariate Chance Constraint:**

  Probability($y \le b$)

  $y \sim \mathcal{N}(\bar{y}, \Sigma_y^2)$

  Mean $\quad \bar{y} = a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n$

  Variance $\quad \Sigma_y^2 = [a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]$

Probability($y \le b$) $\xrightarrow{\;y = \bar{y} + \Sigma_y z \qquad z \sim \mathcal{N}(0,1)\;}$ Probability($z \le \frac{b-\bar{y}}{\Sigma_y}$)= $\phi\left(\frac{b-\bar{y}}{\Sigma_y}\right) = \frac{1}{2}[1 + \mathrm{erf}(\frac{b-\bar{y}}{\sqrt{2}\Sigma_y})]$

➢ Chance constraint Probability($a_1 x_1 + \cdots + a_n x_n \le b$) $\le \Delta$

- Probability($y \le b$) $\le \Delta$ $\longrightarrow$ $\bar{y} \ge \mathrm{b} - \Sigma_y \phi^{-1}(\Delta)$ (Linear in deviation Σ_y)

➢ Chance constraint in terms mean and variance

$$a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n \ge \mathrm{b} - \sqrt{[a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]}\,\phi^{-1}(\Delta)$$

(Nonlinear in Covariance Σ_x^2)

Multivariate Gaussian Linear Chance Constraints

- **Multivariate Chance Constraint:**

Probability$(a_1 x_1 + \cdots + a_n x_n \leq b)$

$[x_1, \ldots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- **Univariate Chance Constraint:**

Probability$(y \leq b)$

$y \sim \mathcal{N}(\bar{y}, \Sigma_y^2)$

Mean $\quad \bar{y} = a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n$

Variance $\quad \Sigma_y^2 = [a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]$

Probability$(y \leq b)$ $\xrightarrow{\quad y = \bar{y} + \Sigma_y z \qquad z \sim \mathcal{N}(0,1) \quad}$ Probability$(z \leq \frac{b-\bar{y}}{\Sigma_y})$= $\phi\left(\frac{b-\bar{y}}{\Sigma_y}\right) = \frac{1}{2}[1 + \mathrm{erf}(\frac{b-\bar{y}}{\sqrt{2}\Sigma_y})]$

➢ Chance constraint Probability$(a_1 x_1 + \cdots + a_n x_n \leq b) \leq \Delta$

- Probability$(y \leq b) \leq \Delta$ $\longrightarrow$ $\bar{y} \geq b + \sqrt{2}\Sigma_y \mathrm{erf}^{-1}(1 - 2\Delta)$ (Linear in deviation Σ_y)

➢ Chance constraint in terms mean and variance

$$a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n \geq b + \sqrt{2[a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]}\mathrm{erf}^{-1}(1 - 2\Delta)$$ (Nonlinear in Covariance Σ_x^2)

Multivariate Gaussian Linear Chance Constraints

- Multivariate Chance constraint

Probability$(a_1x_1 + \cdots + a_nx_n \leq b) \leq \Delta$

$[x_1, \dots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- Linear constraint in mean

$$a_1\bar{x}_1 + \cdots + a_n\bar{x}_n \geq \mathrm{b} - \sqrt{[a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]}\phi^{-1}(\Delta)$$

(or $a_1\bar{x}_1 + \cdots + a_n\bar{x}_n \geq b + \sqrt{2[a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]}\mathrm{erf}^{-1}(1 - 2\Delta)$)

- For a given risk level Δ and **covariance matrix $\boldsymbol{\Sigma_x^2}$** , constraints

$$a_1\bar{x}_1 + \cdots + a_n\bar{x}_n \geq \mathrm{b} - \sqrt{[a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]}\phi^{-1}(\Delta)$$

$$a_1\bar{x}_1 + \cdots + a_n\bar{x}_n \geq b + \sqrt{2[a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]}\mathrm{erf}^{-1}(1 - 2\Delta)$$

are linear in mean vector $[\bar{x}_1, \dots., \bar{x}_n]$.

Multivariate Gaussian Linear Chance Constraints

- Multivariate Chance constraint

Probability$(a_1 x_1 + \cdots + a_n x_n \leq b) \leq \Delta$

$[x_1, \ldots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- Linear constraint in mean

$$a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n \geq \mathrm{b} - \sqrt{[a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]}\,\phi^{-1}(\Delta)$$

- Multivariate Chance constraint

$\phi^{-1}(-a) = -\phi^{-1}(1-a)$

Probability$(a_1 x_1 + \cdots + a_n x_n \geq b) \geq 1 - \Delta$

$[x_1, \ldots, x_n] \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

- Linear constraint in mean

$$a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n \geq \mathrm{b} + \sqrt{[a_1, \ldots, a_n]^T \Sigma_x^2 [a_1, \ldots, a_n]}\,\phi^{-1}(1 - \Delta)$$

Chance Constrained Trajectory Optimization

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0, \dots, u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob\big(x_k \notin \underbrace{\cap_{i=1}^{\ell} \; (a_{1_i} x_1 + \dots + a_{n_i} x_n < b_i)}_{X_{obs}}\big) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$

$$E[x_T] = x_G$$

1

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0, \dots, u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$\cup_{i=1}^{\ell} \; prob\big(a_{1_i} x_1 + \dots + a_{n_i} x_n \geq b_i\big) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$

$$E[x_T] = x_G$$

2

- $prob\big(a_{1_i} x_1 + \dots + a_{n_i} x_n \geq b_i\big)$ $\xrightarrow{\textbf{Replace}}$ Deterministic Linear Constraints

Chance Constrained Trajectory Optimization

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0, \dots, u_{T-1}]} E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob\Big(x_k \notin \underbrace{\cap_{i=1}^{\ell} \left(a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i\right)}_{X_{obs}}\Big) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$

3

$$E[x_T] = x_G$$

$$prob\left(x_k \notin \cap_{i=1}^{\ell} \left(a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i\right)\right) \geq 1 - \Delta_k \longrightarrow \cup_{i=1}^{\ell} \; prob\left(a_{1_i} x_1 + \cdots + a_{n_i} x_n \geq b_i\right) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$

- **Disjunctive Linear Program**

$$\min_{[u_0, \dots, u_{T-1}]} E[J(x_k, u_k)]$$

$$s.t. \quad \bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \bar{\omega}_k \qquad \Sigma^2_{x_{k+1}} = A_k \Sigma^2_{x_k} A_k^T + B_{\omega_k} \Sigma^2_{\omega_k} B^T_{\omega_k}$$

$$\cup_{i=1}^{\ell} \quad a_{1_i} \bar{x}_{1_k} + \cdots + a_{n_i} \bar{x}_{n_k} \geq \mathrm{b_i} + \sqrt{\left[a_{1_i}, \dots, a_{n_i}\right]^T \Sigma^2_{x_k} \left[a_{1_i}, \dots, a_{n_i}\right]} \phi^{-1}(1 - \Delta_\mathrm{k}) \qquad k = 1, \dots T-1$$

4

$$E[x_T] = x_G$$

Chance Constrained Trajectory Optimization

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0, \dots, u_{T-1}]} E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob\Big(x_k \notin \underbrace{\cap_{i=1}^{\ell} (a_{1_i} x_1 + \dots + a_{n_i} x_n < b_i)}_{X_{obs}}\Big) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$ 3

$$E[x_T] = x_G$$

$$prob\big(x_k \notin \cap_{i=1}^{\ell} (a_{1_i} x_1 + \dots + a_{n_i} x_n < b_i)\big) \geq 1 - \Delta_k \longrightarrow \cup_{i=1}^{\ell} prob\big(a_{1_i} x_1 + \dots + a_{n_i} x_n \geq b_i\big) \geq 1 - \Delta_k \qquad k = 1, \dots T-1$$

- **Disjunctive Linear Program**

$$\min_{[u_0, \dots, u_{T-1}]} E[J(x_k, u_k)]$$

$$s.t. \quad \bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \bar{\omega}_k \qquad \Sigma^2_{x_{k+1}} = A_k \Sigma^2_{x_k} A_k^T + B_{\omega_k} \Sigma^2_{\omega_k} B_{\omega_k}^T$$

$$\cup_{i=1}^{\ell} \quad a_{1_i} \bar{x}_{1_k} + \dots + a_{n_i} \bar{x}_{n_k} \geq \mathrm{b_i} + \sqrt{[a_{1_i}, \dots, a_{n_i}]^T \Sigma^2_{x_k} [a_{1_i}, \dots, a_{n_i}]}\,\phi^{-1}(1 - \Delta_\mathrm{k}) \qquad k = 1, \dots T-1$$ 4

$$E[x_T] = x_G$$

Linear in control   Precomputed using covariance dynamics   Known

Chance Constrained Trajectory Optimization

➢ **Disjunctive Linear Program**

$$\min_{[u_0, \dots, u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \qquad \bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \bar{\omega}_k \qquad \Sigma^2_{x_{k+1}} = A_k \Sigma^2_{x_k} A_k^T + B_{\omega_k} \Sigma^2_{\omega_k} B^T_{\omega_k}$$

$$\bigcup_{i=1}^{\ell} \quad a_{1_i} \bar{x}_{1_k} + \cdots + a_{n_i} \bar{x}_{n_k} \geq \mathrm{b_i} + \sqrt{[a_{1_i}, \dots, a_{n_i}]^T \Sigma^2_{x_k} [a_{1_i}, \dots, a_{n_i}]}\, \phi^{-1}(1 - \Delta_\mathrm{k}) \qquad k = 1, \dots T-1$$

$$E[x_T] = x_G$$

Disjunctive Linear Program → • Assign integer (0/1) variable for each constraint → Mixed Integer Linear Program (MILP) **solve** → Gurobi solver

Chance Constrained Trajectory Optimization

Instead of working with $prob\big(x_k \notin \underbrace{\cap_{i=1}^{\ell} \ (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i)}_{X_{obs}}\big) \geq 1 - \Delta_k$

we can work with $prob\big(x_k \in \underbrace{\cap_{i=1}^{\ell} \ (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i)}_{X_{obs}}\big) \leq \Delta_k$

Chance Constrained Trajectory Optimization

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0,\dots,u_{T-1}]} \; E[J(x_k, u_k)]$$

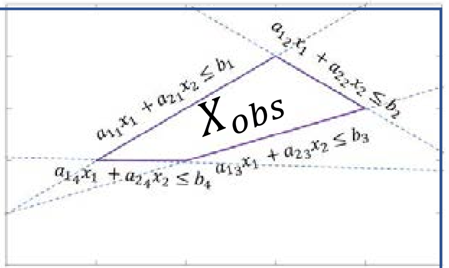
$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob\Big(x_k \in \underbrace{\cap_{i=1}^{\ell} \left(a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i\right)}_{X_{obs}}\Big) \le \Delta_k \qquad k = 1, \dots T-1$$

$$E[x_T] = x_G$$

Probabilistic Safety Constraints

$$X_{obs} = \{\, (x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} \, (a_{1_i} x_1 + \dots + a_{n_i} x_n < b_i) \}$$

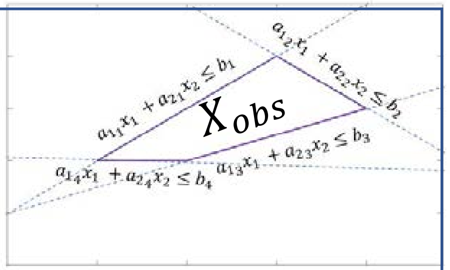


- Probabilistic safety constraints at time step k:

$$prob(x_k \in X_{obs}) \leq \Delta_k$$

$$prob\Big((x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} \, (a_{1_i} x_1 + \dots + a_{n_i} x_n < b_i) \Big) \leq \Delta_k$$

Probabilistic Safety Constraints



$$X_{obs} = \{ (x_1, \dots, x_n) : \cap_{i=1}^{\ell} (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i) \}$$

- Probabilistic safety constraints at time step k:

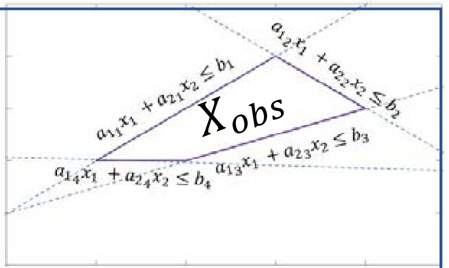
$$prob(x_k \in X_{obs}) \le \Delta_k$$

$$prob\left((x_1, \dots, x_n) : \cap_{i=1}^{\ell} (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i) \right) \le \Delta_k$$

Boole-Frechet Inequality

$$prob\left((x_1, \dots, x_n) : \cap_{i=1}^{l} (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i) \right) \le prob\left((x_1, \dots, x_n) : \; a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i \right) \qquad i = 1, \dots, \ell$$

Probabilistic Safety Constraints



$$X_{obs} = \{ (x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i) \}$$

- Probabilistic safety constraints at time step k:

$$prob(x_k \in X_{obs}) \leq \Delta_k$$

$$prob\left((x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i) \right) \leq \Delta_k$$

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$$prob\left((x_1, \dots, x_n) \colon \cap_{i=1}^{l} (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i) \right) \leq prob\left((x_1, \dots, x_n) \colon a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i \right) \qquad i = 1, \dots, \ell$$

- To satisfy the chance constraint, one of the individual chance constraints should be satisfied

$$prob(x_k \in X_{obs}) \geq \Delta_k \quad \Longrightarrow$$

Disjunction of chance constraints:

$$\cup_{i=1}^{\ell} \; prob\left(a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i \right) \leq \Delta_k$$

Chance Constrained Trajectory Optimization

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \ldots, T-1$

$$\min_{[u_0, \ldots, u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob\Big(x_k \in \underbrace{\cap_{i=1}^{\ell} \; (a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i)}_{X_{obs}}\Big) \le \Delta_k \qquad k = 1, \ldots T-1$$

$$E[x_T] = x_G$$

$$prob\big(x_k \in \cap_{i=1}^{\ell} \; (a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i)\big) \le \Delta_k \longrightarrow \cup_{i=1}^{\ell} \; prob\big(a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i\big) \le \Delta_k \qquad k = 1, \ldots T-1$$

- **Disjunctive Linear Program**

$$\min_{[u_0, \ldots, u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \quad \bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \bar{\omega}_k \qquad \Sigma^2_{x_{k+1}} = A_k \Sigma^2_{x_k} A_k^T + B_{\omega_k} \Sigma^2_{\omega_k} B_{\omega_k}^T$$

$$\cup_{i=1}^{\ell} \; a_{1_i} \bar{x}_{1_k} + \cdots + a_{n_i} \bar{x}_{n_k} > \mathrm{b_i} - \sqrt{[a_{1_i}, \ldots, a_{n_i}]^T \Sigma^2_{x_k} [a_{1_i}, \ldots, a_{n_i}]} \, \phi^{-1}(\Delta_\mathrm{k}) \qquad k = 1, \ldots T-1$$

$$E[x_T] = x_G$$

Chance Constrained Trajectory Optimization

$$prob\big(x_k \notin \underbrace{\cap_{i=1}^{\ell}\ (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i)}_{X_{obs}}\big) \geq 1 - \Delta_k \qquad k = 1, \ldots T - 1$$

$$\cup_{i=1}^{\ell}\quad a_{1_i}\bar{x}_{1_k} + \cdots + a_{n_i}\bar{x}_{n_k} \geq \mathrm{b_i} + \underbrace{\sqrt{[a_{1_i}, \ldots, a_{n_i}]^T \Sigma_{x_k}^2 [a_{1_i}, \ldots, a_{n_i}]}\,\phi^{-1}(1 - \Delta_\mathrm{k})}_{\textbf{Safety Margin}} \qquad k = 1, \ldots T - 1$$

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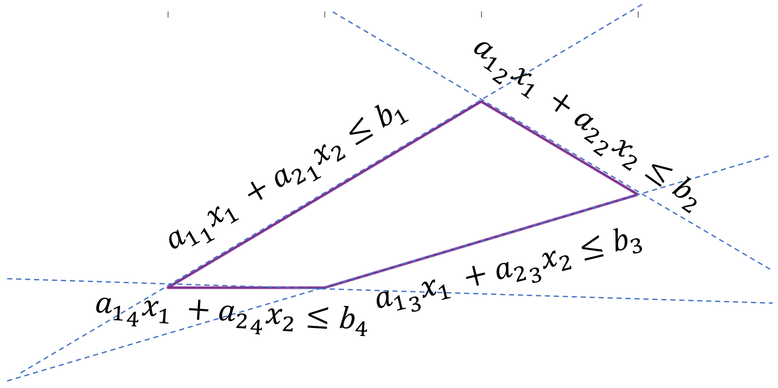
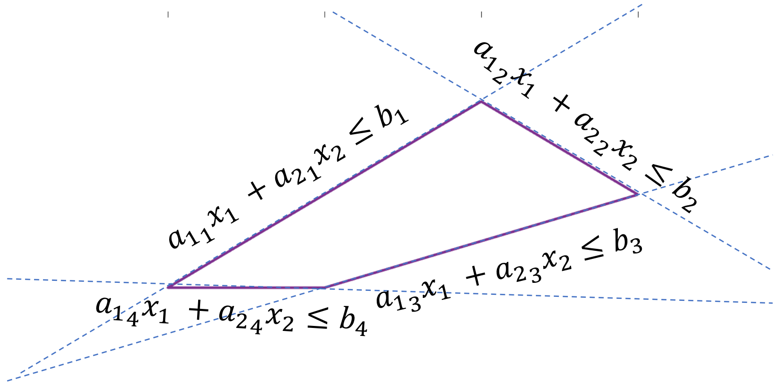
Obstacle:



Deterministic safety constraints:

$a_{1_1}x_{1_k} + a_{2_1}x_2 \ge b_1$ **OR** $a_{1_2}x_1 + a_{2_2}x_2 \ge b_2$ **OR**
$a_{1_3}x_1 + a_{2_3}x_2 \ge b_3$ **OR** $a_{1_4}x_1 + a_{2_4}x_2 \ge b_4$

Obstacle:



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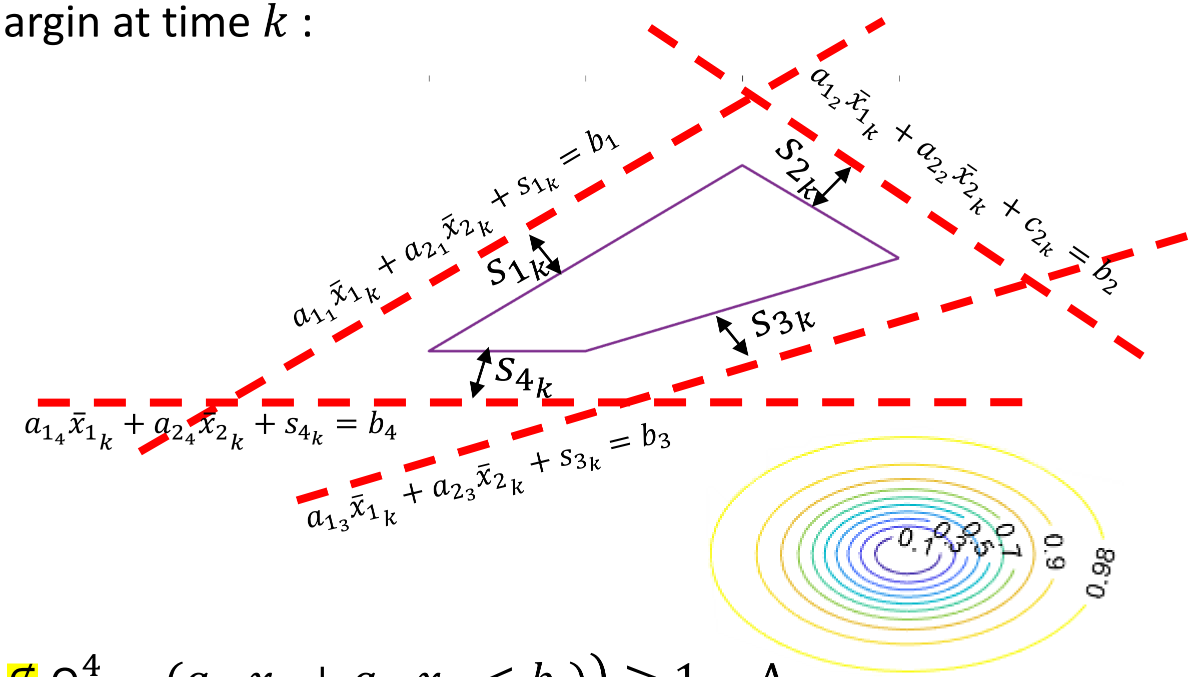
Safety Margin at time k :



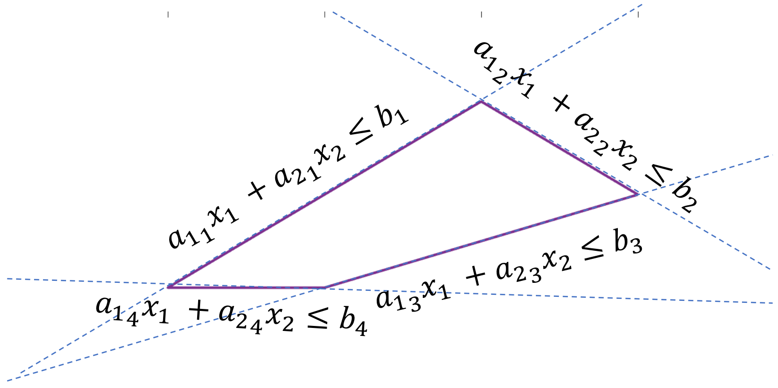
$$prob\left(x_k \notin \underbrace{\cap_{i=1}^{4}\ (a_{1_i}x_1 + a_{2_i}x_n < b_i)}_{X_{obs}}\right) \ge 1 - \Delta_k$$

$a_{1_1}\bar{x}_{1_k} + a_{2_1}\bar{x}_{2_k} \ge b_1 + s_{1_k}$ **OR** $a_{1_2}\bar{x}_{1_k} + a_{2_2}\bar{x}_{2_k} \ge b_2 + s_{2_k}$ **OR**
$a_{1_3}\bar{x}_{1_k} + a_{2_3}\bar{x}_{2_k} \ge b_3 + s_{3_k}$ **OR** $a_{1_4}\bar{x}_{1_k} + a_{2_4}\bar{x}_{2_k} \ge b_4 + s_{4_k}$

Safety margin : $s_{i_k} = \sqrt{[a_{1_i}, a_{2_i}]^T \Sigma_{x_k}^2 [a_{1_i}, a_{2_i}]}\ \phi^{-1}(1 - \Delta_k)$

- At each time k, chance constraint introduces a new obstacle (red) whose size depends on the safety margin and hence covariance of states.
- At each time k, if expected value of state stays out of the new obstacle (red) chance constraint is satisfied.

Obstacle:



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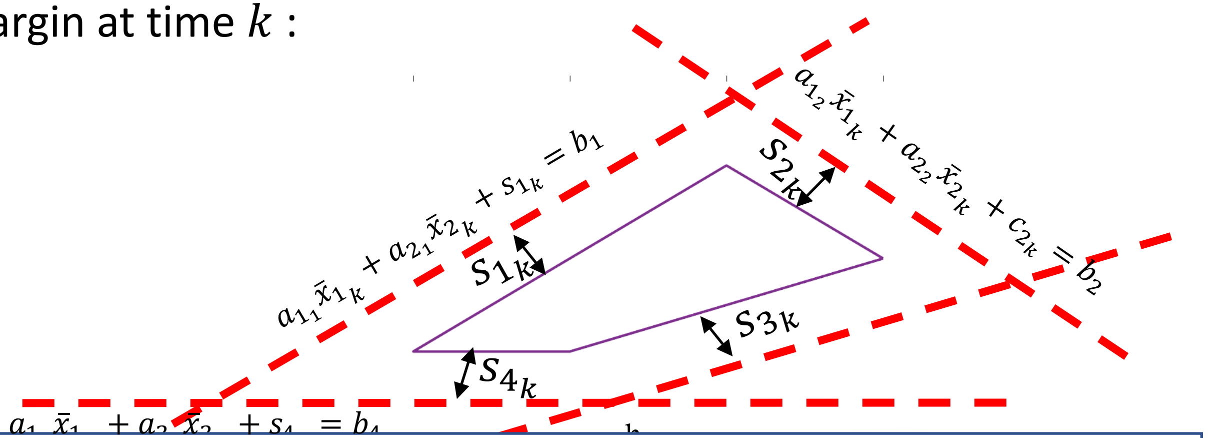

See *risk contours map* in lecture 11 for safety margins of **nonlinear obstacles** in the presence of **arbitrary probabilistic** uncertainties.

Deterministic safety constraints:

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X_{obs}

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$$prob\Big(x_k \notin \underbrace{\cap_{i=1}^{\ell} \; (a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i)}_{X_{obs}}\Big) \geq 1 - \Delta_k \quad k = 1, \ldots T-1$$

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- **Disjunctive Linear Program**

$$\min_{[u_0,\ldots,u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \quad \bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k}\bar{\omega}_k \qquad \Sigma^2_{x_{k+1}} = A_k \Sigma^2_{x_k} A_k^T + B_{\omega_k}\Sigma^2_{\omega_k} B^T_{\omega_k}$$

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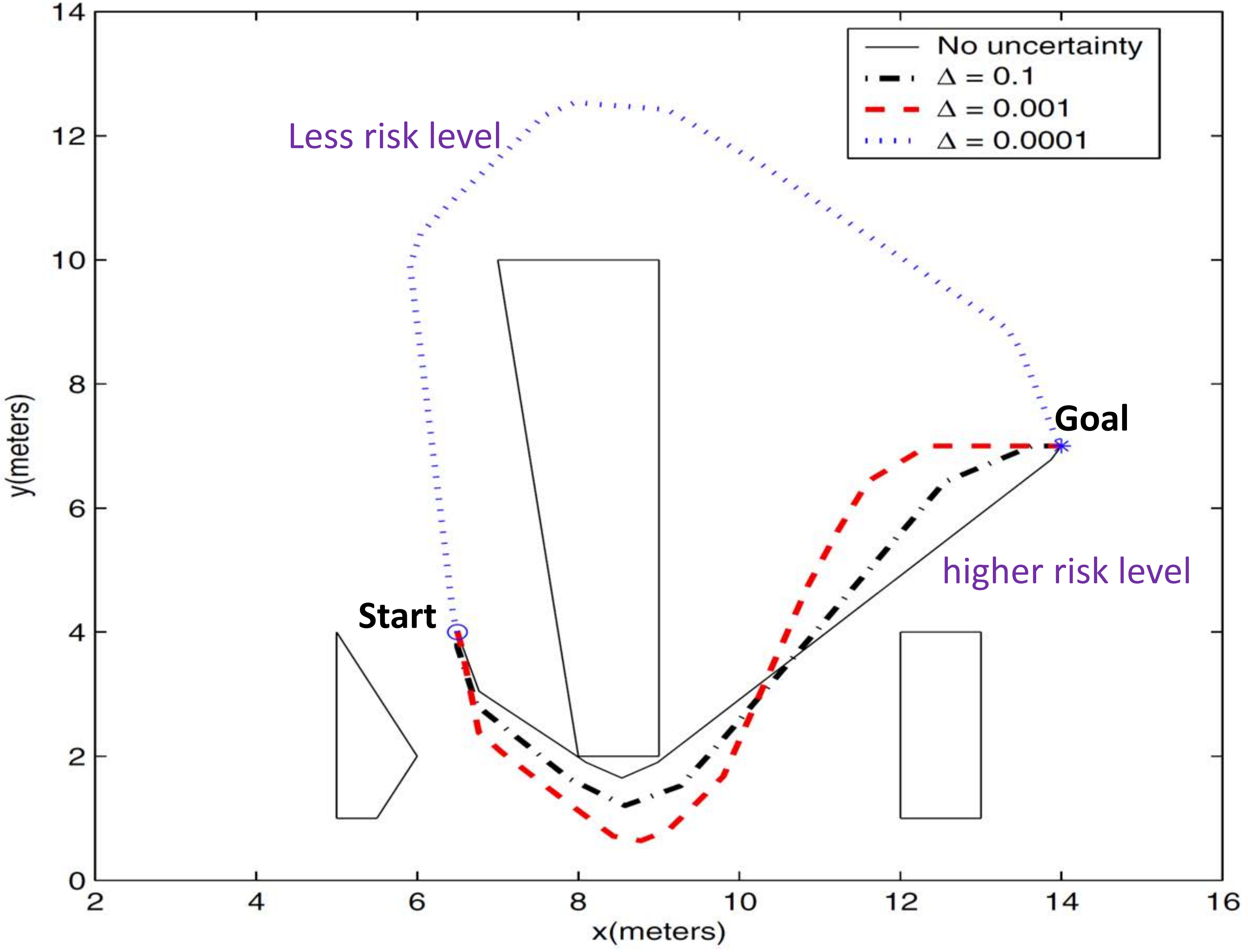
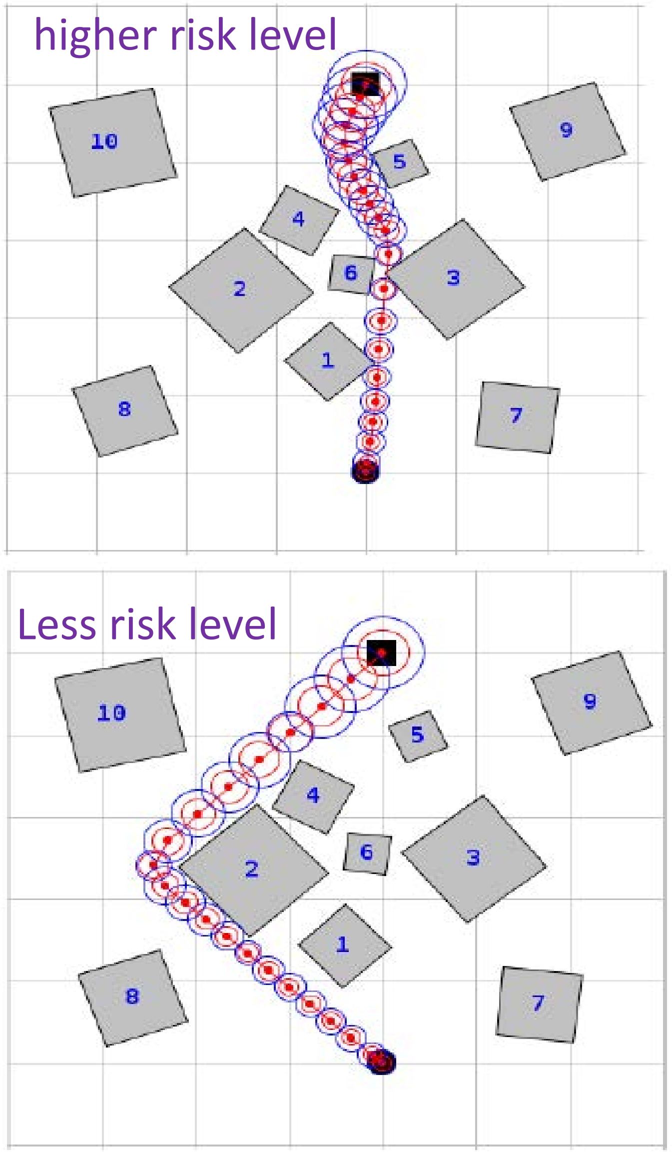
$$E[x_T] = x_G$$

Disjunctive Linear Program → • Assign integer (0/1) variable for each constraint → Mixed Integer Linear Program (MILP) $\xrightarrow{\text{solve}}$ Gurobi solver

- As risk level decrees, safety margins increase. Hence, obtained path stays far from obstacles.





- Each obstacle, introduces as set of disjunctive linear constraints.

Risk Allocation

- We assumed that at each time risk level Δ_k is given

$$prob\left(x_k \notin \underbrace{\cap_{i=1}^{\ell} \ (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i)}_{X_{obs}}\right) \geq 1 - \Delta_k \quad k = 1, \ldots T - 1$$

  - risk level at time k $\leq \Delta_k$

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- In practice, we are given the risk level of the plan over planning horizon $k = 1, \ldots T-1$.

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- We need to allocate risk for each time k

Given Δ $\longrightarrow$ Find Δ_k $k = 1, \ldots T-1.$

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1) Uniform risk allocation

2) Non-Uniform risk allocation

Uniform Risk Allocation

- In practice, we are given the risk level of the plan over planning horizon $k = 1, \dots T - 1$.

  - $\{$risk of the plan over $k = 1, \dots T - 1\} \leq \Delta$

- Risk allocated for each time steps:

$$\text{Given } \Delta \longrightarrow \Delta_k = \frac{\Delta}{T-1} \qquad k = 1, \dots T - 1.$$

Uniform Risk Allocation

- In practice, we are given the risk level of the plan over planning horizon $k = 1, \ldots T-1$.

  - {risk of the plan over $k = 1, \ldots T-1$} $\leq \Delta$

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- Risk allocated for each obstacle

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$$\text{Given } \Delta \longrightarrow \Delta_k = \frac{\Delta}{T-1} \qquad k = 1, \dots T-1.$$

- Risk allocated for each obstacle

$$prob\left(X_{obs_1}\ or X_{obs_2}\ or\ \dots .or\ X_{obs_M}\right) \leq \Delta_k$$

Boole-Frechet Inequality

$$prob\left(X_{obs_1}\ or X_{obs_2}\ or\ \dots .or\ X_{obs_M}\right) \leq \sum_{j=1}^{M} prob(X_{obs_j}) \leq \Delta_k \longrightarrow \Delta_{j_k} = \frac{\Delta_k}{M} \qquad j = 1, \dots M$$

obstacles

Uniform Risk Allocation

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obstacles

- Chance Constraint for obstacle j at time k:

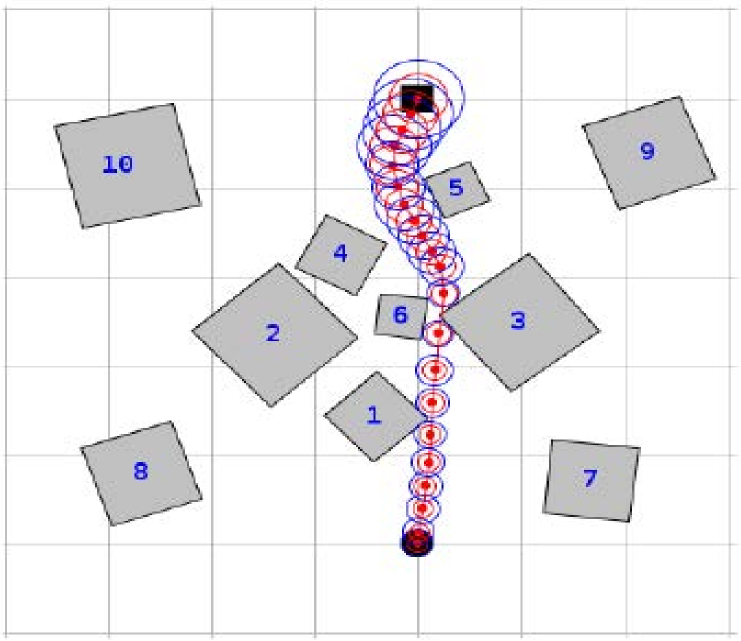
$$prob\left(x_k \notin \underbrace{\cap_{i=1}^{\ell}\ \left(a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i\right)}_{X_{obs_j}}\right) \geq 1 - \Delta_{j_k} \qquad \Delta_{j_k} = \frac{\Delta}{(T-1)M}$$

$k = 1, \dots T-1$ Time steps

$j = 1, \dots M$ obstacles

Optimization Based Risk Allocation

Non-Uniform risk allocation:
Example: Spend less risk when robot is far from obstacles.
Spend more risk when it gets close to obstacles.



Optimization Based Risk Allocation

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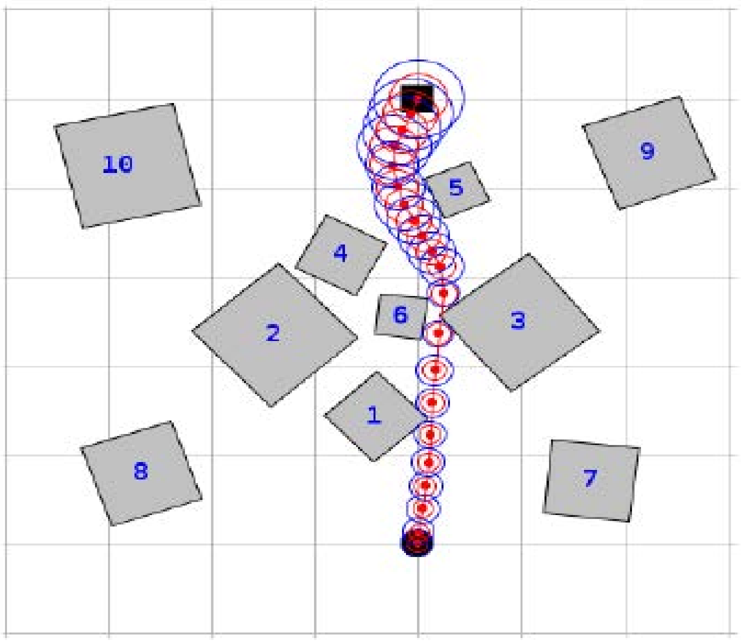
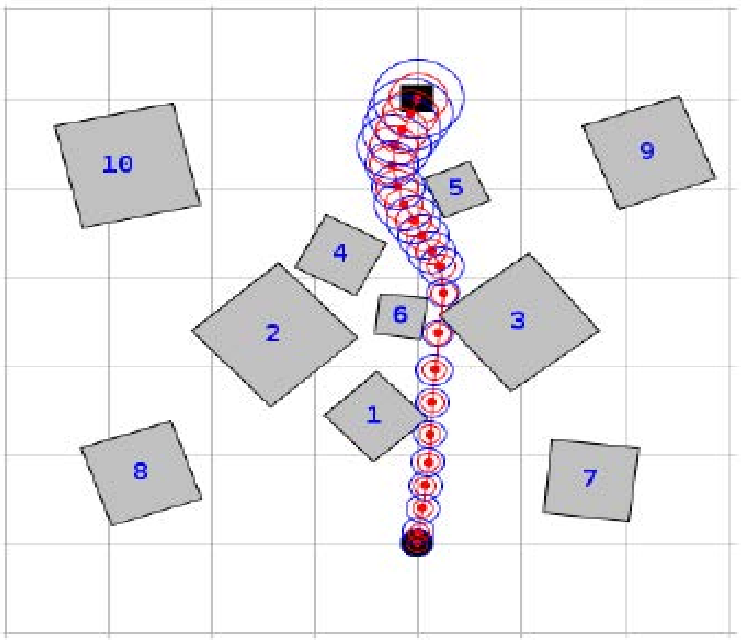
- We treat risk levels as new variables in the optimization.



Optimization Based Risk Allocation



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➢ If risk level Δ_k is unknown, we need to deal with function $\phi^{-1}(\Delta_\mathrm{k})$ in the optimization.

Optimization Based Risk Allocation

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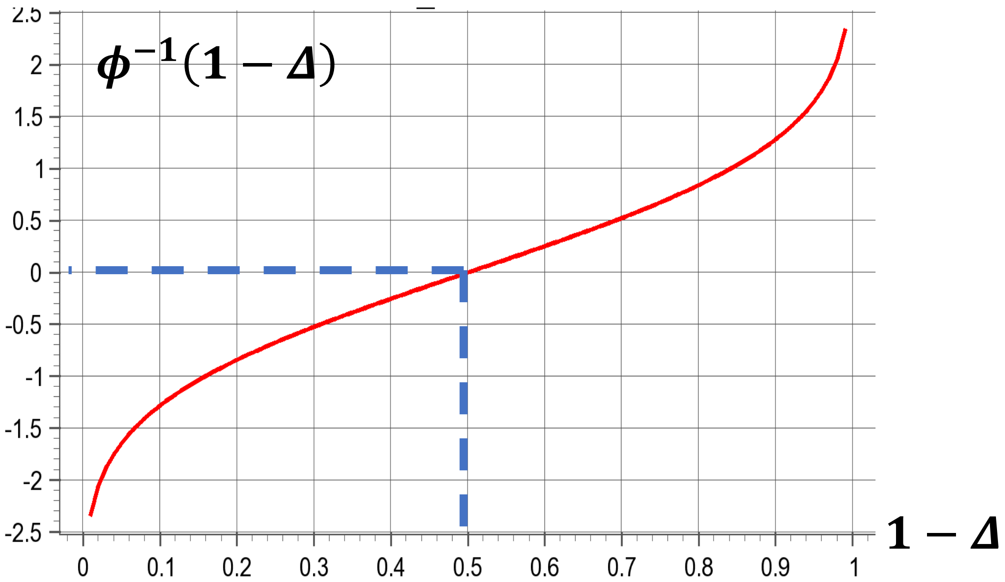
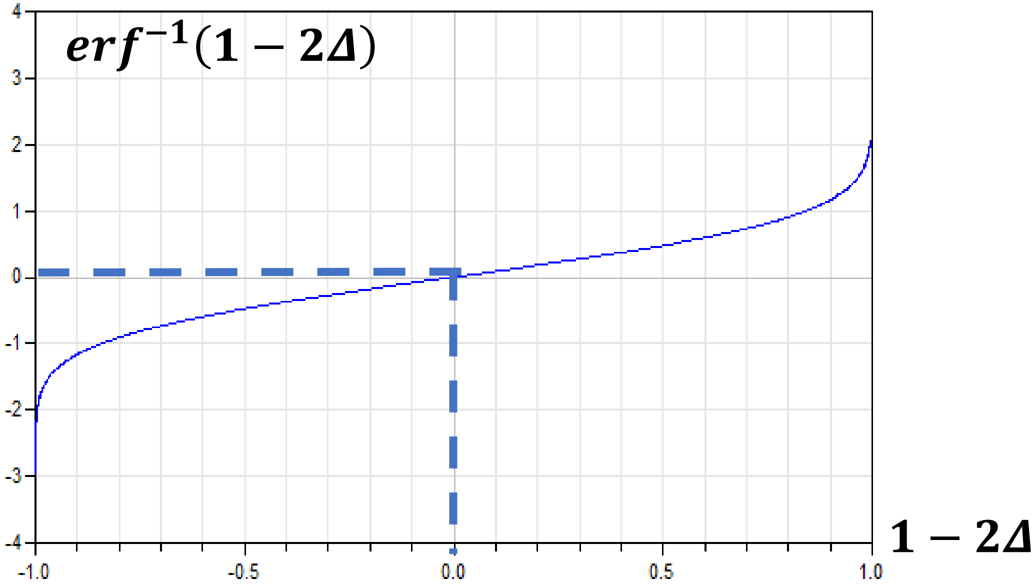


- $\phi^{-1}(1-\Delta)$ is convex function for $\Delta \leq 0.5$



- $\mathrm{erf}^{-1}(1-2\Delta)$ is convex function for $\Delta \leq 0.5$

Optimization Based Risk Allocation

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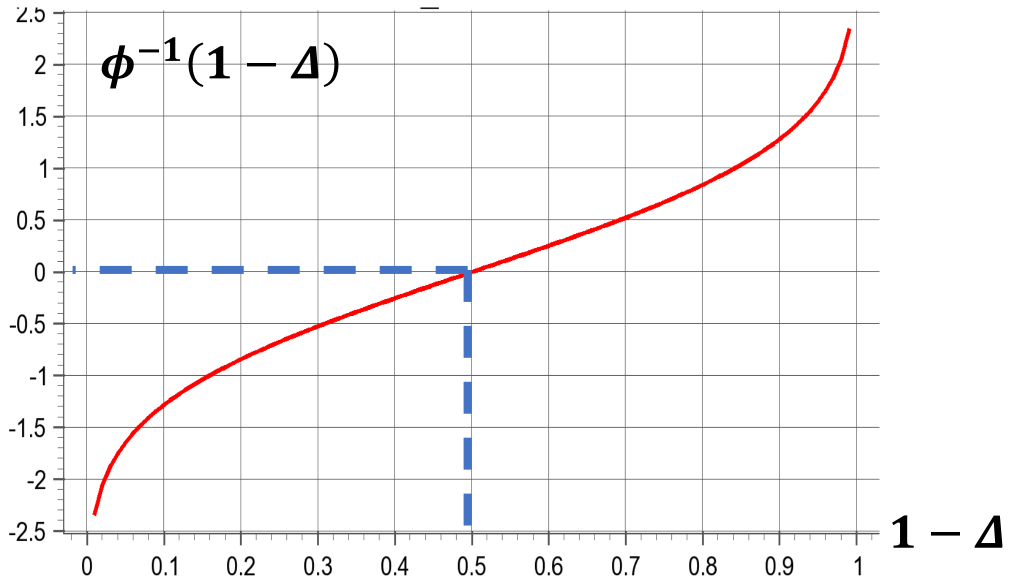

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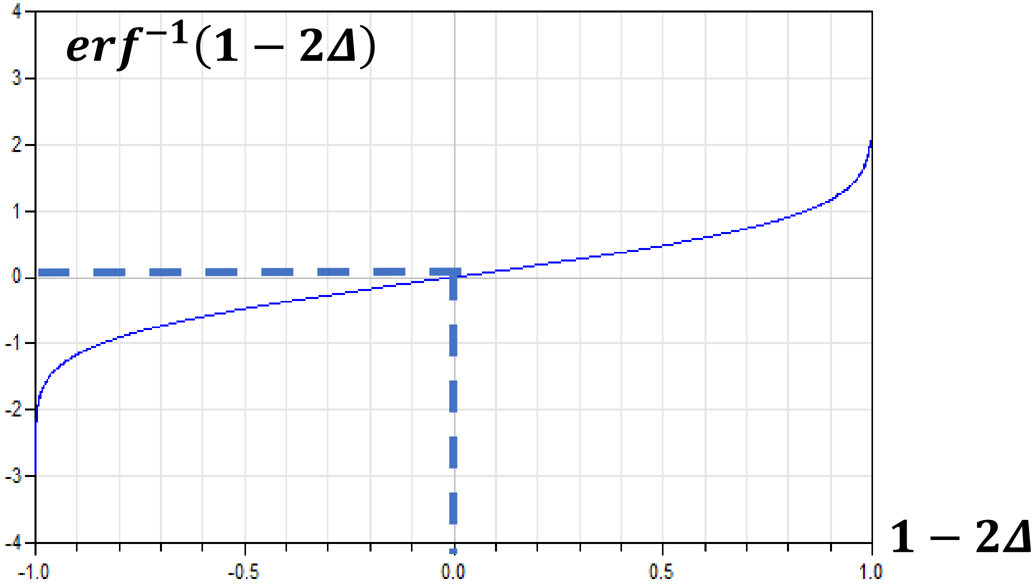

- $\mathrm{erf}^{-1}(1-2\Delta)$ is convex function for $\Delta \leq 0.5$

$$prob\Big(x_k \in \underbrace{\cap_{i=1}^{\ell}\ (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i)}_{X_{obs}}\Big) \leq \Delta_k \qquad \Delta_k \leq 0.5 \qquad k = 1, \dots T-1$$

$$\cup_{i=1}^{\ell}\quad a_{1_i}\bar{x}_{1_k} + \cdots + a_{n_i}\bar{x}_{n_k} \geq \mathrm{b_i} + \sqrt{\left[a_{1_i}, \dots, a_{n_i}\right]^T \Sigma_{x_k}^2 \left[a_{1_i}, \dots, a_{n_i}\right]}\,\phi^{-1}(1-\Delta_\mathrm{k}) \qquad \Delta_k \leq 0.5 \qquad k = 1, \dots T-1$$

Convex

Optimization Based Risk Allocation

- Chance Constraint for obstacle j at time k:

$$prob\big(x_k \in \underbrace{\cap_{i=1}^{\ell}\ (a_{1_i}x_1 + \cdots + a_{n_i}x_n < b_i)}_{X_{obs_j}}\big) \le \Delta_k$$

$k = 1, \ldots T-1$ Time steps

$j = 1, \ldots M$ obstacles

Convex constraints in terms of Δ_k and mean $\bar{x}_k$:

Obstacles j: $j = 1, \ldots M$

$$\cup_{i=1}^{\ell}\quad a_{1_i}\bar{x}_{1_k} + \cdots + a_{n_i}\bar{x}_{n_k} \ge b_i + \sqrt{[a_{1_i}, \ldots, a_{n_i}]^T \Sigma_{x_k}^2 [a_{1_i}, \ldots, a_{n_i}]}\,\phi^{-1}(1 - \Delta_{j_k})$$

$k = 1, \ldots T-1$

Time steps

$$\sum_{k=1}^{T-1}\sum_{j=1}^{M} \Delta_{j_k} \le \Delta$$

$$\Delta_{j_k} \le 0.5$$

Chance Constrained Trajectory Optimization

➢ **Disjunctive Linear Program**

$$\min_{[u_0, \dots, u_{T-1}]} \; E[J(x_k, u_k)]$$

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$$\Delta_{j_k} \leq 0.5$$

$$\sum_{k=1}^{T-1} \sum_{j=1}^{M} \Delta_{j_k} \leq \Delta \qquad k = 1, \dots T-1 \qquad j = 1, \dots M$$

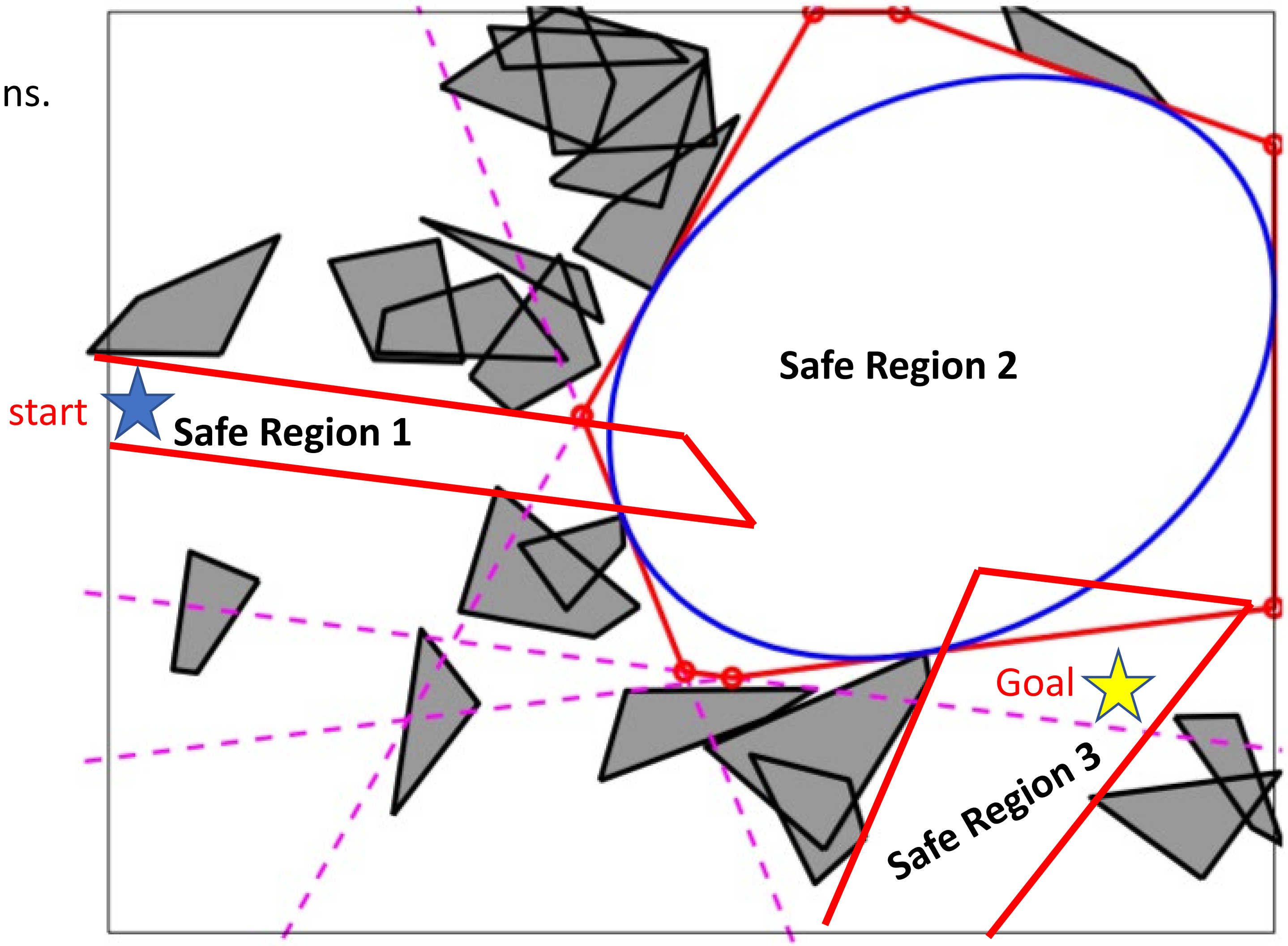
Disjunctive Linear Program → • Assign integer (0/1) variable for each constraint → Mixed Integer Linear Program (MILP) $\xrightarrow{\text{solve}}$ Gurobi solver

Chance Constrained Trajectory Optimization

- In the presence of large number of obstacles, chance constraints generates large number of constraints.
- To reduce the number of constraints, we can describe the safety in terms of the safe regions.

A Large Number of Obstacles

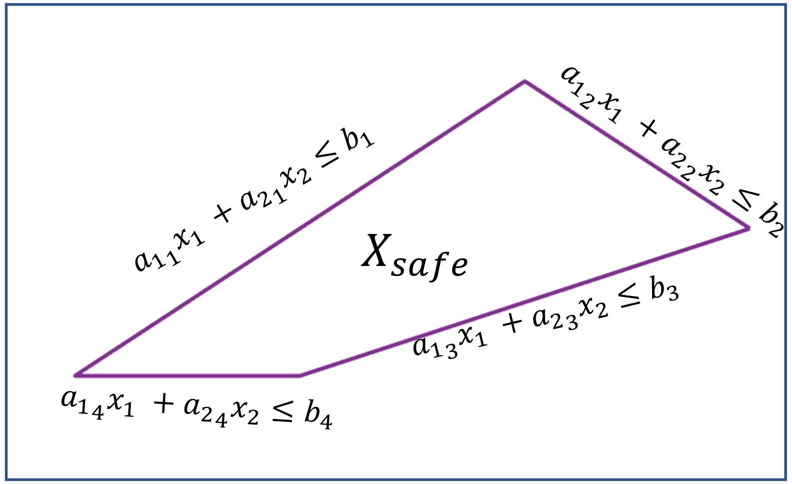
Chance constraints on safety in terms of safe regions.



- Robin Deits and Russ Tedrake, ,"Computing Large Convex Regions of Obstacle-Free Space through Semidefinite Programming", Workshop on the Algorithmic Foundations of Robotics (WAFR), 2014.
- Robin Deits, Russ Tedrake, "Efficient Mixed-Integer Planning for UAVs in Cluttered Environments", IEEE International Conference on Robotics and Automation (ICRA), 2015.

- **Safe Set:** Conjunction of linear constraints:

$$X_{safe} = \{ \ (x_1, \ldots, x_n): \cap_{i=1}^{\ell} (a_{1_i} x_1 + \cdots + a_{n_i} x_n \le b_i) \ \}$$ (convex linear set)



➢ **Chance Constraint:** probability of remaining o the safe region

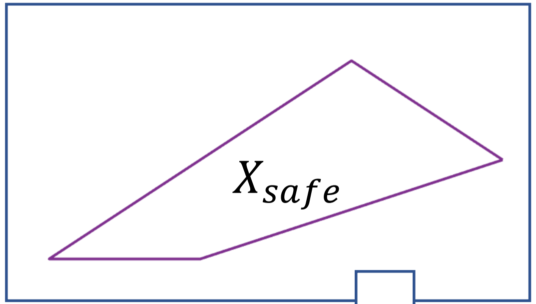
- $prob(x_k \notin X_{safe}) \le \Delta_k \quad k = 1, \ldots, T-1$

Probabilistic Safety Constraints

$$X_{safe} = \{\, (x_1, \dots, x_n) : \cap_{i=1}^{\ell} \, (a_{1_i} x_1 + \cdots + a_{n_i} x_n \leq b_i) \}$$

- Probabilistic safety constraints at time step k:

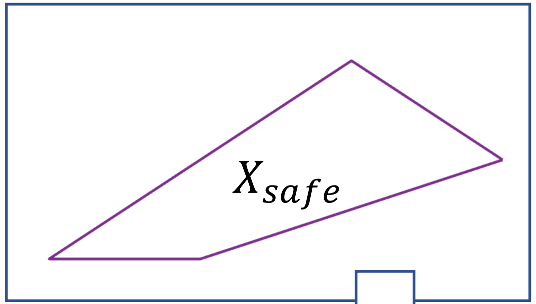
$$prob\left(x_k \notin X_{safe}\right) \leq \Delta_k$$



$$prob\left((x_1, \dots, x_n) : \cup_{i=1}^{\ell} \, (a_{1_i} x_1 + \cdots + a_{n_i} x_n > b_i) \right) \leq \Delta_k$$

Probabilistic Safety Constraints

$$X_{safe} = \{\, (x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} \, (a_{1_i} x_1 + \dots + a_{n_i} x_n \le b_i) \}$$



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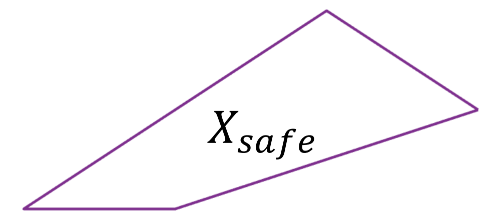
Boole-Frechet inequality

$$prob\left((x_1, \dots, x_n) \colon \cup_{i=1}^{l} \, (a_{1_i} x_1 + \dots + a_{n_i} x_n > b_i) \right) \le \sum_{i=1}^{\ell} prob\left((x_1, \dots, x_n) \colon \; a_{1_i} x_1 + \dots + a_{n_i} x_n > b_i \right) \le \Delta_k$$

Probabilistic Safety Constraints

$$X_{safe} = \{ (x_1, \dots, x_n) \colon \cap_{i=1}^{\ell} (a_{1_i} x_1 + \cdots + a_{n_i} x_n \leq b_i)\}$$



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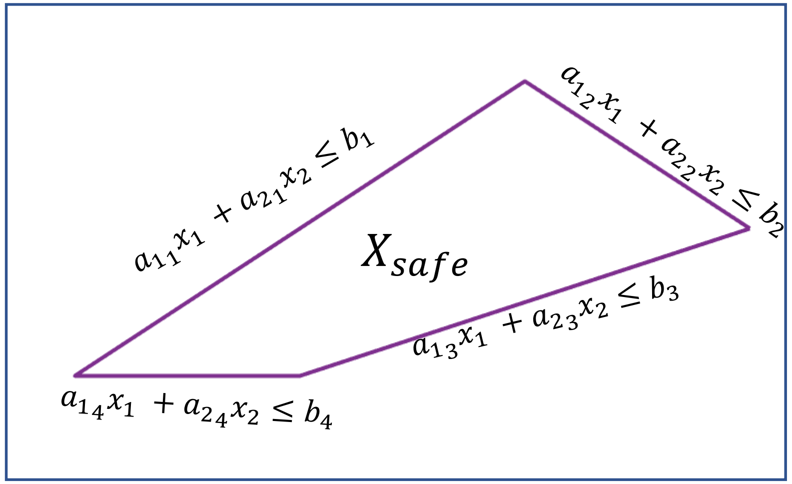
$$prob(x_k \notin X_{safe}) \leq \Delta_k \quad \Rightarrow \quad \begin{array}{l} prob(a_{1_i} x_1 + \cdots + a_{n_i} x_n > b_i) \leq \Delta_{i_k} \qquad i = 1, \dots, \ell \\ \sum_{i=1}^{\ell} \Delta_{i_k} \leq \Delta_k \end{array}$$

- **Safe Set:** Conjunction of linear constraints:

$$X_{safe} = \{ \ (x_1, \dots, x_n) : \cap_{i=1}^{\ell} (a_{1_i} x_1 + \dots + a_{n_i} x_n \le b_i) \ \}$$ (convex linear set)



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- $prob(x_k \notin X_{safe}) \le \Delta_k \quad k = 1, \dots, T-1$

$$prob(a_{1_i} x_1 + \dots + a_{n_i} x_n > b_i) \le \Delta_{i_k} \quad i = 1, \dots, \ell$$

$$\sum_{i=1}^{\ell} \Delta_{i_k} \le \Delta_k$$

Set of linear constraints:

$$a_1 \bar{x}_1 + \dots + a_n \bar{x}_n < \mathrm{b} + s_k \qquad i = 1, \dots, \ell$$

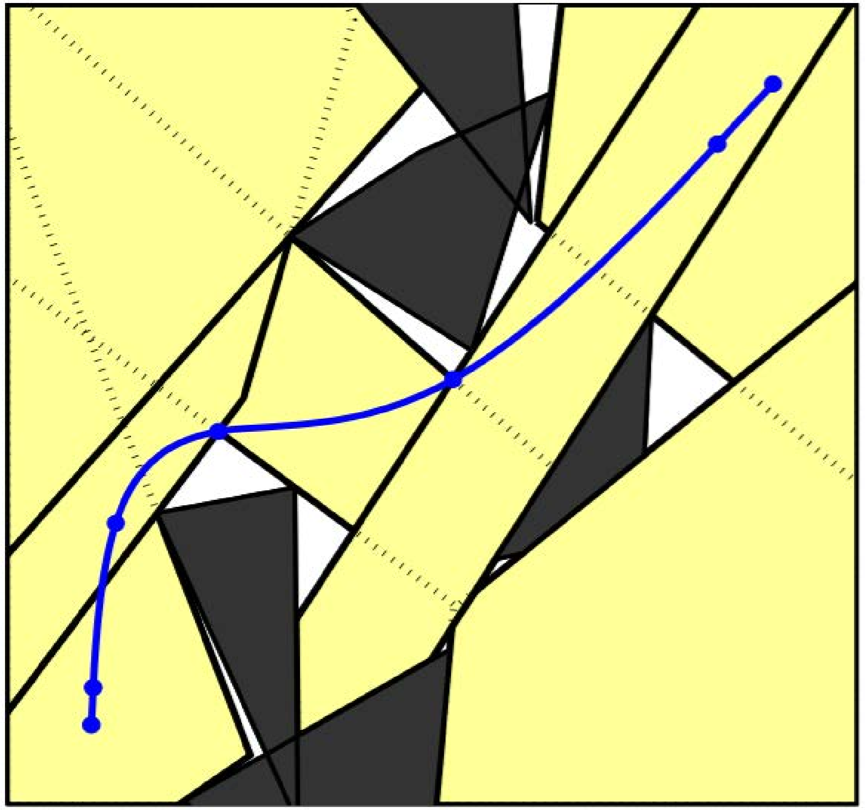
$$s_k = \sqrt{[a_1, \dots, a_n]^T \Sigma_x^2 [a_1, \dots, a_n]}\ \phi^{-1}(\Delta)$$

$$\sum_{i=1}^{\ell} \Delta_{i_k} \le \Delta_k$$

➢ Given, i) a set of convex regions that covers the obstacle-free space, and ii) initial and goal point

We can formulate the trajectory planning problem as mixed-integer Convex optimization.

- Integer variable for each convex region to choose the sequence of regions to construct the trajectory.
- Convex optimization for each convex region.



Given convex obstacles, computes set of convex free sets:

- Robin Deits and Russ Tedrake, ,"Computing Large Convex Regions of Obstacle-Free Space through Semidefinite Programming", Workshop on the Algorithmic Foundations of Robotics (WAFR), 2014.

Summary of Chance Constrained Trajectory Optimization:

- Replace joint safety chance constraints with disjoint chance constraints
- Replace disjoint chance constraints with deterministic convex constraints in terms of mean and variance of the states.
- Solve Mixed-Integer Convex optimization.

- Lars Blackmore and Hui Li and Brian Williams ,"A Probabilistic Approach to Optimal Robust Path Planning with Obstacles", American Control Conference, 2006.
- Lars Blackmore, Masahiro Ono, "Convex Chance Constrained Predictive Control Without Sampling" AIAA Guidance, Navigation, and Control Conference Chicago, Illinois, 2009.
- Masahiro Ono and Brian C. Williams, "Iterative Risk Allocation: A New Approach to Robust Model Predictive Control with a Joint Chance Constraint", IEEE Conference on Decision and Control, 2008.
- Masahiro Ono, "Closed-Loop Chance-Constrained MPC with Probabilistic Resolvability" 51st IEEE Conference on Decision and Control, December 10-13, 2012. Maui, Hawaii, USA
- Lars Blackmore,, "Robust Path Planning and Feedback Design under Stochastic Uncertainty", AIAA Guidance, Navigation and Control Conference and Exhibit, 18 - 21 August 2008, Honolulu, Hawaii
- Marcio da Silva Arantes, Claudio Fabiano Motta Toledo , Brian Charles Williams, and Masahiro Ono,"Collision-Free Encoding for Chance-Constrained Nonconvex Path Planning "IEEE TRANSACTIONS ON ROBOTICS, VOL. 35, NO. 2, APRIL 2019
- Masahiro Ono, "Joint Chance-Constrained Model Predictive Control with Probabilistic Resolvability", American Control Conference Fairmont Queen Elizabeth, Montréal, Canada, June 27-June 29, 2012

We can use these results in the different setup:

i) RRT*, ii) PRM, iii) Motion Primitive

Chance constrained RRT*

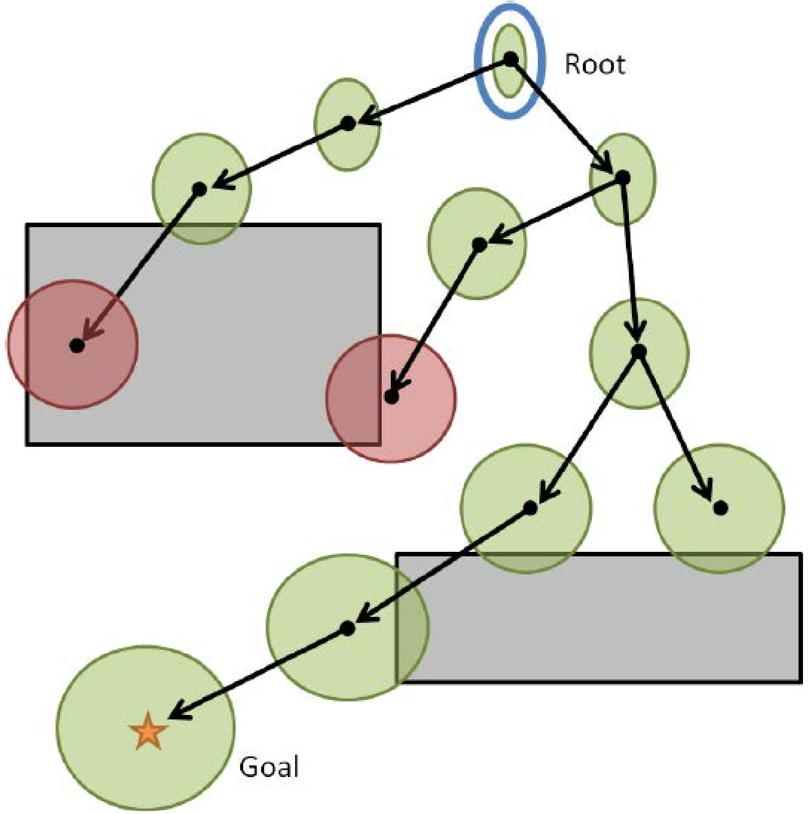
Chance constrained RRT*

Uncertain System:

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

$\omega_k \sim$ iven Probability distribution

u_k: Given controller to steer the system toward the sampled point x



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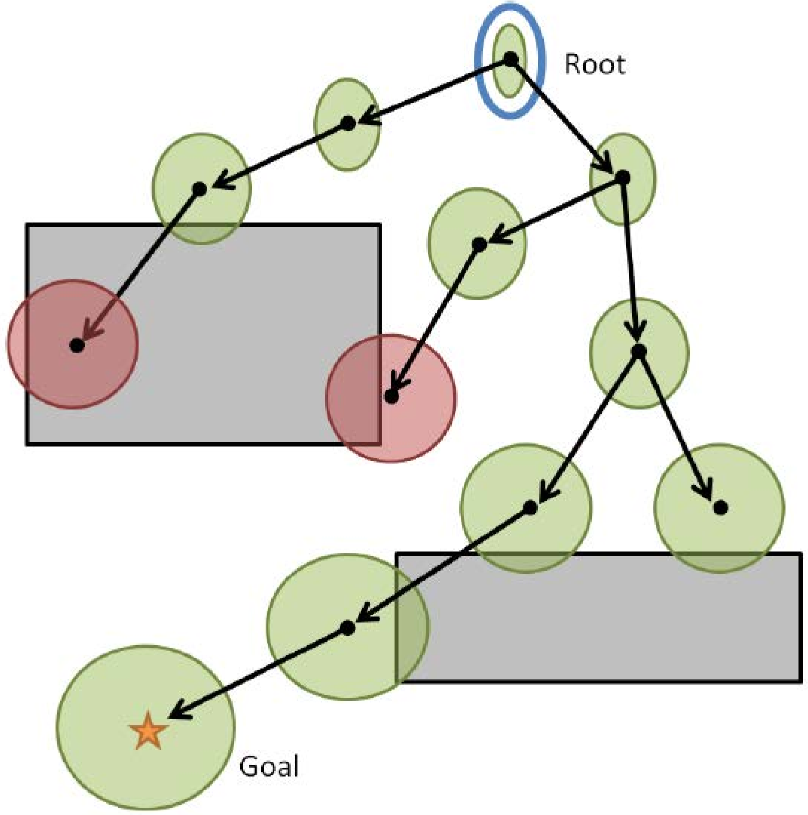


- Linearize the nonlinear system around the sampled points

  - $A_k = \frac{\partial f}{\partial x}(x_k^*, u_k^*, 0)$
  - $B_{u_k} = \frac{\partial f}{\partial u}(x_k^*, u_k^*, 0)$
  - $B_{\omega_k} = \frac{\partial f}{\partial \omega}(x_k^*, u_k^*, 0)$

$$\hat{x}_k = x_k - x_k^* \qquad \hat{u}_k = u_k - u_k^*$$

Linearized Dynamic: $\hat{x}_{k+1} = A_k \hat{x}_k + B_{u_k} \hat{u}_k + B_{\omega_k} \omega_k$

Chance constrained RRT*

Uncertain System:

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

$\omega_k \sim$ iven Probability distribution

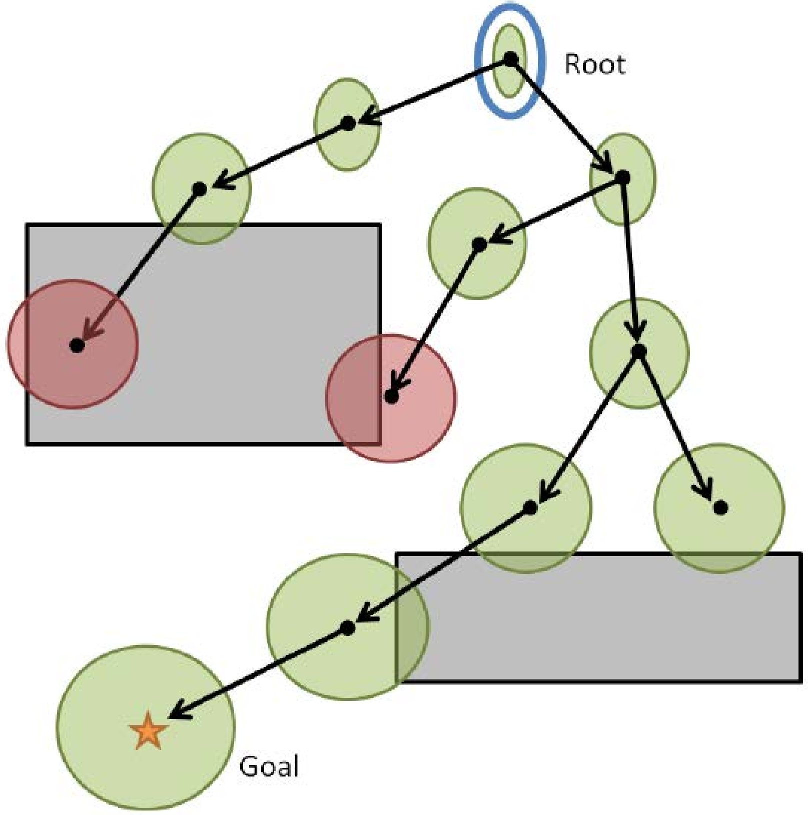
u_k: Given controller to steer the system toward the sampled point x

- Linearize the nonlinear system around the sampled points
- Propagate the mean and variance

We model distribution of the states with normal distribution

$$\bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \bar{\omega}_k$$

$$\Sigma^2_{x_{k+1}} = A_k \Sigma^2_{x_k} A_k^T + B_{\omega_k} \Sigma^2_{\omega_k} B_{\omega_k}^T$$



Chance constrained RRT*

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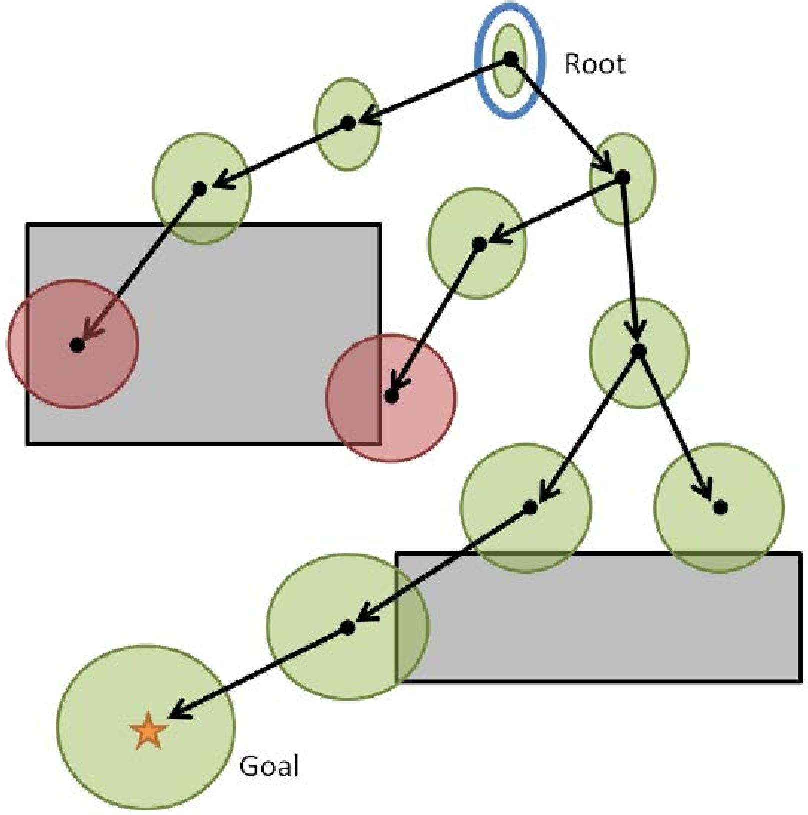


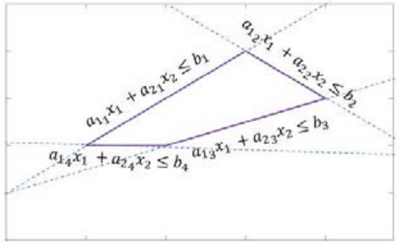
- Linearize the nonlinear system around the sampled points
- Propagate the mean and variance
- Check Probabilistic Safety

Probabilistic Safety Constraints

- **Obstacle set:** Conjunction of linear constraints: $X_{obs} = \{ \ (x_1, \dots, x_n) : \cap_{i=1}^{\ell} \left(a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i\right) \ \}$

- $prob(x_k \in X_{obs}) \leq \Delta_k$

- Safety Constraint: $\cup_{i=1}^{\ell} \ a_{1_i} \bar{x}_{1_k} + \cdots + a_{n_i} \bar{x}_{n_k} > \mathrm{b_i} - \sqrt{\left[a_{1_i}, \dots, a_{n_i}\right]^T \Sigma_{x_k}^2 \left[a_{1_i}, \dots, a_{n_i}\right]} \phi^{-1}(\Delta_\mathrm{k})$



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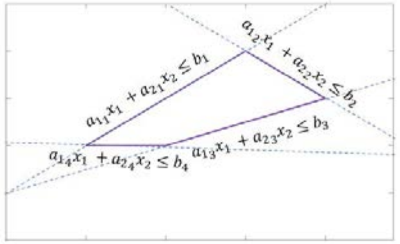

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- **Obstacle set:** $X_{obs} = \{x\colon \ \|x - x_c\| \leq r\}$

- $prob(x_k \in X_{obs}) \geq \Delta_k$

- Safety Constraint: $\|\bar{x} - x_c\| \leq r - \sqrt{\chi_n^2(\Delta_k)\lambda_{\max}(\Sigma_{x_k}^2)}$

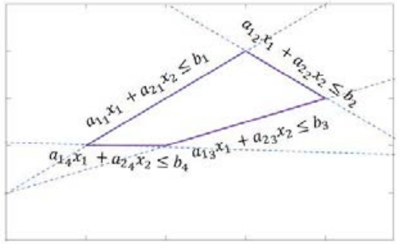
Quantile function of chi-squared distribution with n degrees freedom   Maximum eigenvalue

Lemma 1: L. Hewing, A. Liniger, and M. N. Zeilinger,"Cautious NMPC with Gaussian Process Dynamics for Autonomous Miniature Race Cars", European Control Conference (ECC), 2018

- For tighter probability bound: Huan Liua , Yongqiang Tang b, Hao Helen Zhang ,"A new chi-square approximation to the distribution of non-negative definite quadratic forms in non-central normal variables" Computational Statistics and Data Analysis 53 (2009) 853–856

Probabilistic Safety Constraints

- **Obstacle set:** Conjunction of linear constraints: $X_{obs} = \{\ (x_1, \ldots, x_n) \colon \cap_{i=1}^{\ell} \left(a_{1_i} x_1 + \cdots + a_{n_i} x_n < b_i\right)\ \}$



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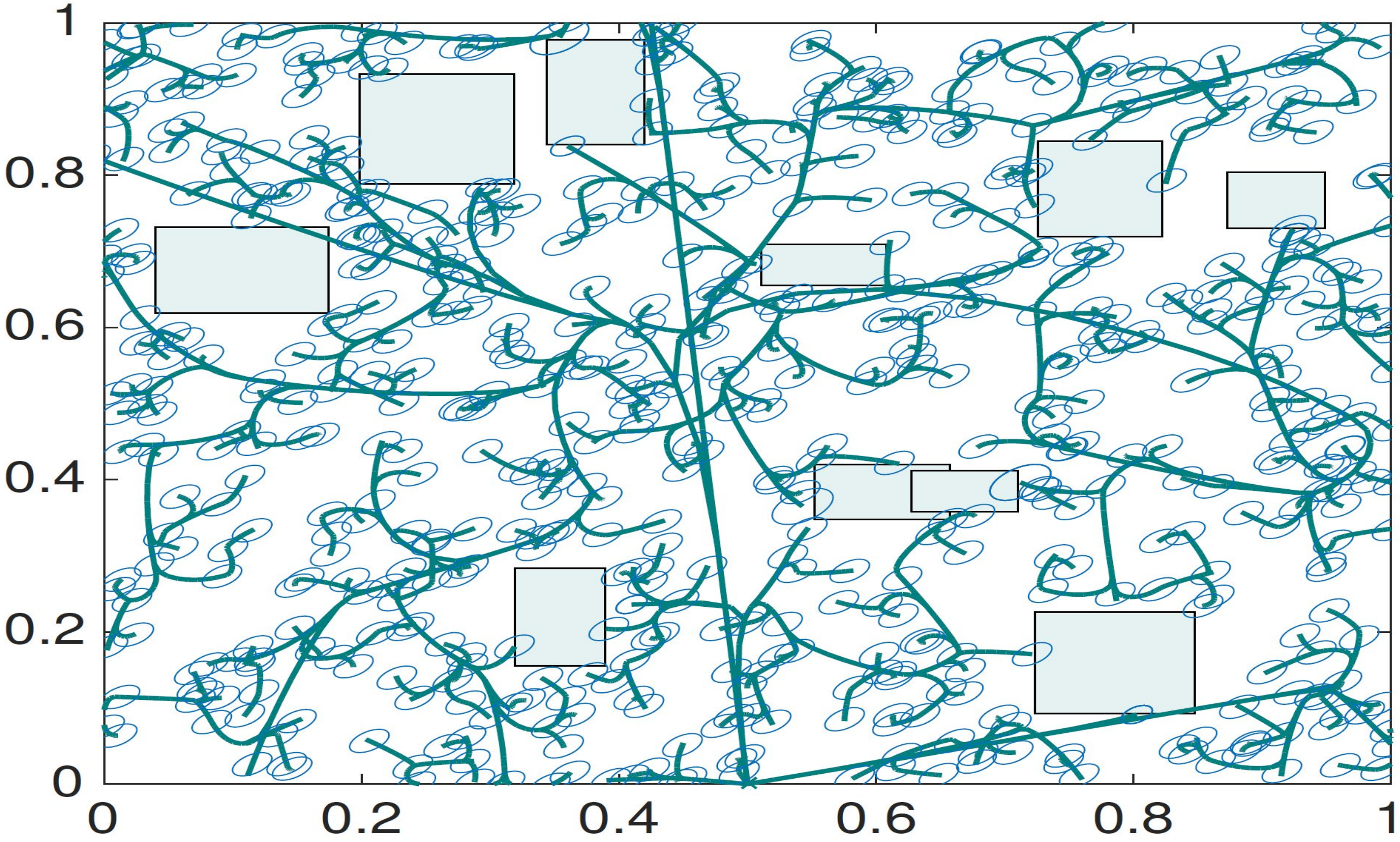
- **Nonlinear Obstacles and Arbitrary Probabilistic Uncertainties**   Lecture 10, section 1 on uncertainty propagation and section 2 on risk estimation



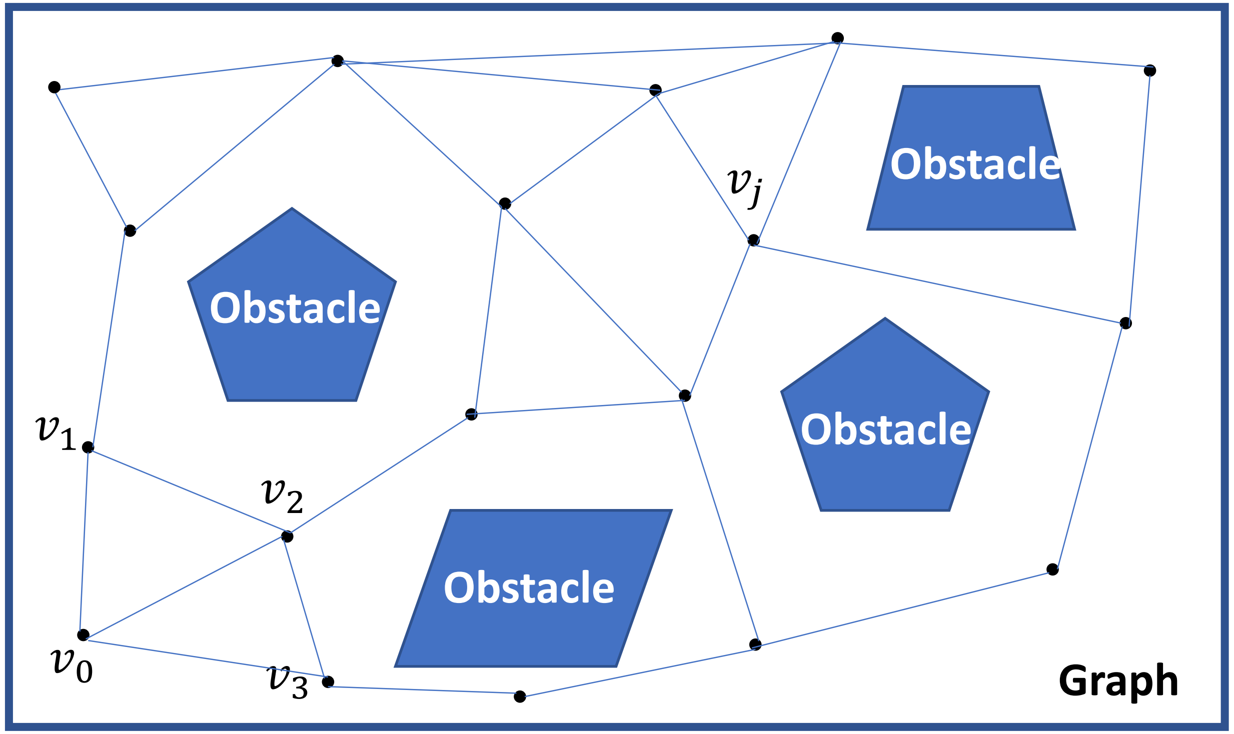
- Tyler Summers , Distributionally Robust Sampling-Based Motion Planning Under Uncertainty, International Conference on Intelligent Robots (IROS), 2018.
- Luders, Brandon J., Mangal Kothariyand and Jonathan P. How. "Chance Constrained RRT for Probabilistic Robustness to Environmental Uncertainty." In Proceedings of the AIAA Guidance, Navigation, and Control Conference, 2010.

Chance constrained PRM

PRM

- Instead of working with tree (e.g., rrt*), we like to work with graph.

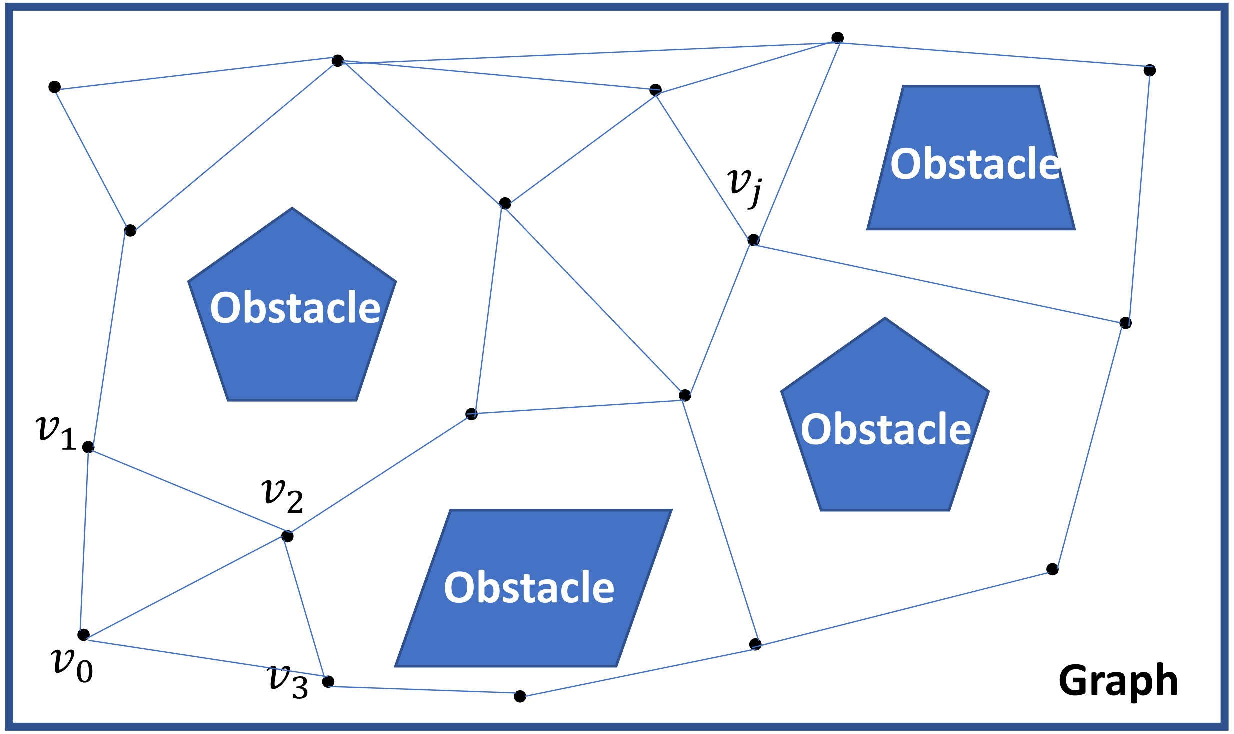
- We can construct the graph and controllers to drive the robot between the nodes in the offline step, and use the graph search to obtain the safe path in real-time.



PRM

- Instead of working with tree (e.g., rrt*), we like to work with graph.
- We can construct the graph and controllers to drive the robot between the nodes in the offline step, and use the graph search to obtain the safe path in real-time.

- In the presence of uncertainties, PRM results in a tee in belief space (space of probability distributions).



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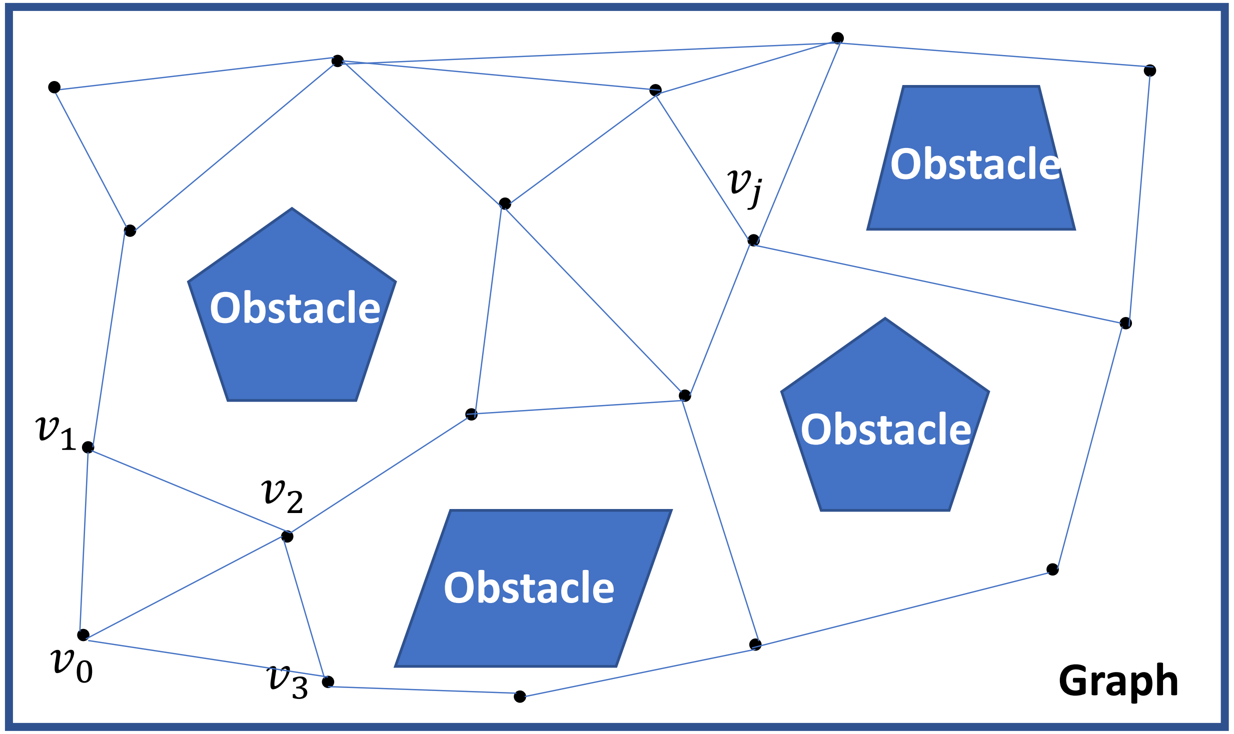
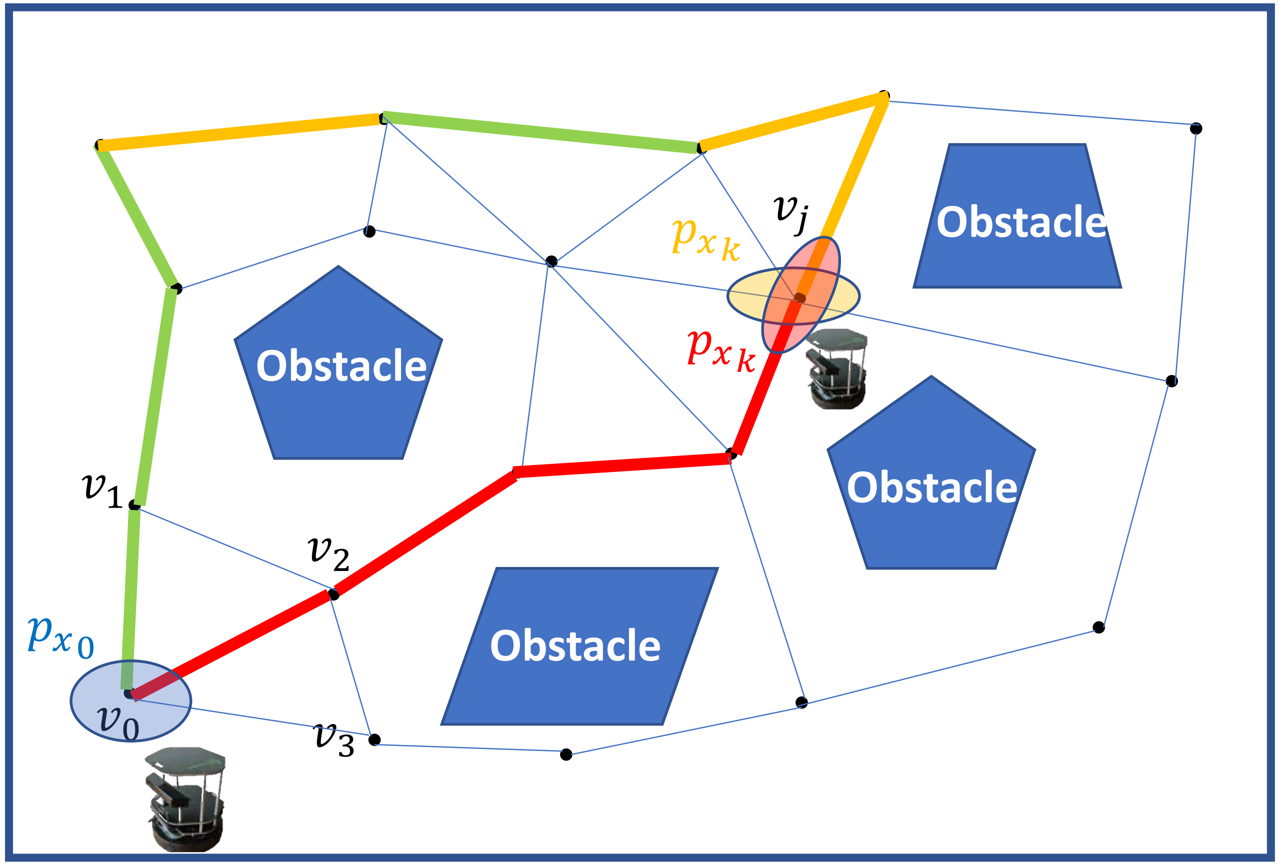
Uncertain System:

$$x_{k+1} = f(x_k, u_k, \omega_k)$$

$\omega_k \sim$ iven Probability distribution

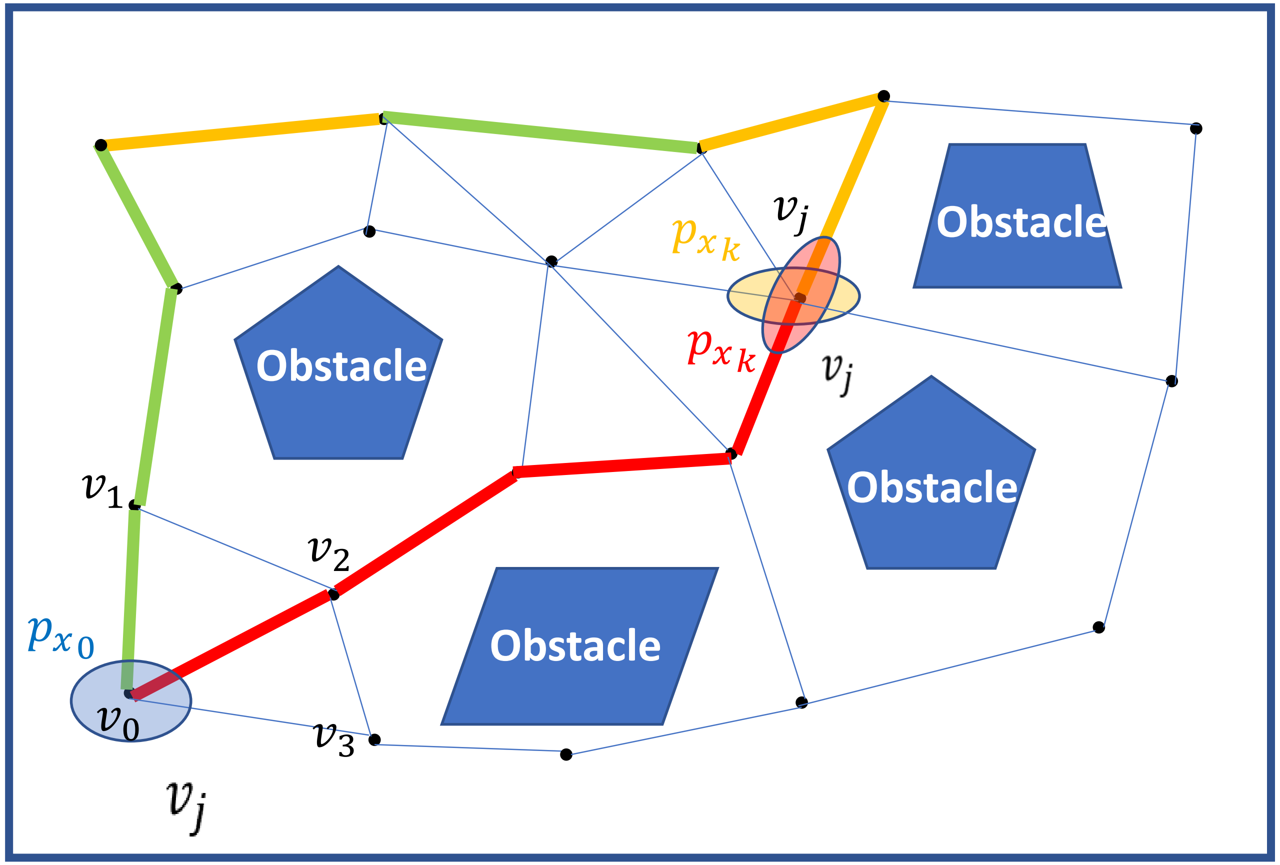
u_k: Given controller to derive the robot between the nodes



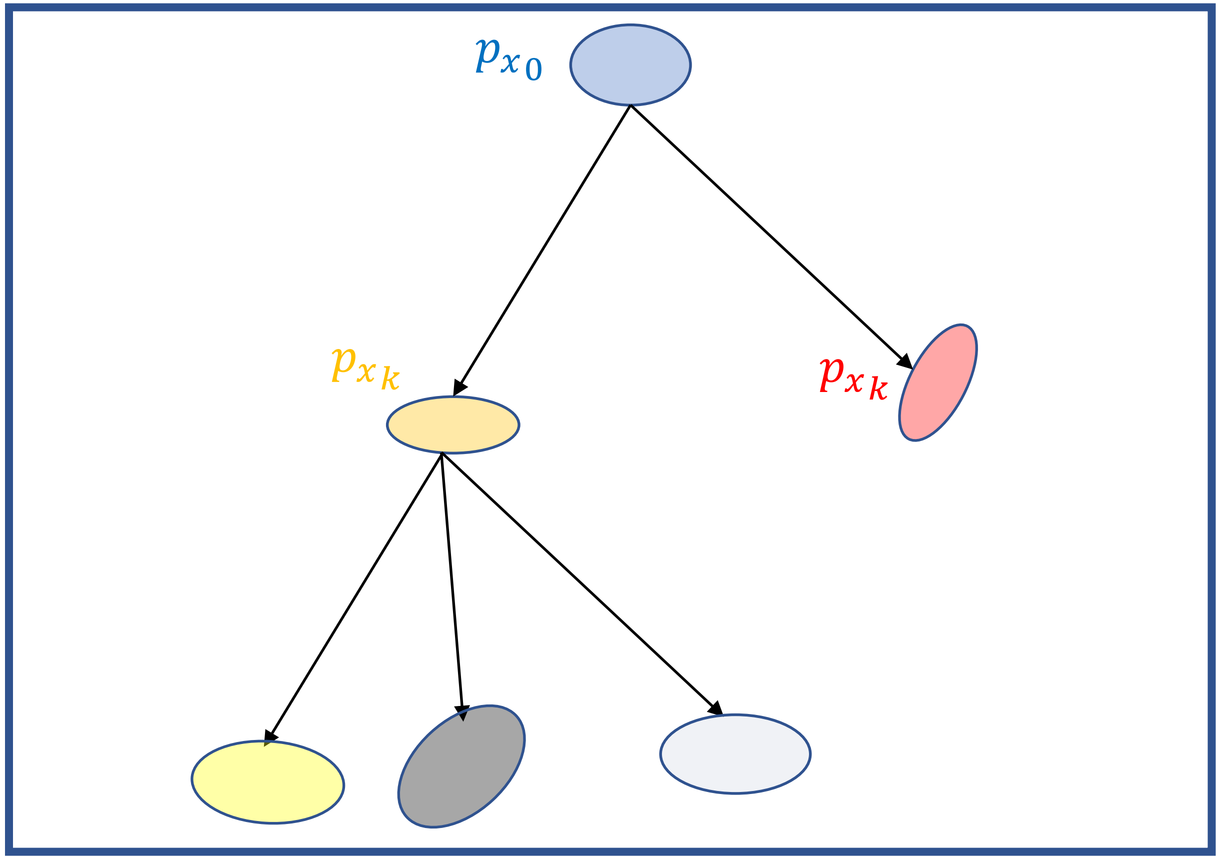


Graph in the state Space

Final distribution of the states depends on the traversed path



Graph in the state Space



Tree in the belief space

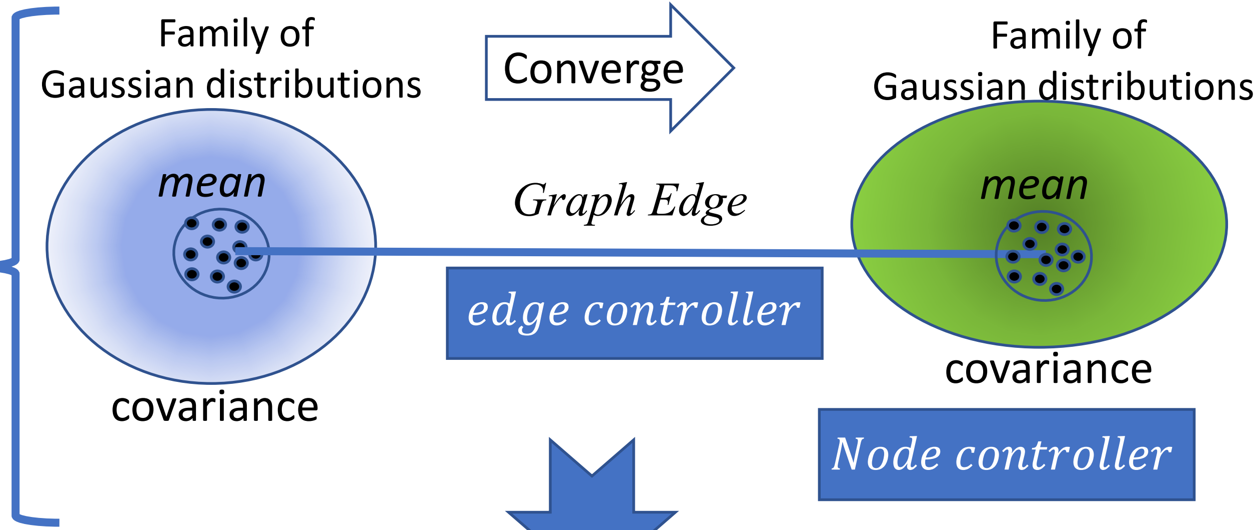
curse of history

- To construct the graph in the belief space, one can use stabilizing controllers to stabilize the distributions in the nodes.

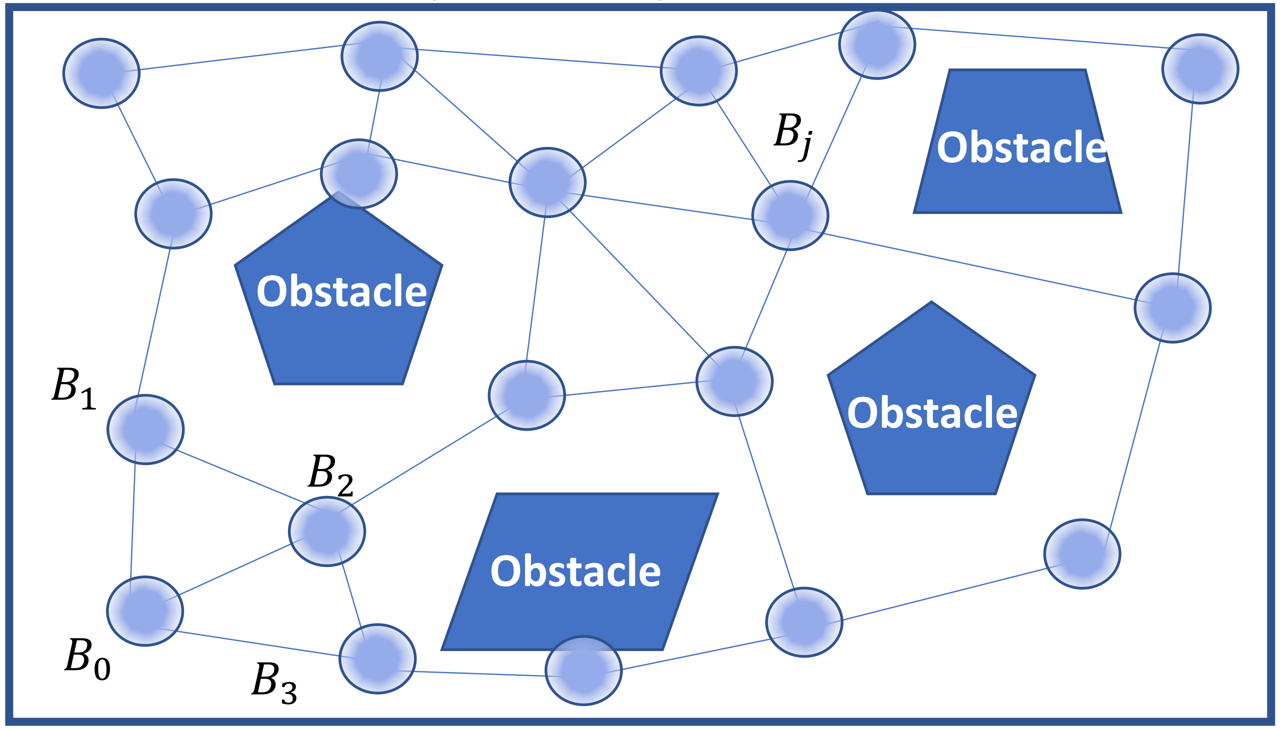
- A Agha-mohammadi, S Chakravorty, N Amato , “FIRM: Sampling-based Feedback Motion Planning Under Motion Uncertainty and Imperfect Measurements”, International Journal of Robotics Research (IJRR) 33 (2), 268 – 304, 2014.

- A Agha-mohammadi, S Chakravorty, NM Amato, , “FIRM: Feedback controller-based Information-state Roadmap - A framework for motion planning under uncertainty “ IEEE/RSJ International conference on Intelligent Robots and Systems (IROS), 2011.

Edge Controller: Time varying LQG
Node Controller: stationary LQG (belief stabilizer)



B_j: Ball in the belief space (Family of Gaussian distributions)



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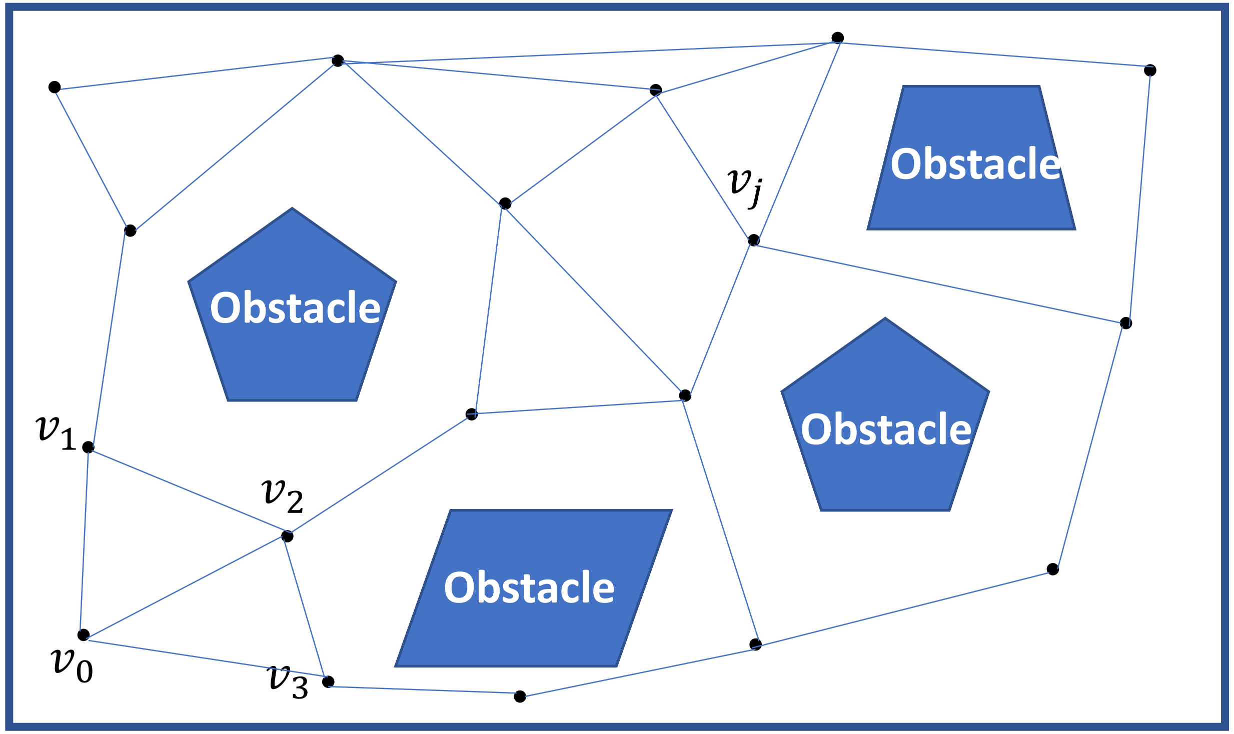
Alternative Approach:

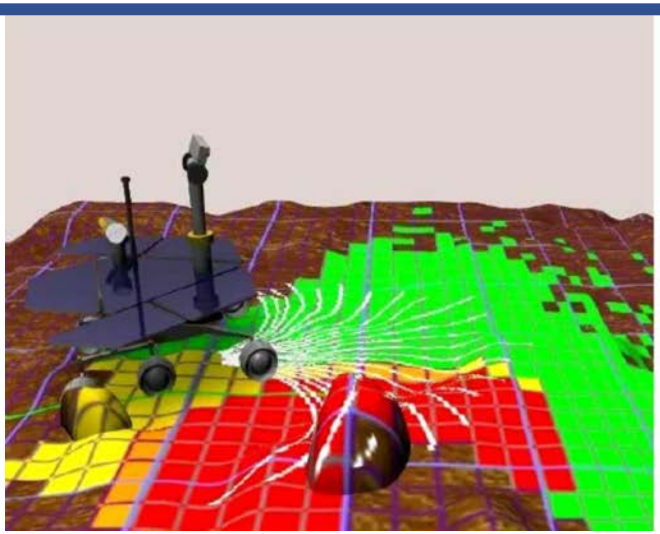
At each planning step:

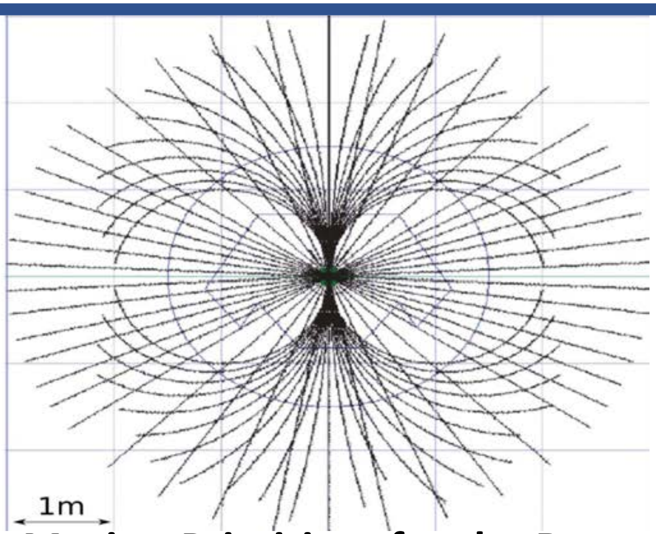
- Identify the $k-th$ best paths between the current and goal nodes.
- Propagate the uncertainty along the paths.
- Choose one path that satisfies probabilistic safety constraints and drive the robot to the next node.



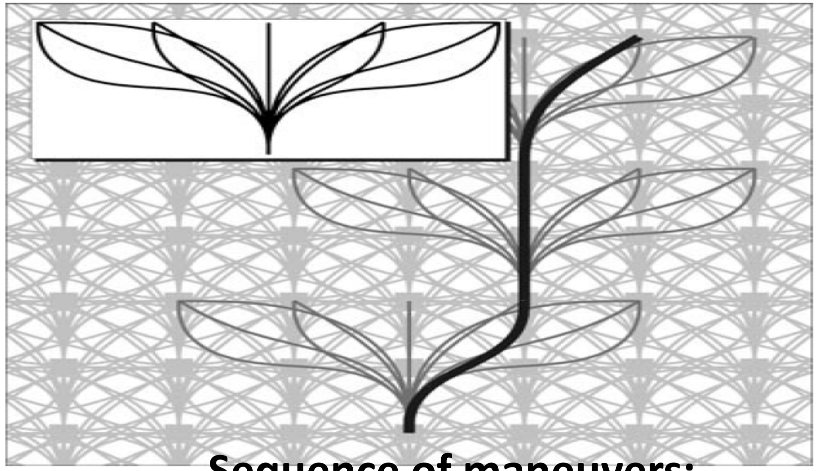
Chance Constrained Motion Primitive

State lattice/Motion primitive Based Path Planning





Motion Primitives for the Rover



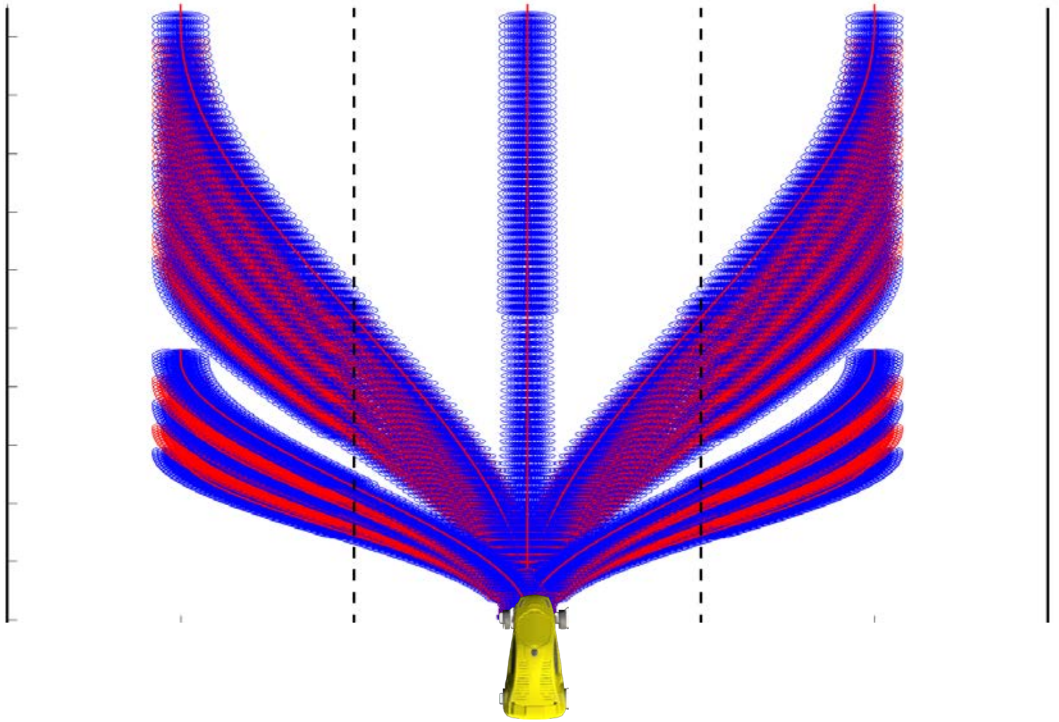
Sequence of maneuvers:

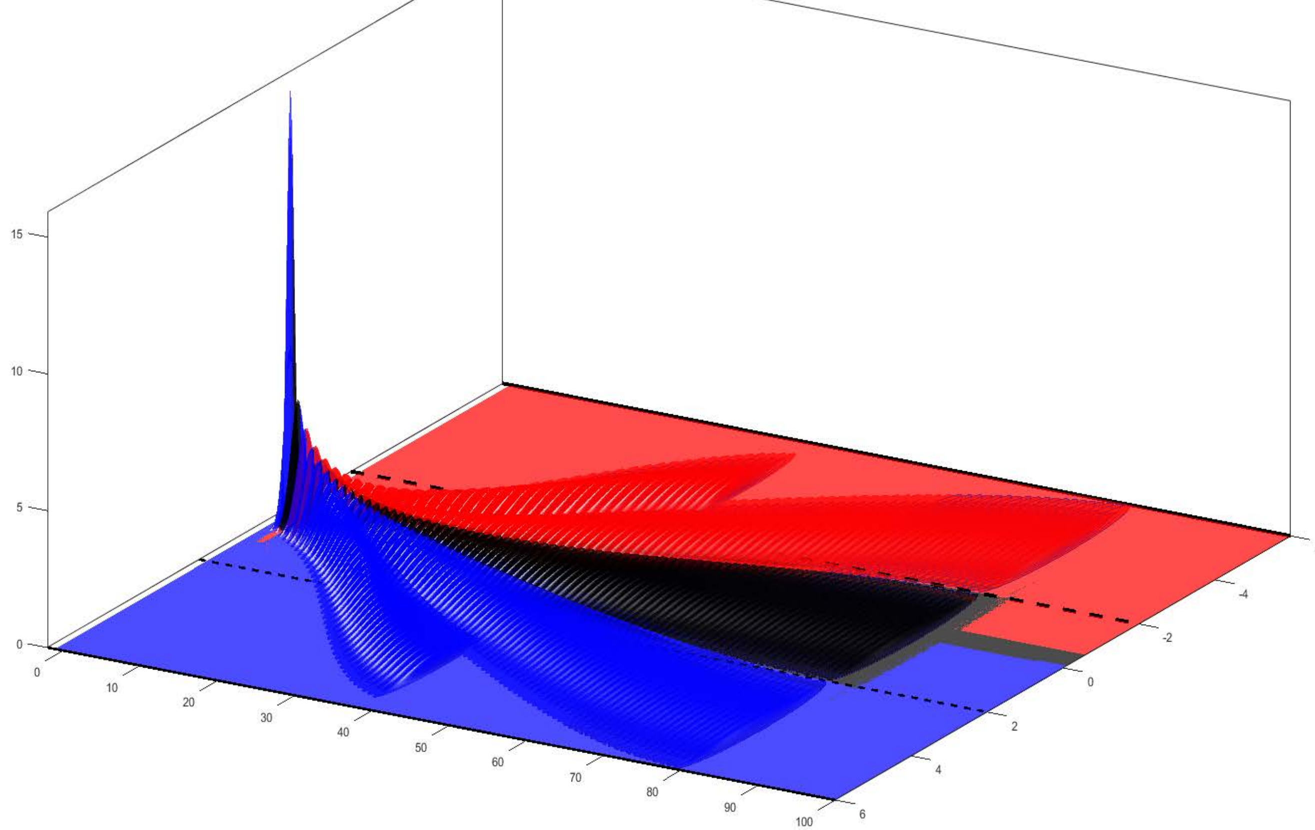
- Construct the library of maneuvers in the offline step
- Choose the right maneuver to execute in real-time.

- Dave Ferguson, Thomas M. Howard, Maxim Likhachev, “Motion Planning in Urban Environments” IEEE/RSJ International Conference on Intelligent Robots and Systems, 2008
- Mihail Pivtoraiko, Ross A. Knepper, and Alonzo Kelly, “Differentially Constrained Mobile Robot Motion Planning in State Lattices”, IEEE/RSJ International Conference on Intelligent Robots and Systems, 2008.
- Alexandru Rusu, Sabine Moreno, Yoko Watanabe, Mathieu Rognant, Michel Devy, “State lattice generation and nonholonomic path planning for a planetary exploration rover” 65th International Astronautical Congress 2014
- Peter R. Florence, John Carter, and Russ Tedrake. Integrated perception and control at high speed: Evaluating collision avoidance maneuvers without maps. In WAFR: Workshop on the Algorithmic Foundations of Robotics, 2016.
- Anirudha Majumdar, Russ Tedrake, “Funnel libraries for real-time robust feedback motion planning”, The International Journal of Robotics Research, Vol. 36(8) 947–982, 2017

- Propagate the uncertainty along the maneuvers .
- Choose the maneuver that satisfies probabilistic safety constraints.

Propagated uncertainties along the motion primitives





Topics:

- Chane Constrained Control

    i) Trajectory optimization, ii) RRT*, iii) PRM, iv) Motion Primitive

    v) Continuous-Time Safety Guarantees

- Distributionally Robust Chane Constrained Control
- Chance Constrained Covariance Control
- Sum-of-Squares based Probabilistic Safety Verification in Continuous-Time
- Distributionally Robust Chance Constrained Set for Moment Ambiguity Sets

Continuous-Time Safety Guarantees

Chance Constrained Trajectory Optimization

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0, \dots, u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob(x_k \in \chi_{obs}) \le \Delta_k \qquad k = 1, \dots T-1$$

$$E[x_T] = x_G$$

- Given Normal distributions of uncertainties, goal state x_G, and risk levels Δ_k , Find u_k, $k = 0, \dots, T-1$

$$\min_{[u_0, \dots, u_{T-1}]} \; E[J(x_k, u_k)]$$

$$s.t. \quad x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k$$

$$prob\left(x_t \Big|_{k \le t \le k+1} \in \chi_{obs} \right) \le \Delta_k \qquad k = 1, \dots T-1$$

Continuous-Time Safety Guarantees

$$E[x_T] = x_G$$

- To deal with chance constraint $prob\left(x_t\Big|_{k\leq t\leq k+1} \in \chi_{obs}\right) \leq \Delta_k$

  we assume that trajectory $x_t,\, k \leq t \leq k+1$ is a **Brownian motion.**

➢ To deal with chance constraint $prob\left(x_t\Big|_{k\le t\le k+1} \in \chi_{obs}\right) \le \Delta_k$

we assume that trajectory $x_t, k \le t \le k+1$ is a **Brownian motion.**

Brownian motion: is a continuous-time stochastic process

$W_0 = 0$

$$\left.\begin{array}{l} W_t \text{ has independent increments} \\ W_t - W_s \sim \mathcal{N}(0, t-s) \text{ (for } 0 \le s \le t\text{).} \end{array}\right\} \text{Gaussian independent increments}$$

- Since Brownian motion has Gaussian independent increments with mean zero, its time derivative is a **Gaussian stochastic process** with mean zero whose values at different times are independent.

- Brownian motion is the integral of a Gaussian process.

- Ordinary Differential Equation (ODE)

$$\frac{dx(t)}{dt} = f\big(x(t)\big)$$

- **Stochastic Differential Equation** (SDE) to describe probabilistic systems in **continuous-time**

$$\frac{dx(t)}{dt} = f\big(x(t)\big) + g\big(x(t)\big)\underbrace{\zeta(t)}_{\text{Noise: Gaussian Process}}$$

- Ordinary Differential Equation (ODE)

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- **Stochastic Differential Equation** (SDE) to describe probabilistic systems in **continuous-time**

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Noise: Gaussian Process

➢ Brownian motion is the integral of a Gaussian process.

- $x(t)$ (location of robot) is obtained by integration of continuous-time **linear** differential equitation of motion subjected to **Gaussian noise.**
- Hence, $x(t)$ is assumed to be a **Brownian motion.**.

- To deal with chance constraint $prob\left(x_t\Big|_{k\leq t\leq k+1} \in \chi_{obs}\right) \leq \Delta_k$

  we assume that trajectory $x_t, k \leq t \leq k+1$ is a gaussian stochastic process.

Reflection Principle for Brownian Motions:

- To deal with chance constraint $prob\left(x_t\Big|_{k\le t\le k+1} \in \chi_{obs}\right) \le \Delta_k$

  we assume that trajectory $x_t,\, k \le t \le k+1$ is a gaussian stochastic process.

Reflection Principle for Brownian Motions:

Let $w(t) : t(\ge 0)$ be a one-dimensional Brownian motion, and $a(\ge 0)$ is a threshold value. Then,

$$prob\left(\sup_{0\le s\le t} w(s) > a\right) = 2P(w(t) > a)$$

This theory can also be relaxed for an arbitrary time segment.

$$prob\left(\max_{t-1\le s\le t} w(s) \ge a\right) \le P\left(\max_{0\le s\le t} w(s) \ge a\right) = 2P\left(w(t) \ge a\right)$$

- This indicates that the probability of the constraint violation in a single segment and in the continuous time model can be guaranteed only by checking the violation probability of the final point.

- To deal with chance constraint $prob\left(x_t\Big|_{k\le t\le k+1} \in \chi_{obs}\right) \le \Delta_k$

Reflection Principle for Brownian Motions:

$$prob\left(\max_{t-1\leq s\leq t} w(s) \geq a\right) \leq P\left(\max_{0\leq s\leq t} w(s) \geq a\right) = 2P\left(w(t) \geq a\right)$$

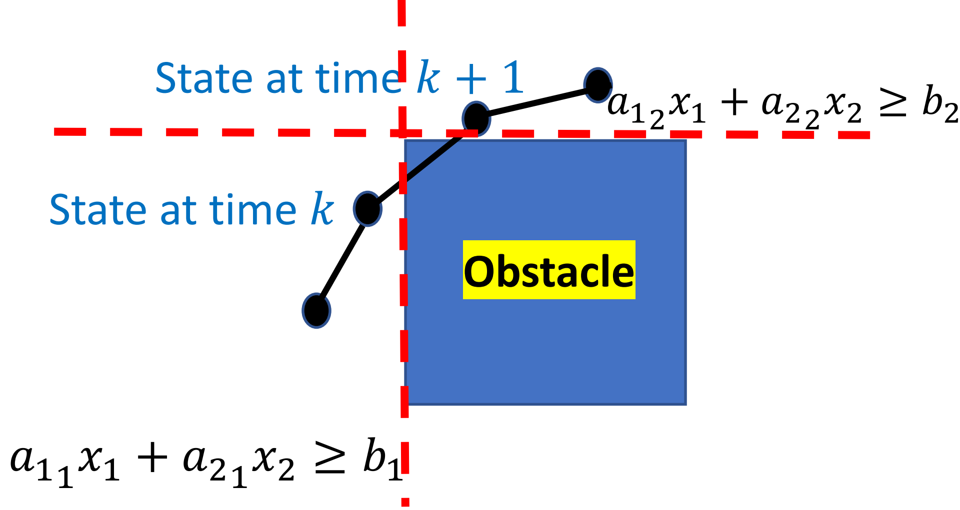
$$prob\left(x_t\Big|_{k-1\le t\le k} \in \chi_{obs}\right) \le \ 2\, prob(x(k) \in \chi_{obs})$$

Chance constraint:

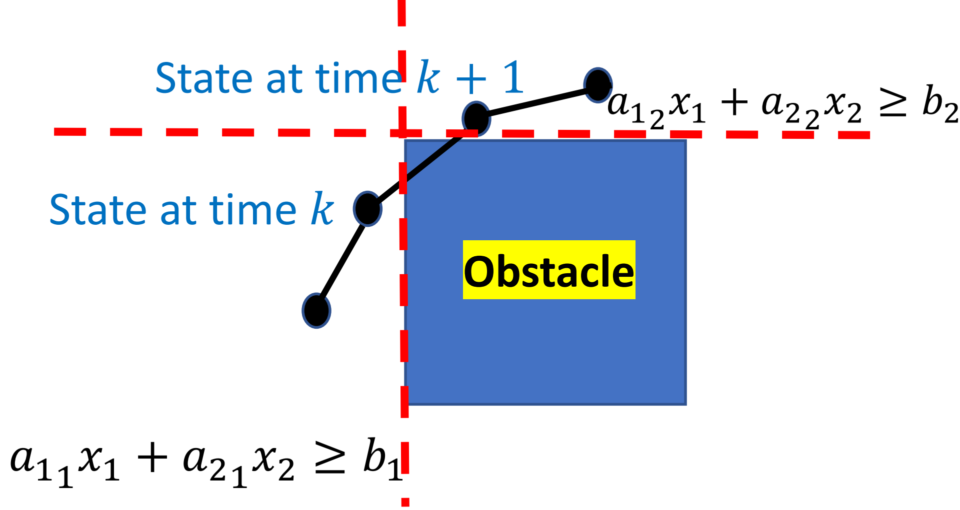
$$prob\left(x_t\Big|_{k\le t\le k+1} \in \chi_{obs}\right) \le \Delta_k \quad \Rightarrow \quad prob(x(k) \in \chi_{obs}) \le \frac{\Delta_k}{2}$$

Kaito Ariu, Cheng Fang, Marcio Arantes, Claudio Toledo, Brian Williams"Chance-Constrained Path Planning with Continuous Time Safety Guarantees", Thirty-First AAAI Conference on Artificial Intelligence, 2017

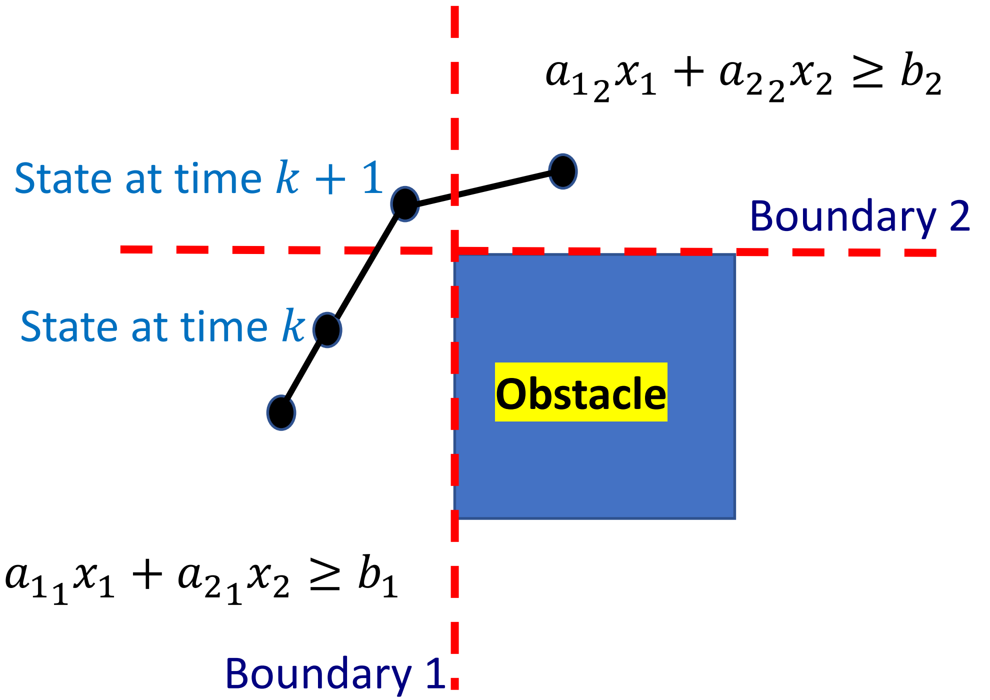
➢ To avoid such scenarios:



- To avoid such scenarios:



Two consecutive time steps to share a single active boundary for each obstacle.



Kaito Ariu, Cheng Fang, Marcio Arantes, Claudio Toledo, Brian Williams“Chance-Constrained Path Planning with Continuous Time Safety Guarantees”, Thirty-First AAAI Conference on Artificial Intelligence, 2017

Arantes, M.; Toledo, C.; Williams, B.; and Ono, M.2016. “Collision-free encoding for chance-constrained, non-convex path planning”

Continuous-Time Safety Guarantees

- **Chance constraint:**

$$prob\left(x_t\Big|_{k\leq t\leq k+1} \in \chi_{obs}\right) \leq \Delta_k \quad \Rightarrow \quad prob(x(k) \in \chi_{obs}) \leq \frac{\Delta_k}{2}$$

- Probability(Two consecutive time steps share a single active boundary for each obstacle) $\geq 1 - \Delta_k$

Kaito Ariu, Cheng Fang, Marcio Arantes, Claudio Toledo, Brian Williams"Chance-Constrained Path Planning with Continuous Time Safety Guarantees", Thirty-First AAAI Conference on Artificial Intelligence, 2017

Topics:

- Chane Constrained Control

  i) Trajectory optimization,   ii) RRT*,   iii) PRM,   iv) Motion Primitive

  v) Continuous-Time Safety Guarantees

- Distributionally Robust Chane Constrained Control
- Chance Constrained Covariance Control
- Sum-of-Squares based Probabilistic Safety Verification in Continuous-Time
- Distributionally Robust Chance Constrained Set for Moment Ambiguity Sets

Distributionally Robust Chane Constrained

- For given first and second moments, mean m and variance $\sum^2$

$\mathcal{M}$: Family of probability distribution whose first and second moments are m and $\sum^2$

Chance Constraint: $\underbrace{\max_{pr \in \mathcal{M}} \text{Probability}\{x_k \in \chi_{obs}\}}_{\text{Worst-case risk}} \leq \Delta_k$

- For given first and second moments, mean m and variance Σ^2

$\mathcal{M}$: Family of probability distribution whose first and second moments are m and Σ^2

Chance Constraint: $\underbrace{\max_{pr \in \mathcal{M}} \text{Probability}\{x_k \in \chi_{obs}\}}_{\text{Worst-case risk}} \leq \Delta_k$

Worst-case probability $\max_{pr \in \mathcal{M}} \text{Probability}\{x \geq b\} \leq \Delta_k \longrightarrow \bar{x} \leq b + \Sigma_x \sqrt{\frac{1 - \Delta_k}{\Delta_k}}$

G. C. Calafiore and L. E. Ghaoui, "On distributionally robust chance-constrained linear programs," Journal of Optimization Theory and Applications, vol. 130, no. 1, pp. 1–22, 2006.

T. Summers, "Distributionally robust sampling-based motion planning under uncertainty," IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), 2018

- For given first and second moments, mean m and variance $\sum^2$

$\mathcal{M}$: Family of probability distribution whose first and second moments are m and $\sum^2$

Chance Constraint: $\underbrace{\max_{pr\in\mathcal{M}} \text{Probability}\{x_k \in \chi_{obs}\}}_{\text{Worst-case risk}} \leq \Delta_k$

Worst-case probability $\max_{pr\in\mathcal{M}} \text{Probability}\{x \geq b\} \leq \Delta_k$ $\longrightarrow$ $\bar{x} \leq b + \Sigma_x \sqrt{\frac{1-\Delta_k}{\Delta_k}}$

➢ Probability($x \geq b$) $\leq \Delta$ $\longrightarrow$ $\bar{x} \leq b + \Sigma_x \phi^{-1}(\Delta_k)$

$x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

Gaussian Linear Chance Constraint

$$prob\left(a_1 x_{1_k} + \cdots + a_n x_{n_k} \geq b\right) \geq 1 - \Delta_k$$

Hard Constraint

$$a_1 \bar{x}_{1_k} + \cdots + a_n \bar{x}_{n_k} \geq b + c$$

Safety Margin

$$c = \sqrt{[a_1, \ldots, a_n]' \Sigma^2_{x_k} [a_1, \ldots, a_n]}\, \phi^{-1}(1 - \Delta_k)$$

$$\Sigma_{x_{k+1}} = A_k \Sigma_{x_k} A'_k + B_{\omega_k} \Sigma_{\omega_k} B'_{\omega_k}$$

Distributionally Robust Linear Chance Constraints

$$prob\left(a_1 x_{1_k} + \cdots + a_n x_{n_k} \geq b\right) \geq 1 - \Delta_k$$

Hard Constraint

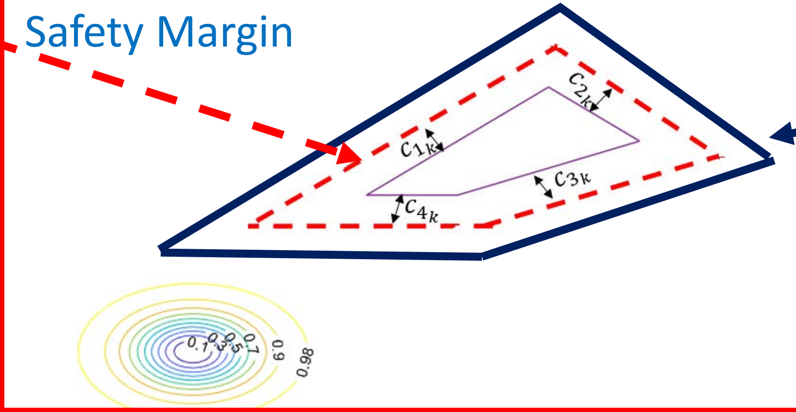
$$a_1 \bar{x}_{1_k} + \cdots + a_n \bar{x}_{n_k} \geq b + c$$

Safety Margin

$$c = \sqrt{[a_1, \ldots, a_n]' \Sigma^2_{x_k} [a_1, \ldots, a_n]}\, \sqrt{\frac{1 - \Delta_k}{\Delta_k}}$$

$$\Sigma_{x_{k+1}} = A_k \Sigma_{x_k} A'_k + B_{\omega_k} \Sigma_{\omega_k} B'_{\omega_k}$$

$$\Delta_k \in (0,1)$$



Safety Margin

Topics:

- Chane Constrained Control

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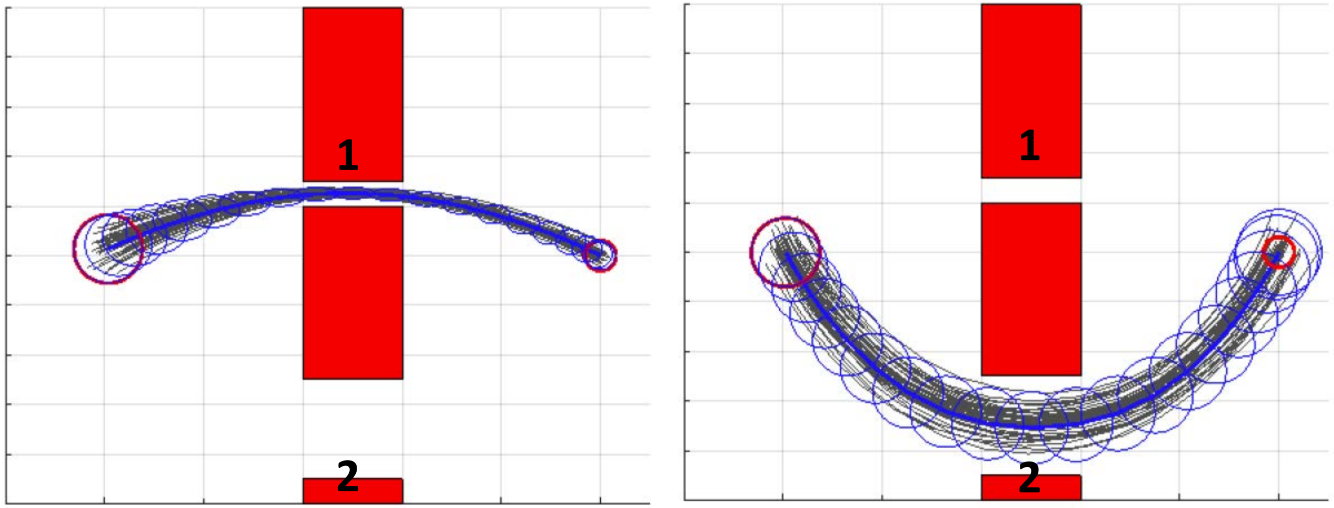
Chance Constrained Covariance Control



With Covariance Control **Without Covariance Control**

$$\Sigma_{x_{k+1}}^2 = A_k \Sigma_{x_k}^2 A_k^T + B_{\omega_k} \Sigma_{\omega_k}^2 B_{\omega_k}^T$$

- K. Okamoto, P. Tsiotras, “Stochastic Model Predictive Control for Constrained Linear Systems Using Optimal Covariance Steering”, 2019.
- K. Okamoto, P. Tsiotras, “Optimal Stochastic Vehicle Path Planning Using Covariance Steering”, 2018.
- K. Okamoto, M. Goldshtein, and Panagiotis Tsiotras“Optimal Covariance Control for Stochastic Systems Under Chance Constraints”, IEEE CONTROL SYSTEMS LETTERS, VOL. 2, NO. 2, APRIL 2018
- M. Goldshtein and P. Tsiotras,“Finite-Horizon Covariance Control of Linear Time-Varying Systems” 2017 IEEE 56th Annual Conference on Decision and Control (CDC), Australia, 2017

Linear systems and Gaussian Uncertainty Propagation:

$$x_{k+1} = A_k x_k + B_{u_k} u_k + B_{\omega_k} \omega_k \qquad x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma_{x_0}^2) \qquad \omega_k \sim \mathcal{N}(\overline{\omega}, \Sigma_{\omega_k}^2)$$

- By the assumption that u_k is deterministic i.e., $\mathrm{E}[u_k] = u_k$

$$\bar{x}_{k+1} = A_k \bar{x}_k + B_{u_k} u_k + B_{\omega_k} \overline{\omega}_k$$

$$\Sigma_{x_{k+1}}^2 = A_k \Sigma_{x_k}^2 A_k^T + B_{\omega_k} \Sigma_{\omega_k}^2 B_{\omega_k}^T$$

$B_{u_k} u_k$ term in x_{k+1} and $B_{u_k} \mathrm{E}[u_k]$ term in $\bar{x}_{k+1}$ cancels out

- We can control the covariance using non-deterministic control input u_k.
- State feedback $\quad u_k = K_0 + Kx$
- We need to look for a **feedback structure** that results in a **convex optimization.**

- **Gaussian Linear System:**

  - $x_{k+1} = A_k x_k + B_k u_k + D_k w_k,$ $\quad x_0 \sim \mathcal{N}(\mu_0, \Sigma_0^2)$ $\quad \omega_k \sim \mathcal{N}(0,1)$ $\quad x_0 \sim \mathcal{N}(\mu_0, \Sigma_0).$

- **Safety Constraints:**

  - $x_k \in \mathcal{X},$ $\quad \mathcal{X} \triangleq \bigcap_{i=0}^{N_s-1} \left\{x : \alpha_{x,i}^\top x \leq \beta_{x,i}\right\},$
  - $u_k \in \mathcal{U},$ $\quad \mathcal{U} \triangleq \bigcap_{j=0}^{N_c-1} \left\{u : \alpha_{u,j}^\top u \leq \beta_{u,j}\right\}$

- **Probabilistic Safety Constraints**

  - $\Pr(x_k \in \mathcal{X}) \geq 1 - \epsilon_x \qquad \text{where } \epsilon_x \in [0, 0.5)$

- Probabilistic Safety Constraints for Nondeterministic control

  - $\Pr(u_k \in \mathcal{U}) \geq 1 - \epsilon_u,$

Chance Constrained Covariance Control

Given initial and final normal distributions for the states $x_0 \sim \mathcal{N}(\mu_0, \Sigma_0^2)$, $x_f \sim \mathcal{N}(\mu_f, \Sigma_f^2)$
Find the sequence of control inputs

$$\min_{u_0,\ldots,u_{N-1}} J(u_0,\ldots,u_{N-1}) = \mathbb{E}\left[\sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k\right],$$

subject to

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \quad x_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$$

$$x_N = x_f \sim \mathcal{N}(\mu_f, \Sigma_f),$$

we assume that $\Sigma_0 \succeq 0$ and $\Sigma_f \succ 0$.

Feedback Control

$$U = [u_0^\top, u_1^\top, \ldots, u_{N-1}^\top]^\top$$

$$u_k = \underbrace{v_k}_{\text{constant}} + \underbrace{K_k}_{\text{Feedback Gain}} y_k,$$

Design parameters

where $v_k \in \mathbb{R}^{n_u}$, $K_k \in \mathbb{R}^{n_u \times n_x}$, and $y_k \in \mathbb{R}^{n_x}$ is given by

$$y_{k+1} = A_k y_k + D_k w_k,$$
$$y_0 = x_0 - \mu_0,$$

Probabilistic Safety Constraints

- $\Pr(x_k \in \mathcal{X}) \geq 1 - \epsilon_x$ $\xrightarrow{\text{Using Boole's inequality}}$ $\Pr\left(\alpha_{x,i}^\top x_k \leq \beta_{x,i}\right) \geq 1 - p_{x,i}, \quad \sum_{i=0}^{N_s-1} p_{x,i} \leq \epsilon_x.$

$$\mathcal{X} \triangleq \bigcap_{i=0}^{N_s-1} \left\{x : \alpha_{x,i}^\top x \leq \beta_{x,i}\right\},$$

Probabilistic Safety Constraints for Nondeterministic control

- $\Pr(u_k \in \mathcal{U}) \geq 1 - \epsilon_u,$ $\longrightarrow$ $\Pr\left(\alpha_{u,j}^\top u_k \leq \beta_{u,j}\right) \geq 1 - p_{u,j}, \quad \sum_{j=0}^{N_c-1} p_{u,j} \leq \epsilon_u.$

$$\mathcal{U} \triangleq \bigcap_{j=0}^{N_c-1} \left\{u : \alpha_{u,j}^\top u \leq \beta_{u,j}\right\}$$

Chance Constrained Covariance Control

Given initial and final normal distributions for the states $x_0 \sim \mathcal{N}(\mu_0, \Sigma_0^2)$, $x_f \sim \mathcal{N}(\mu_f, \Sigma_f^2)$
Find the sequence of control inputs

$$\min_{u_0,\ldots,u_{N-1}} J(u_0,\ldots,u_{N-1}) = \mathbb{E}\left[\sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k\right],$$

subject to

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \quad x_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$$

$$\Pr\left(\alpha_{x,i}^\top x_k \leq \beta_{x,i}\right) \geq 1 - p_{x,i}, \quad i = 0,\ldots,N_s-1,$$

$$\Pr\left(\alpha_{u,j}^\top u_k \leq \beta_{u,j}\right) \geq 1 - p_{u,j}, \quad j = 0,\ldots,N_c-1,$$

$$x_N = x_f \sim \mathcal{N}(\mu_f, \Sigma_f),$$

$$\sum_{i=0}^{N_s-1} p_{x,i} \leq \epsilon_x. \quad \sum_{j=0}^{N_c-1} p_{u,j} \leq \epsilon_u.$$

we assume that $\Sigma_0 \succeq 0$ and $\Sigma_f \succ 0$.

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k,$$

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W,$$

$$X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{(N+1)n_x}, \quad U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{Nn_u}, \quad W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} \in \mathbb{R}^{Nn_w},$$

$$\mathbb{E}[x_0 x_0^\top] = \Sigma_0 + \mu_0 \mu_0^\top,$$
$$\mathbb{E}[x_0 W^\top] = 0,$$
$$\mathbb{E}[WW^\top] = I_{Nn_w}$$

Chance Constrained Covariance Control

Given initial and final normal distributions for the states $x_0 \sim \mathcal{N}(\mu_0, \Sigma_0^2)$, $x_f \sim \mathcal{N}(\mu_f, \Sigma_f^2)$

Find the sequence of control inputs

$$\min_U \; J(U) = \mathbb{E}\left[X^\top \bar{Q} X + U^\top \bar{R} U\right],$$

subject to

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W, \quad x_0 \sim \mathcal{N}(\mu_0, \Sigma_0),$$
$$\Pr\left(\alpha_{x,i}^\top E_k X \le \beta_{x,i}\right) \ge 1 - p_{x,i},$$
$$\Pr\left(\alpha_{u,j}^\top F_k U \le \beta_{u,j}\right) \ge 1 - p_{u,j},$$
$$\mu_f = E_N \mathbb{E}[X],$$
$$\Sigma_f = E_N \left(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top\right) E_N^\top,$$

where $\bar{Q} = \mathtt{blkdiag}(Q, \ldots, Q, 0) \in \mathbb{R}^{(N+1)n_x \times (N+1)n_x}$ and $\bar{R} = \mathtt{blkdiag}(R, \ldots, R) \in \mathbb{R}^{Nn_u \times Nn_u}$,

$$E_k = \left[0_{n_x, kn_x}, I_{n_x}, 0_{n_x, (N-k)n_x}\right] \in \mathbb{R}^{n_x \times (N+1)n_x}, \quad k = 0, \ldots, N,$$
$$F_k = \left[0_{n_u, kn_u}, I_{n_u}, 0_{n_u, (N-k-1)n_u}\right] \in \mathbb{R}^{n_u \times Nn_u}, \quad k = 0, \ldots, N-1,$$

and thus $x_k = E_k X$ and $u_k = F_k U$.

Feedback Control:

$$U = [u_0^\top, u_1^\top, \ldots, u_{N-1}^\top]^\top$$

$$\underbrace{u_k = \underset{\text{constant}}{v_k} + \underset{\text{Feedback Gain}}{K_k} y_k,}_{\text{Design parameters}}$$

where $v_k \in \mathbb{R}^{n_u}$, $K_k \in \mathbb{R}^{n_u \times n_x}$, and $y_k \in \mathbb{R}^{n_x}$ is given by

$$y_{k+1} = A_k y_k + D_k w_k,$$
$$y_0 = x_0 - \mu_0,$$

Vector of Control over planning Horizon $\quad U = V + K(\mathcal{A} y_0 + \mathcal{D} W)$

$$Y = \mathcal{A} y_0 + \mathcal{D} W \qquad V = [v_0^\top, \ldots, v_{N-1}^\top]^\top$$

$\mathbb{E}[y_0] = 0$ and $\mathbb{E}[W] = 0$ that $\mathbb{E}[U] = V$.

- We describe the objective function and variance of the stares in terms of the design parameters V and K.

- By applying control input $U = V + K\left(\mathcal{A}y_0 + \mathcal{D}W\right)$ to the system $X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W,$ the following hold:

$$\bar{X} = \mathbb{E}[X] = \mathcal{A}\mu_0 + \mathcal{B}V,$$

$$\tilde{X} = X - \mathbb{E}[X] = \mathcal{A}(x_0 - \mu_0) + \mathcal{B}(U - V) + \mathcal{D}W = (I + \mathcal{B}K)\left(\mathcal{A}y_0 + \mathcal{D}W\right)$$

$$\mathbb{E}[\tilde{X}\tilde{X}^\top] = (I + \mathcal{B}K)\left(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top\right)(I + \mathcal{B}K)^\top$$

Mean and the covariance at time N:

$$\mu_N = E_N\left(\mathcal{A}\mu_0 + \mathcal{B}V\right),$$

$$\Sigma_N = E_N(I + \mathcal{B}K)\left(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top\right)(I + \mathcal{B}K)^\top E_N^\top.$$

- V steers the mean and K steers the covariance $U = V + K\left(\mathcal{A}y_0 + \mathcal{D}W\right)$

Final Covariance Constraint:

Relaxation: $\Sigma_N \succeq \underbrace{E_N(I + \mathcal{B}K)\left(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top\right)(I + \mathcal{B}K)^\top E_N^\top}_{\text{covariance of states at time N}}$

Σ_N ↓ Desired Covariance

$$1 - \|(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)^{1/2}(I + \mathcal{B}K)^\top E_N^\top \Sigma_N^{-1/2}\|_2 \geq 0$$

Chance Constraints:

$$\Pr(\alpha_j^\top E_k X > \beta_j) \leq p_j$$

$$\alpha_j^\top E_k X \sim \mathcal{N}(\alpha_j^\top E_k \bar{X}, \alpha_j^\top E_k \Sigma_X E_k^\top \alpha_j)$$

$$\Sigma_X = (I + \mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)(I + \mathcal{B}K)^\top.$$

- Probability($x > b$) $\leq \Delta$ → $\bar{x} \leq b + \Sigma_x \phi^{-1}(\Delta_k)$

  $x \sim \mathcal{N}(\bar{x}, \Sigma_x^2)$

  $\bar{x} \leq b - \Sigma_x \phi^{-1}(1 - \Delta_k)$

$$\underbrace{\alpha_j^\top E_k (\mathcal{A}\mu_0 + \mathcal{B}V)}_{\text{Mean at time k}} + \underbrace{\|(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)^{1/2} (I + \mathcal{B}K)^\top E_k^\top \alpha_j\| \Phi^{-1}(1 - p_{j,\text{fail}})}_{\text{Safety margin at time k}} - \beta_j \leq 0,$$

States Chance Constraint:

$$\Pr(\alpha_j^\top E_k X > \beta_j) \leq p_j \quad \Longrightarrow \quad \alpha_j^\top E_k(\mathcal{A}\mu_0 + \mathcal{B}V) + \|(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top)^{1/2}(I + \mathcal{B}K)^\top E_k^\top \alpha_j\| \Phi^{-1}(1 - p_{j,\text{fail}}) - \beta_j \leq 0,$$

Input Chance Constraint:

$$\Pr\left(\alpha_{u,j}^\top F_k U \leq \beta_{u,j}\right) \geq 1 - p_{u,j}, \quad \Longrightarrow \quad \alpha_{u,j}^\top F_k V - \beta_{u,j} + \|\Sigma_y^{1/2} K^\top F_k^\top \alpha_{u,j}\| \Phi^{-1}(1 - p_{u,j}) \leq 0,$$

Objective function:

$$J(U) = \mathbb{E}\left[X^\top \bar{Q} X + U^\top \bar{R} U\right],$$

Using the state and control vectors

$$J(\bar{X}, \tilde{X}, V, \tilde{U}) = \mathtt{tr}(\bar{Q}\mathbb{E}[\tilde{X}\tilde{X}^\top]) + \bar{X}^\top \bar{Q} \bar{X} + \mathtt{tr}(\bar{R}\mathbb{E}[\tilde{U}\tilde{U}^\top]) + V^\top \bar{R} V,$$

where $\tilde{U} = U - V$

$$\mathbb{E}[y_0 y_0^\top] = \Sigma_0,\ \mathbb{E}[y_0 W^\top] = 0, \text{ and } \mathbb{E}[WW^\top] = I_{N n_w}$$

$$\mathbb{E}[\tilde{X}\tilde{X}^\top] = (I + \mathcal{B}K)\left(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top\right)(I + \mathcal{B}K)^\top$$

$$\mathbb{E}[\tilde{U}\tilde{U}^\top] = K\left(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top\right)K^\top$$

$$J(V, K) = \mathtt{tr}\left(\left((I + \mathcal{B}K)^\top \bar{Q}(I + \mathcal{B}K) + K^\top \bar{R} K\right)\left(\mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top\right)\right) + (\mathcal{A}\mu_0 + \mathcal{B}V)^\top \bar{Q}(\mathcal{A}\mu_0 + \mathcal{B}V) + V^\top \bar{R} V.$$

Chance Constrained Covariance Control

Convex Deterministic Optimization:

$$\min_{V,K} \; J(V,K) = \mathtt{tr}\left[\left((I+\mathcal{B}K)^\top \bar{Q}(I+\mathcal{B}K) + K^\top \bar{R}K\right)\Sigma_y\right]$$

$$+ \, (\mathcal{A}\mu_0 + \mathcal{B}V)^\top \bar{Q}(\mathcal{A}\mu_0 + \mathcal{B}V) + V^\top \bar{R}V.$$

subject to

State Chane Constraints

$$\alpha_{x,i}^\top E_k(\mathcal{A}\mu_0 + \mathcal{B}V) - \beta_{x,i} + \|\Sigma_y^{1/2}(I+\mathcal{B}K)^\top E_k^\top \alpha_{x,i}\|\Phi^{-1}(1-p_{x,i}) \le 0,$$

$$\alpha_{u,j}^\top F_k V - \beta_{u,j} + \|\Sigma_y^{1/2}K^\top F_k^\top \alpha_{u,j}\|\Phi^{-1}(1-p_{u,j}) \le 0,$$

Input Chane Constraints

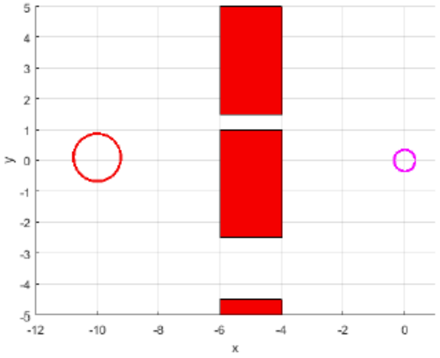
$$\mu_f = E_N(\mathcal{A}\mu_0 + \mathcal{B}V),$$

$$\Sigma_f \succeq E_N(I+\mathcal{B}K)\Sigma_y(I+\mathcal{B}K)^\top E_N^\top,$$

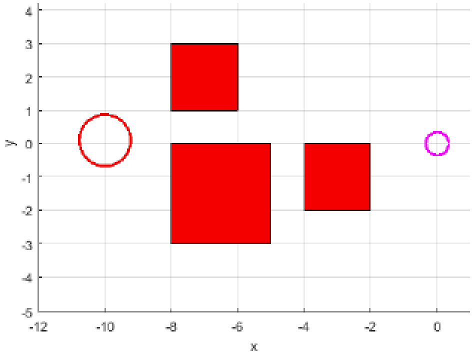
Final Covariance

for $i = 0, \ldots, N_s - 1$ and $j = 0, \ldots, N_c - 1$, where $\Sigma_y = \mathcal{A}\Sigma_0\mathcal{A}^\top + \mathcal{D}\mathcal{D}^\top$

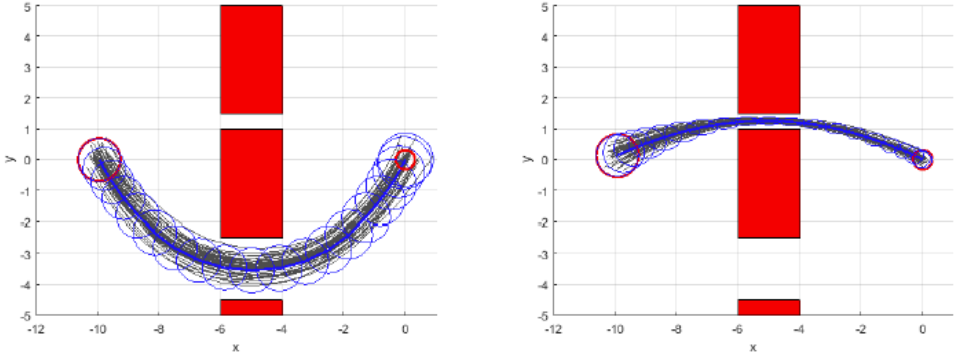
Example: Receding Horizon

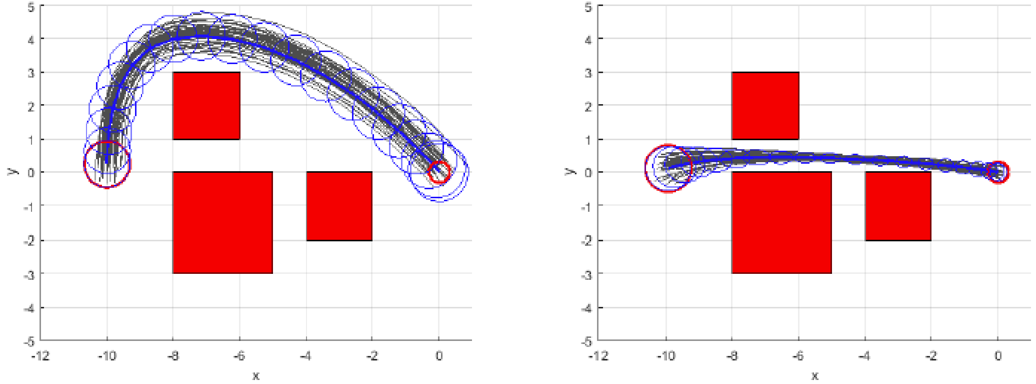


(a) Problem setup.



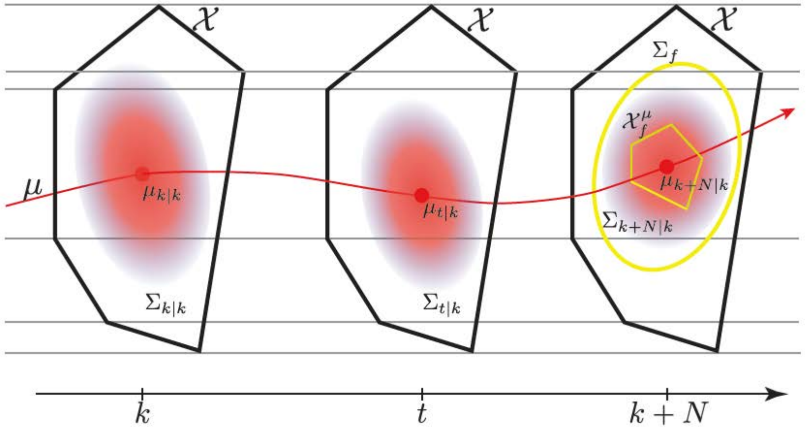
(a) Problem Setup.



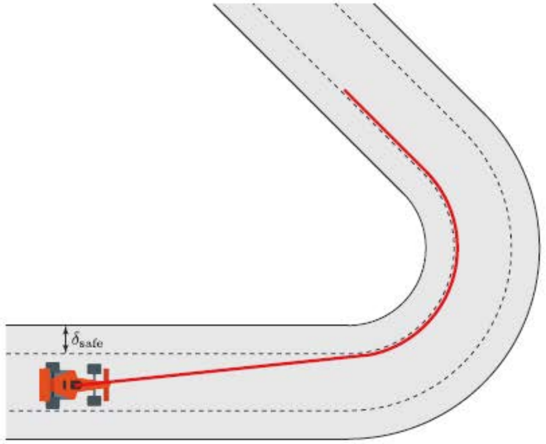


- K. Okamoto, P. Tsiotras, “Optimal Stochastic Vehicle Path Planning Using Covariance Steering”, 2018.

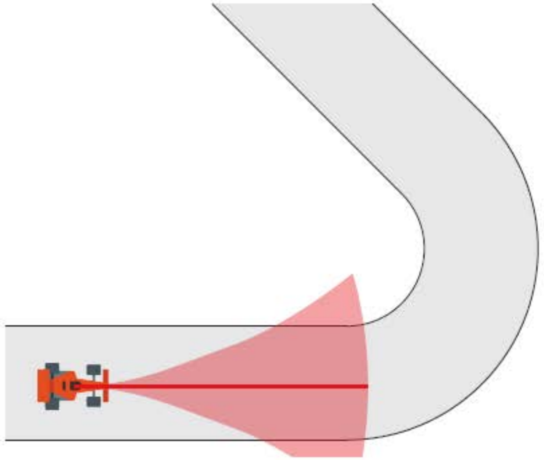
Example: Tube based Receding Horizon



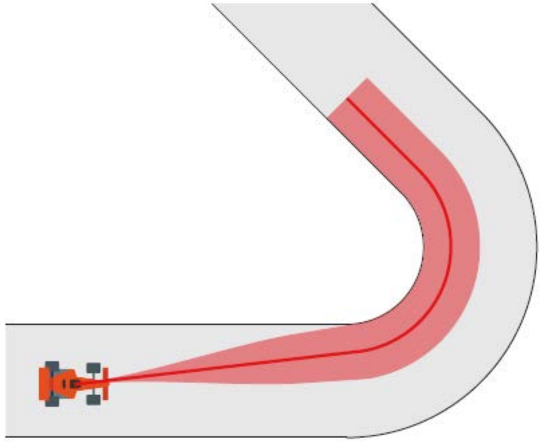
At the end of the horizon, the state **mean** has to be in a **yellow polytope**, and the system **covariance** has to be smaller a **yellow ellipse**.



(a) Deterministic MPC.



(b) Stochastic MPC with open-loop vehicle dynamics.



(c) Stochastic Tube-MPC.

- K. Okamoto, P. Tsiotras, "Stochastic Model Predictive Control for Constrained Linear Systems Using Optimal Covariance Steering", 2019.

Topics:

- Chane Constrained Control

  i) Trajectory optimization, ii) RRT*, iii) PRM, iv) Motion Primitive

  v) Continuous-Time Safety Guarantees

- Distributionally Robust Chane Constrained Control
- Chance Constrained Covariance Control
- Sum-of-Squares based Probabilistic Safety Verification in Continuous-Time
- Distributionally Robust Chance Constrained Set for Moment Ambiguity Sets

Sum-of-Squares based Probabilistic Safety Verification of Continuous-Time Nonlinear Stochastic Systems

- For safety of probabilistic continuous-time dynamical systems
  We look for **Barrier functions,** similar to robust case (Lecture 8)

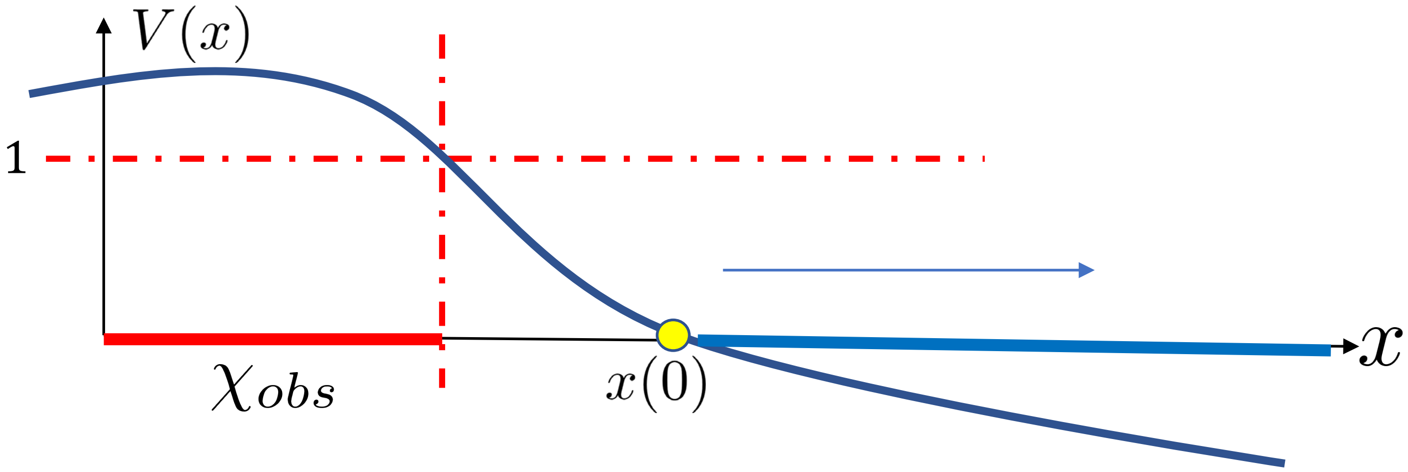
- S. Prajna, A. Jadbabaie, and G. J. Pappas "A Framework for Worst-Case and Stochastic Safety Verification Using Barrier Certificates", IEEE Transaction on Automatic Control, VOL. 52, NO. 8, 2007
- S. Prajna, A. Jadbabaie, and G. J. Pappas, "Stochastic Safety Verification Using Barrier Certificates" 43rd IEEE Conference on Decision and Control December 14-17, 2004 .
- J. Steinhardt, R. Tedrake, "Finite-time regional verification of stochastic non-linear systems"Journal International Journal of Robotics Research archive Volume 31 Issue 7, June 2012 Pages 901-923

➢ In Safety Verification

- Uncertain nonlinear dynamical system $\dot{x} = f(x, u, \omega)$
- Bounded Uncertainty $\omega \in \Omega$
- Unsafe Set $\mathcal{X}_{obs}$
- Initial state $x(0)$
- Policy $u(x)$

➢ Given policy $u(x)$ is safe if there exist a function $V(x)$ (Barrier function)

$V(x(0)) = 0$ $\quad$ $V(x) \geq 1 \;\; \forall x \in \mathcal{X}_{obs}$ $\quad$ $\dot{V}(x,\omega) = \frac{\partial V(x)}{\partial x} f(x, u(x), \omega) \leq 0 \; \forall x, \quad \forall \omega \in \Omega$



$u(x): \;\; \dot{V}(x,\omega) \leq 0 \; \forall x, \quad \forall \omega \in \Omega$

➢ x is constrained to evolve within the $\{x: V(x) \leq 0\}$

(0-level set of the function $V(x)$)

- Ordinary Differential Equation (ODE)

$$\frac{dx(t)}{dt} = f\big(x(t)\big)$$

- Stochastic Differential Equation (SDE)

$$\frac{dx(t)}{dt} = f\big(x(t)\big) + g\big(x(t)\big)\zeta(t)$$

Gaussian stochastic process

Given:

- SDE $\dfrac{dx(t)}{dt} = f\big(x(t)\big) + g\big(x(t)\big)\zeta(t)$
- χ: compact state space
- χ_0: compact initial state
- χ_u: compact unsafe set

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➢ **Stopped Process**

- In general stochastic process $x(t)$ is not guaranteed to always remain inside the set χ
- We define the stopped process with respect to $x(t)$ and χ as follows

$$\tilde{x}(t) = \begin{cases} x(t) & \text{For } t < \tau \\ x(\tau) & \text{For } t \geq \tau \end{cases}$$

τ: is the first time of exit of $x(t)$ form the set χ

Given:

- SDE $\frac{dx(t)}{dt} = f\big(x(t)\big) + g\big(x(t)\big)\zeta(t)$
- χ: compact state space
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- χ_u: compact unsafe set

Stochastic Safety verification: Compute the upper bound for the probability of a process $\tilde{x}(t)$ starting at χ_0 to reach χ_u

$$\text{Probability}\{\, \tilde{x}(t) \in \chi_u \text{ for some } t \geq 0 \mid \tilde{x}(0) \,\} \leq \gamma \quad \text{for all } \tilde{x}(0) \in \chi_0$$

➢ To satisfy the probabilistic safety constraints, γ should be less that acceptable risk level.

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- Similar to the robust case, we look for Barrier function $\mathrm{B}(x)$
  - Instead of requiring the value of $\mathrm{B}(x)$ to decreases along the trajectories of the system, we want $\mathrm{E}[\mathrm{B}(x)]$ decrease or stay constant over time.

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- Similar to the robust case, we look for Barrier function $\mathrm{B}(x)$
  - Instead of requiring the value of $\mathrm{B}(x)$ to decreases along the trajectories of the system, we want $\mathrm{E}[\mathrm{B}(x)]$ decrease or stay constant over time.
  - A function satisfying such property is called "**supermartingale**" i.e, $\mathrm{E}\big[\,\mathrm{B}\big(\tilde{x}(t_2)\big) \;\big|\; \tilde{x}(t_1)\,\big] \leq \mathrm{B}\big(\tilde{x}(t_1)\big) \quad 0 < t_1 < t_2 < \infty$

Nonnegative Supermartingale

- Let $\mathrm{B}(\tilde{x}(t))$ be a supermartingale of the process $\tilde{x}(t)$ and be nonnegative on χ

  Then, for any initial condition $\tilde{x}(t) \in \chi$

$$\text{Probability}\{ \sup_{0 \le t < \infty} \mathrm{B}\big(\tilde{x}(t)\big) \ge \lambda \mid \tilde{x}(0) \} \le \frac{B\big(\tilde{x}(0)\big)}{\lambda}$$

(Similar to Chebyshev bound)

B. Øksendal, *Stochastic Differential Equation: An Introduction with Applications*. Berlin: Springer-Verlag, 2000.

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(Similar to Chebyshev bound)

Barrier function-based Safety Verification:

- If there exist a function $\mathrm{B(x)}$ such that

1) $\mathrm{B}(x) \ge 0 \quad \forall x \in \chi$

2) $\mathrm{B}(x) \ge 1 \quad \forall x \in \chi_u$

3) $\mathrm{B}(x) \le \gamma \quad \forall x \in \chi_0$

4) $\dfrac{\partial \mathrm{B}}{\partial x} f(x) + \dfrac{1}{2} Trace\left(g^T(x) \dfrac{\partial^2 \mathrm{B}}{\partial x^2} g(x) \right) \le 0 \quad \forall x \in \chi$

Then $\text{Probability}\{ \tilde{x}(t) \in \chi_u$ for some $t \ge 0 \mid \tilde{x}(0) \} \le \gamma$ for all $\tilde{x}(0) \in \chi_0$

- We need to show that $\mathrm{B}(x)$ is a nonnegative supermartingale and hence "$\text{Probability}\{ \sup_{0 \le t < \infty} \mathrm{B}\big(\tilde{x}(t)\big) \ge \lambda \mid \tilde{x}(0) \} \le \frac{B(\tilde{x}(0))}{\lambda}$"

- S. Prajna, A. Jadbabaie, and G. J. Pappas, "Stochastic Safety Verification Using Barrier Certificates" 43rd IEEE Conference on Decision and Control December 14-17, 2004

- To show that $B(x)$ is suppermartingale, we need to describe its conditional expected value.
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$$AB(x_0) = \ Lim_{t \to 0} \frac{\mathrm{E}[\ B(x(t))\ |x(0)\,] \ - \ B(x(0))}{t}$$

- Dynkin's formula: $\mathrm{E}\big[\,B\big(\tilde{x}(t_2)\big)\ \big|\ \tilde{x}(t_1)\,\big] = B\big(\tilde{x}(t_1)\big) + \mathrm{E}[\int_{t1}^{t_2} AB\big(\tilde{x}(t)\big)dt] \qquad 0 < t_1 < t_2 < \infty$

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$\mathrm{B}(x)$ is a supermartingale

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(1) $\mathrm{B}(x) \geq 0 \quad \forall x \in \chi$ ⇒ B(x): Nonnegative

(1) and (4) ⇒ B(x): Nonnegative supermartingale $\quad \text{Probability}\{ \sup_{0 \leq t < \infty} \mathrm{B}(\tilde{x}(t)) \geq \lambda \mid \tilde{x}(0) \} \leq \frac{B(\tilde{x}(0))}{\lambda}$

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Safety: $\text{Probability}\{ \tilde{x}(t) \in \chi_u$ for some $t \geq 0 \mid \tilde{x}(0) \} = \text{Probability}\{ \sup_{0 \leq t < \infty} \mathrm{B}(\tilde{x}(t)) \geq 1 \mid \tilde{x}(0) \}$

2) $\mathrm{B}(x) \geq 1 \quad \forall x \in \chi_u$

$$\leq \frac{B(\tilde{x}(0))}{1} \quad \leq \gamma$$

3) $\mathrm{B}(x) \leq \gamma \quad \forall x \in \chi_0$

$\boxed{\text{Probability}\{ \tilde{x}(t) \in \chi_u \text{ for some } t \geq 0 \mid \tilde{x}(0) \} \leq \gamma \quad \text{for all } \tilde{x}(0) \in \chi_0}$

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Then Probability$\{ \tilde{x}(t) \in \chi_u$ for some $t \geq 0 \mid \tilde{x}(0) \} \leq \gamma$ for all $\tilde{x}(0) \in \chi_0$

SOS Program

$$\min_{\gamma, \mathrm{B}(x)} \quad \gamma$$

$$\begin{aligned}
\textit{subject to} \quad & \mathrm{B}(x) \geq 0 \quad \forall x \in \chi \\
& \mathrm{B}(x) \geq 1 \quad \forall x \in \chi_u \\
& \mathrm{B}(x) \leq \gamma \quad \forall x \in \chi_0 \\
& \frac{\partial \mathrm{B}}{\partial x} f(x) + \frac{1}{2} Trace\left(g^T(x) \frac{\partial^2 \mathrm{B}}{\partial x^2} g(x) \right) \leq 0 \quad \forall x \in \chi
\end{aligned}$$

Example:

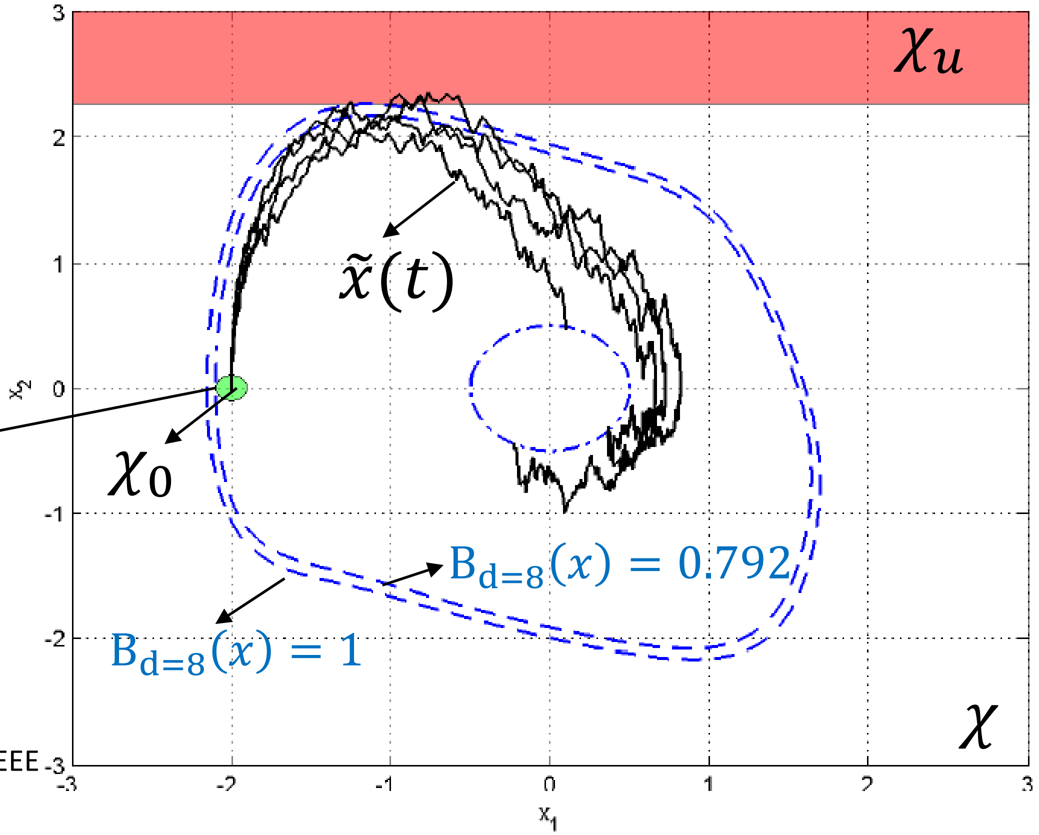
$$dx_1(t) = x_2(t)dt,$$
$$dx_2(t) = (-x_1(t) - x_2(t) - 0.5x_1^3(t))dt + \sigma dw(t),$$

diffusion coefficient

$$\mathcal{X} = \{(x_1, x_2) : -3 \le x_1 \le 3, -3 \le x_2 \le 3, x_1^2 + x_2^2 \ge 0.5^2\}$$
$$\mathcal{X}_0 = \{(x_1, x_2) : (x_1 + 2)^2 + x_2^2 \le 0.1^2\}$$
$$\mathcal{X}_u = \{(x_1, x_2) \in \mathcal{X} : x_2 \ge 2.25\}$$

	Degree= 4	Degree= 6	Degree= 8	Degree= 10
$\sigma = 0.5$	$\gamma = 1$	$\gamma = 0.847$	$\gamma = 0.792$	$\gamma = 0.771$
$\sigma = 0.25$	$\gamma = 0.848$	$\gamma = 0.616$	$\gamma = 0.472$	$\gamma = 0.412$
$\sigma = 0.1$	$\gamma = 0.824$	$\gamma = 0.450$	$\gamma = 0.257$	$\gamma = 0.157$

$B(x) \le \gamma \quad \forall x \in \chi_0$



- S. Prajna, A. Jadbabaie, and G. J. Pappas, "Stochastic Safety Verification Using Barrier Certificates" 43rd IEEE Conference on Decision and Control December 14-17, 2004.

Modified SOS-based approach to look for **exponential Barrier functions** (tighter probability bound)

- J. Steinhardt, R. Tedrake, "Finite-time regional verification of stochastic non-linear systems"Journal International Journal of Robotics Research archive Volume 31 Issue 7, June 2012 Pages 901-923

Topics:

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Distributionally Robust Chance Constrained Set for Moment Ambiguity Sets

- Jean B. Lasserre, Tillmann Weisser, “Distributionally robust polynomial chance-constraints under mixture ambiguity sets” Mathematical Programming, 2019

❖ Uncertain information of first and second moments :

First order moments (mean vector) $m \in [\underline{m}, \overline{m}]$

Second order moments (Covariance Matrix) $\Sigma \in [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \bar{\delta} I]$ for some given $\underline{\delta}, \bar{\delta} > 0$

➢ $\mathcal{M}$: Family of probability distribution supported on Ω whose first and second moments belongs to the set $\mathcal{A}$

- Moment uncertainty set $\mathcal{A} = [\underline{m}, \overline{m}] \times [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \bar{\delta} I]$

Probabilistic Safety Constraints: $\text{Probability}(\ g_i(x, \omega) \geq 0,\ i = 1, \ldots, n_g\ ) \geq 1 - \Delta$

Design parameter   Probabilistic uncertainty

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Design parameter Probabilistic uncertainty

Distributionally Robust Chane Constrained Set:

$$\chi_{DR} = \{x \in \chi : \text{Probability}(\ g_i(x, \omega) \geq 0,\ i = 1, ..., n_g\ ) \geq 1 - \Delta, \forall \text{pr}(\omega) \in \mathcal{M}\}$$

- We leverage on the results of lecture 8 (Distributionally Robust Chance Constrained Optimization)

In lecture 8: Sum-of-Squares Program to construct the set

Distributionally Robust Set

Set of all design variable "x" that satisfies "safety/design constraints" for all **possible probability distribution of uncertainty** "ω".

$$\chi_{DR} = \{x \in \chi : \text{Probability}(\ g_i(x,\omega) \geq 0,\ i = 1, ..., n_g\ ) \geq 1 - \Delta, \forall \mathrm{pr}_a(\omega),\ a \in \mathcal{A}\}$$

Uncertain Parameters of probability distribution

In lecture 8: Sum-of-Squares Program to construct the set

Distributionally Robust Set

Set of all design variable "x" that satisfies "safety/design constraints" for all **possible probability distribution of uncertainty** "ω".

$$\chi_{DR} = \{x \in \chi : \text{Probability}(\ g_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_g\ ) \geq 1-\Delta, \forall \text{pr}_a(\omega),\ \ a \in \mathcal{A}\}$$

Uncertain Parameters of probability distribution

- In this lecture, Uncertainty set of probability distribution ,i.e., $\mathcal{A}$ is defined based on the moments.

$$\mathcal{A} = [\underline{m}, \overline{m}] \times [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \overline{\delta} I]$$

- We will look at dual of SOS program provided in lecture 8. (moment SDP)
- Set $\mathcal{A}$ will introduce new set of moment constraints to moment SDP.

Distributionally Robust Chance Constrained Set with Moment Ambiguity Set

- Uncertainty ω has unknown probability distribution on uncertainty set Ω
- We have the uncertain information of the first and second order moments

First order moments (mean vector)

$$m \in [\underline{m}, \overline{m}]$$

Second order moments (Covariance Matrix)

$$\Sigma \in [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \overline{\delta} I]$$

for some given $\underline{\delta}, \overline{\delta} > 0$

- Moment uncertainty set $\mathcal{A} = [\underline{m}, \overline{m}] \times [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \overline{\delta} I]$

Example: $\omega \in \Omega \subset \mathbb{R}^2$ $\quad \boldsymbol{m} = (m_1, m_2); \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix},$

$$\mathbf{A} = \{ \quad \underline{m}_i \leq m_i \leq \overline{m}_i,\ i = 1, 2; \quad 2\underline{\delta} \leq \sigma_{11} + \sigma_{22} \leq 2\overline{\delta};$$
$$(\overline{\delta} - \sigma_{11})(\overline{\delta} - \sigma_{22}) - \sigma_{12}^2 \geq 0;\ (\sigma_{11} - \underline{\delta})(\sigma_{22} - \underline{\delta}) - \sigma_{12}^2 \geq 0\}.$$

➢ $\mathcal{M}$: Family of probability distribution supported on Ω whose first and second moments belongs to the set $\mathcal{A}$

➢ Chance constraints should be satisfied for the all probability distributions in $\mathcal{M}$

Distributionally Robust Chance Constrained Set with Moment Ambiguity Set

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$$\chi_{DR} = \{x \in \chi : \mathrm{Probability}(\ g_i(x,\omega) \geq 0,\ i = 1,...,n_g\ ) \geq 1 - \Delta, pr(\omega) \in \mathcal{M}\ \}$$

- $\mathcal{M}$: Family of probability distribution supported on Ω whose first and second moments belongs to the set $\mathcal{A}$
  - Moment uncertainty set $\quad \mathcal{A} = [\underline{m}, \overline{m}] \times [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \bar{\delta} I]$

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Dual of moment SDP

Primal Conic Program

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x \rangle_{V_1} \\ \text{subject to} \quad & A^*(x) = b \\ & x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & \langle y, b \rangle_{V_2} \\ \text{subject to} \quad & c - A(y) = s \\ & s \in K. \end{aligned}$$

measure space

continuous function space

$$\begin{aligned} \bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\bar{\mathcal{W}}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \bar{\mathcal{W}}(x,\omega) pr(\omega) d\omega dx \\ \text{subject to} \quad & \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}} \\ & \bar{\mathcal{W}}(x,\omega) \geq 0 \end{aligned}$$

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$\mu : \mu_x \times \mu_\omega$ given probability measure of uncertainties

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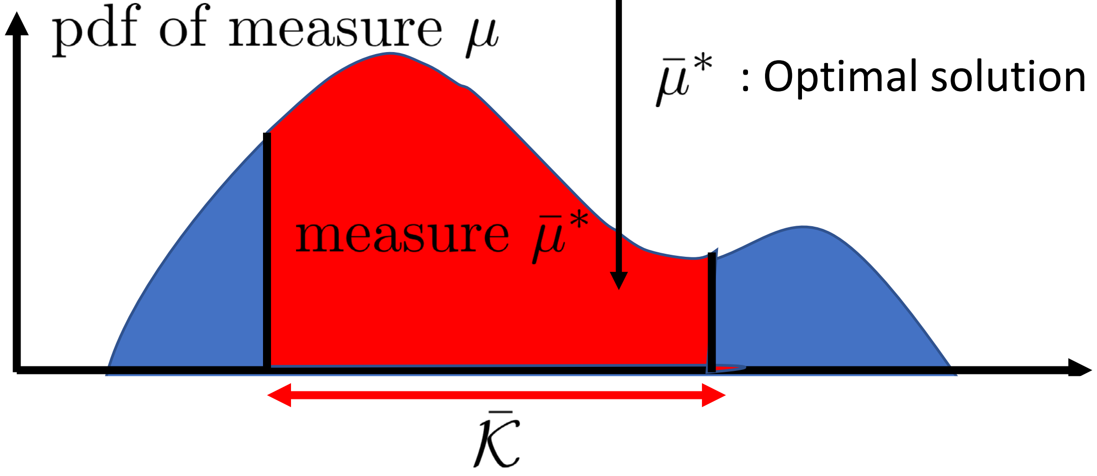

$\bar{\mu}^*$: Optimal solution

Dual of moment SDP

Primal Conic Program

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Moment Representation

- Looks for moments of a measure $\bar{\mu}$ defined in the space of (x, ω)
- Using duality, we can obtain the coefficient of polynomial $\bar{\mathcal{W}}$ from the solution of moment SDP.

Polynomial Representation

- Looks for coefficients of the polynomial $\bar{\mathcal{W}}$ in the space of (x, ω)

$$h(x) = \int \bar{\mathcal{W}}(x,\omega)\ pr(\omega)d\omega \qquad \bar{\chi}_{CC} = \{x \in \chi : \bar{h}(x) \leq \Delta\}$$

https://github.com/jasour/rarnop19/blob/master/Lecture11_Probabilistic_Nonlinear_Control/Risk_Contours_Map/Example_1_RiskContour_Inner.m

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Measure LP:

$$\begin{aligned} \underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \mu_x \times \mu_\omega \\ & supp(\bar{\mu}) \subset \bar{\mathcal{K}} \end{aligned}$$

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- In **distributionally Robust chance constrained optimization**, probability measure of uncertainty $\mu_\omega \in \mathcal{M}$

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- In **distributionally Robust chance constrained optimization**, probability measure of uncertainty $\mu_\omega \in \mathcal{M}$
  - ➢ $\mathcal{M}$: Family of probability distribution supported on Ω whose first and second moments belongs to the set $\mathcal{A}$
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- Measure LP for chance constrained optimization is defined in terms of measure $\mu_x \times \mu_\omega$ where, μ_x is the Lebesgue measure.

$$
\begin{aligned}
\underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\
\text{subject to} \quad & \bar{\mu} \preccurlyeq \mu_x \times \mu_\omega \\
& supp(\bar{\mu}) \subset \bar{\mathcal{K}}
\end{aligned}
$$

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$$\begin{aligned} \underset{\bar{\mu}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \mu_x \times \mu_\omega \\ & supp(\bar{\mu}) \subset \bar{\mathcal{K}} \end{aligned}$$

- Measure LP for distributionally robust chance constrained optimization is defined in terms of measure $\mu_x \times \mu_\omega \times \mu_a$ and its marginal measures $\mu_x \times \mu_\omega$ and $\mu_x \times \mu_a$ where, μ_a is probability measure on $\mathcal{A}$.

$$\begin{aligned} \underset{\bar{\mu}, \psi_{xa}}{\text{maximize}} \quad & \int d\bar{\mu} \\ \text{subject to} \quad & \bar{\mu} \preccurlyeq \psi_{x\omega} \\ & supp(\bar{\mu}) \subset \bar{\mathcal{K}} \\ & supp(\psi_{xa}) \subset \chi \times \mathcal{A} \end{aligned}$$

where $\psi_{xa}, \psi_{x\omega}$ are marginal measures of $\psi_{x\omega a}$ and ψ_x is the Lebesgue measure on χ.

- Measure LP for distributionally robust chance constrained optimization for $\mathcal{A} = [\underline{m}, \overline{m}] \times [\underline{\delta} I \preccurlyeq \Sigma \preccurlyeq \overline{\delta} I]$

$$
\begin{aligned}
\underset{\bar{\mu}, \psi_{xa}, \psi_{x\omega}}{\text{maximize}} \quad & \int d\bar{\mu} \\
\text{subject to} \quad & \bar{\mu} \preccurlyeq \ \psi_{x\omega} \\
& supp(\bar{\mu}) \subset \bar{\mathcal{K}} \\
& supp(\psi_{xa}) \subset \chi \times \mathcal{A} \\
& supp(\psi_{x\omega}) \subset \chi \times \Omega \\
& \textstyle\int x^\alpha \omega_i d\bar{\mu} + \int x^\alpha \omega_i d\phi_{x\omega} = \int x^\alpha m_i d\phi_{xa} \quad i = 1, ..., p \\
& \textstyle\int x^\alpha \omega_i \omega_j d\bar{\mu} + \int x^\alpha \omega_i \omega_j d\phi_{x\omega} = \int x^\alpha \sigma_{ij} d\phi_{xa} \quad 1 \leq i < j \leq p
\end{aligned}
$$

where $\psi_{xa}, \psi_{x\omega}$ are marginal measures of $\psi_{x\omega a}$ and ψ_x is the Lebesgue measure on χ .

- Jean B. Lasserre, Tillmann Weisser, “Distributionally robust polynomial chance-constraints under mixture ambiguity sets” Mathematical Programming, 2019

- Alternative conservative approaches:

- Delage, E., Ye, Y.: Distributionally robust optimization under moment uncertainty with applications to data-driven problems. Oper. Res. 58, 595–612 , 2010
- de Klerk, E., Kuhn, D., Postek, K.: Distributionally robust optimization with polynomial densities: theory, models and algorithms. arXiv:1805.03588
- El Ghaoui, L., Oks, M., Oustry, F.:Worst-case value-at-risk and robust portfolio optimization: a conic programming approach. Oper. Res. 51, 543–556 , 2003.
- Hanasusanto, G.A., Roitch, V., Kuhn, D., Wiesemann, W.: Ambiguous joint chance constraints under mean and dispersion information. Oper. Res. 65, 751–767,2017.
- Hanasusanto, G.A., Roitch, V., Kuhn, D., Wiesemann, W.: A distributionally robust perspective on uncertainty quantification and chance constrained programming. Math. Program. 151, 35–62 , 2015.
- Zymler, S., Kuhn, D., Rustem, B.: Distributionally robust joint chance constraints with second-order moment information. Math. Program. 137, 167–198, 2013.

Topics:

- Chane Constrained Control

  i) Trajectory optimization,   ii) RRT*,   iii) PRM,   iv) Motion Primitive

  v) Continuous-Time Safety Guarantees

- Distributionally Robust Chane Constrained Control
- Chance Constrained Covariance Control
- Sum-of-Squares based Probabilistic Safety Verification in Continuous-Time
- Distributionally Robust Chance Constrained Set for Moment Ambiguity Sets