Lecture 5

Duality of SOS and Moment Based SDPs

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

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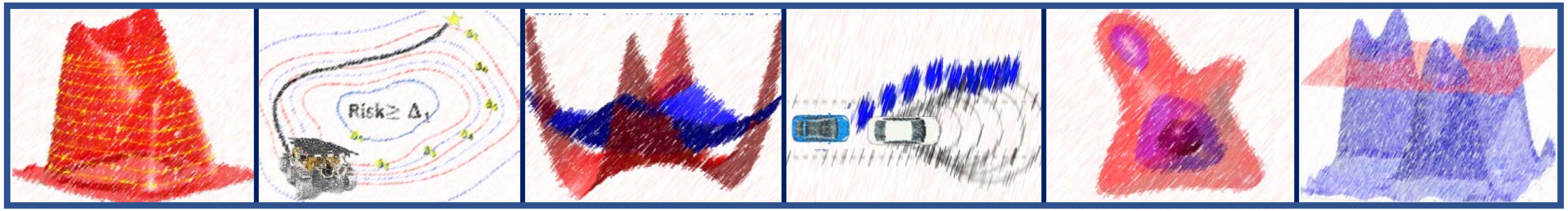


Nonlinear (nonconvex) Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \dots, m \end{aligned}$$

Objective function and constraints are polynomial functions.

Convex Relaxation: i) SOS Based SDP ii) Moment Based SDP

Topics:

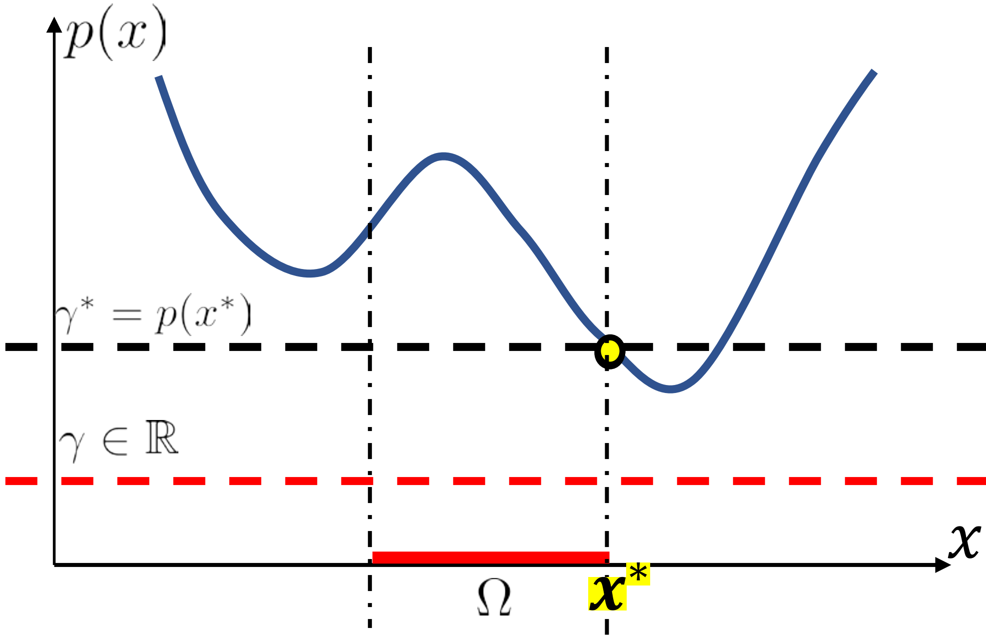
- Brief Review of SOS and Moments Approaches (Lectures 3 and 4)
- Review of Dual Optimization (Lecture 2)
- Duality of SOS and Moments Approaches
- Primal-Dual Interior Point Methods for SDPs
- Appendix I: Conic Duality
- Appendix II: Alternative Representations

Brief Review of SOS and Moments Approaches

1. Sum of Squares Approach

$$P^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$
- Constrained Optimization $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}$



Reformulation in terms of Nonnegative polynomials

$$\begin{aligned} &\underset{\gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x) - \gamma \geq 0, \quad \forall x \in \Omega \end{aligned}$$

Linear constraint

Polynomial Nonnegativity constraint

We are looking for γ such that $p(x) - \gamma$ be a nonnegative polynomial on the set Ω

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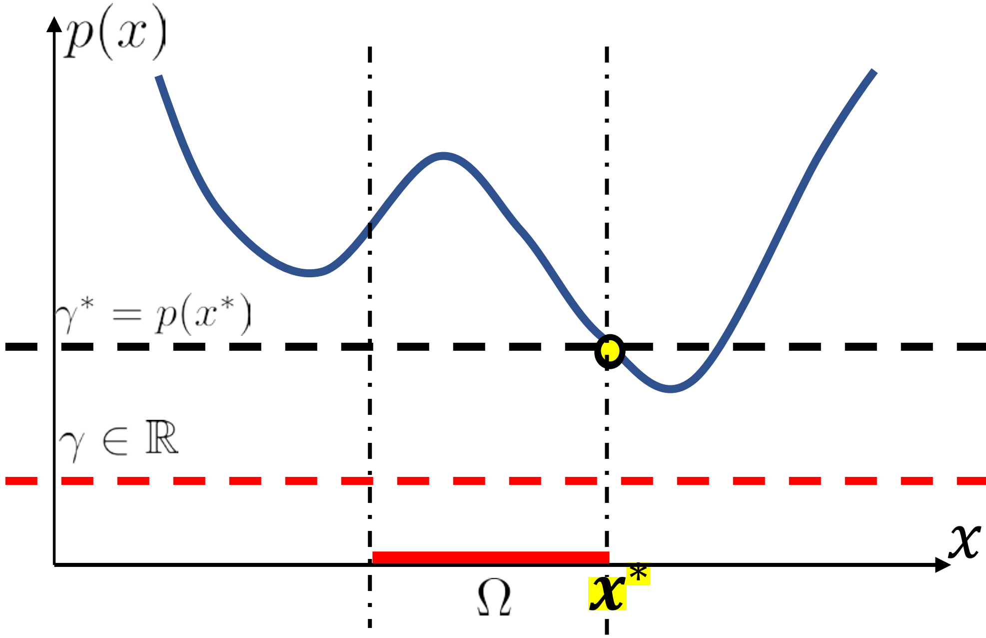


Reformulation in terms of Nonnegative polynomials

$$\underset{\gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma$$ → Linear constraint

$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Polynomial Nonnegativity constraint

$p(x) - \gamma$ be a nonnegative polynomial on the set Ω

SOS Relaxation

Nonnegativity Condition

Unconstrained Case:

If polynomial $p(x)$ is **SOS**, then it is $\boldsymbol{p(x) \geq 0}$ for all $x \in \mathbb{R}^n$.

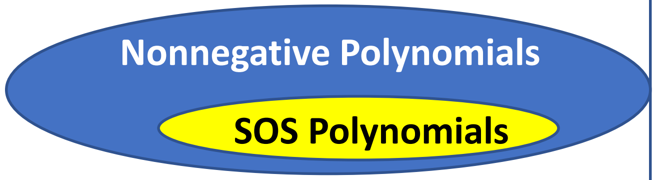
SOS Condition

$$p(x) \in SOS \;\begin{cases} p(x) = \sum_{i=1}^{\ell} h_i^2(x) \\ h_i(x) \in \mathbb{R}[x], \quad i = 1, \ldots, \ell \end{cases} \Longrightarrow$$

Nonnegative polynomial

$$p(x) \geq 0, \;\; \forall x \in \mathbb{R}^n$$

- Not every nonnegative polynomial has a SOS representation.



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SOS Condition

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Nonnegative polynomial

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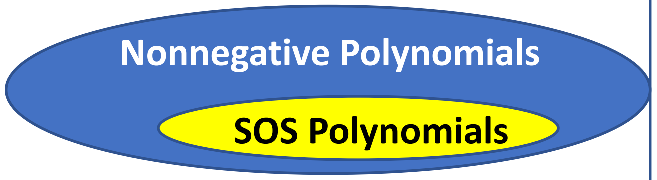

- Not every nonnegative polynomial has a SOS representation.

Let the $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, \dots, m\}$ be a compact set (Archimedean).

Constrained Case: Putinar's Certificate (Positivstellensatze)

If Polynomial $p(x)$ is positive on the set $\mathbf{K}$ then,

$$p(x) = \sigma_0(x) + \sum_{i=1}^{m} \sigma_i(x) g_i(x)$$

for some $\sigma_i(x) \in SOS_{2d_i}, \ i = 0, \dots, m,$

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS_{2d}$$

for some $d \in \mathbb{N}$

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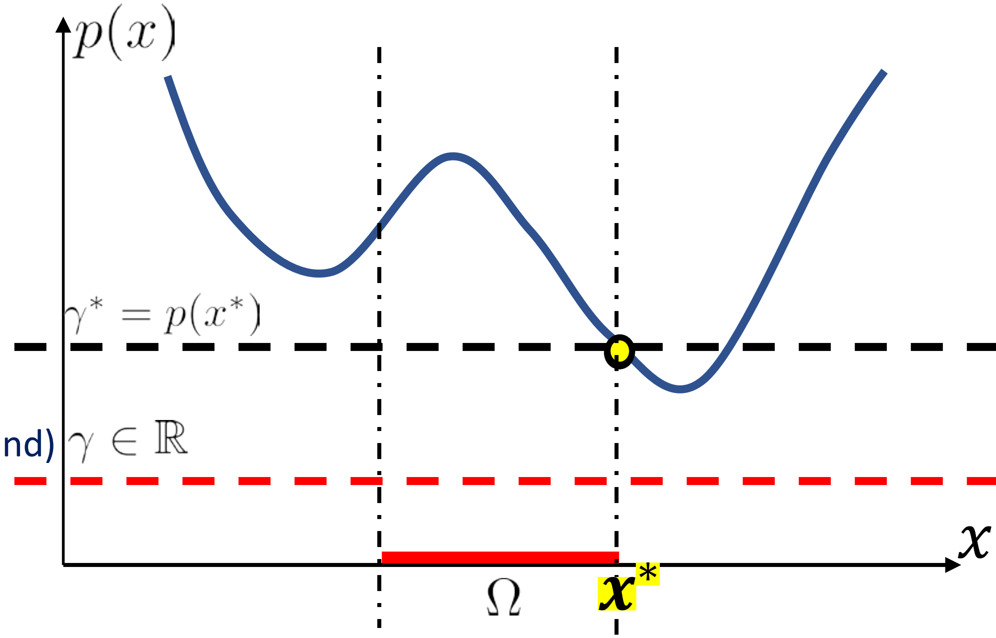

Reformulation in terms of Nonnegative polynomials

$$\underset{\gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma$$ → Linear constraint

$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Polynomial Nonnegativity constraint

We are looking for γ such that $p(x) - \gamma$ be a nonnegative polynomial on the set Ω

SOS Relaxation

$$\underset{\gamma, \sigma_i}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS_{2d} \quad \sigma_i(x) \in SOS_{2d_i}, \; i = 1, ..., m$$

Sum of Squares Condition

1. Sum of Squares Approach

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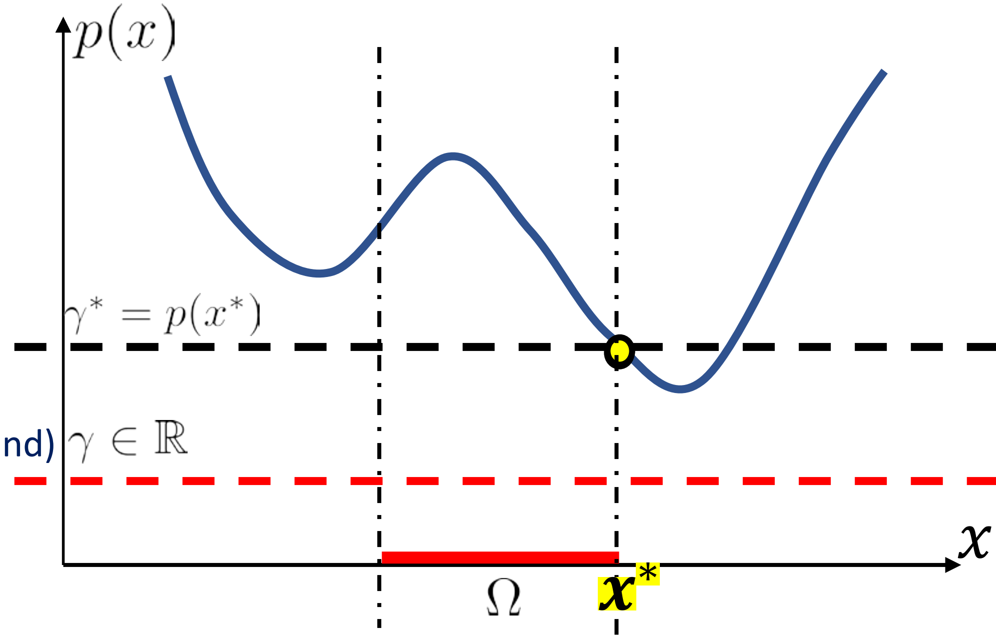

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SOS Relaxation

$\sigma(x) \in SOS \rightarrow \sigma(x) = \sum_{i=1}^{\ell} h_i^2(x) = B^T(x)QB(x), Q \succcurlyeq 0$

$$\begin{aligned} &\underset{\gamma,\sigma_i}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x)g_i(x) \in SOS_{2d} \quad \sigma_i(x) \in SOS_{2d_i},\ i = 1, ..., m \end{aligned}$$

Sum of Squares Condition

SOS SDP

$$\begin{aligned} &\underset{\gamma,Q_i|_{i=0}^{m}}{\text{maximize}} \quad \gamma \\ &\text{subject to} \\ &\text{coefficients of polynomial } p(x) - \gamma - \textstyle\sum_{i=1}^{m} \sigma_i g_i(x) = \text{coefficients of } B^T(x)Q_0B(x) \\ &\quad \sigma_i(x) = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, ..., m \qquad Q_i \succcurlyeq 0 \ i = 0, ..., m \end{aligned}$$

2. Measure and Moment Approach

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Optimization in terms of Probability distributions (measures):

➢ treat x as a random variable.

- **Decision variable:** μ Probability measure associated with x
- Look for μ to minimize $\mathrm{E}[p(x)]$

➢ Linear Program: Objective function is a linear function of the decision variables

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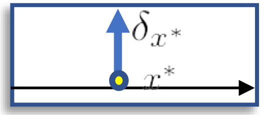
$$\text{subject to} \quad \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

- Linear Program: Objective function is a linear function of the decision variables

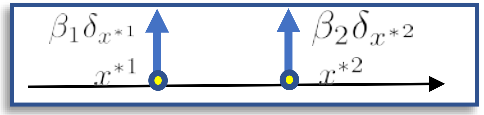
- $x^* \in \mathbf{\Omega}, p(x^*) = \mathrm{P}^*$: Unique global optimal solution of the original problem.

$\mu^* = \delta_{x^*}$



- $x^{*i} \in \mathbf{\Omega}, \quad i = 1, ..., r, p(x^{*i}) = \mathrm{P}^*$: r global optimal solution of the original problem.

$\mu^* = \sum_{i=1}^r \beta_i \delta_{x^{*i}}, \ \beta_i > 0, \sum_{i=1}^r \beta_i = 1$



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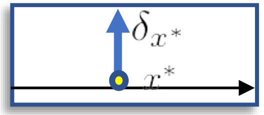

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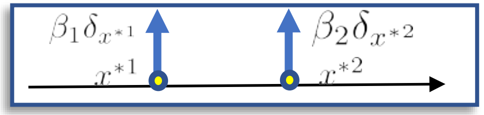

Optimization in Truncated Moment Space

➢ Approximate measure with a finite moment sequence. $y_\alpha = \mathrm{E}_\mu[x^\alpha]$

Moment Condition

Unconstrained Case:

Moments of every (nonnegative) measure μ in $\mathbb{R}^n$ satisfies :

Moment Matrix $\quad \mathbf{M}_\infty(y) \succcurlyeq 0 \quad$ (PSD)

➢ Not every sequence y that satisfies moment condition, has a representing measure μ.

$$\mu \text{ , } y\text{:moments} \;\; \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \;\; \mathbf{M}_\infty(y) \succcurlyeq 0$$

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Constrained Case:

- Sequence y has a representing measure with support contained in the set $\mathbf{K}$, **if and only if,** it satisfies:

$$\mathbf{M}_\infty(y) \succcurlyeq 0, \quad \forall d \qquad\qquad \mathbf{M}_{\infty_{g_i}}(g_i y) \succcurlyeq 0$$

Moment Matrix $\qquad$ Localizing Matrix

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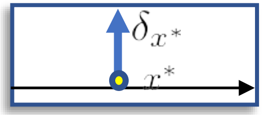

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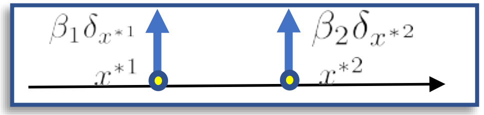

Optimization in Truncated Moment Space

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$$\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha = 0, ..., 2d}{\text{minimize}} \ \sum_\alpha p_\alpha y_\alpha$$

Moment SDP

$$\text{subject to} \quad y_0 = 1$$

Moment matrix $\quad \mathbf{M}_d(y) \succcurlyeq 0$

Localizing matrix $\quad \mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m.$

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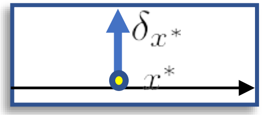

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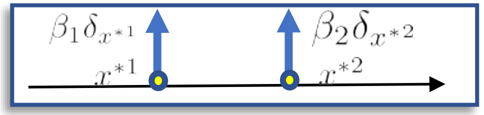

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➢ Optimal solution is the moment sequence of Dirac measures.

Moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = & \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} && \sum_\alpha p_\alpha y_\alpha \\ & \text{subject to} && y_0 = 1 \\ & \text{Moment matrix} && \mathbf{M}_d(y) \succcurlyeq 0 \\ & \text{Localizing matrix} && \mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m. \end{aligned}$$

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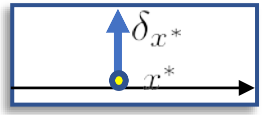

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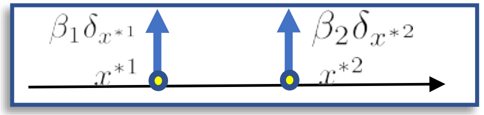

Optimization in Truncated Moment Space

- Approximate measure with a finite moment sequence. $y_\alpha = \mathrm{E}_\mu[x^\alpha]$
- Optimal solution is the moment sequence of Dirac measures.
- Moments of Dirac measure satisfies $r = \text{Rank } M_d(y^*) = \text{Rank } M_{d-v}(y^*)$
- We can extract x^* from the moments of Dirac measure.

$$\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,...,2d}{\text{minimize}} \ \sum_\alpha p_\alpha y_\alpha$$

Moment SDP

subject to $y_0 = 1$

Moment matrix $\mathbf{M}_d(y) \succcurlyeq 0$

Localizing matrix $\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m.$

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Equivalent

Equivalent

Space Nonnegative Continuous functions

$$\mathrm{P}_f^* = \underset{\gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Measure Space

$$\mathrm{P}_\mu^* = \underset{\mu \in \mathcal{M}(\Omega)}{\text{minimize}} \quad \mathrm{E}_\mu[p(x)] = \int p(x) d\mu$$
$$\text{subject to} \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Relaxation

Relaxation

Space of SOS Polynomials

Approximate nonnegative functions with finite degree polynomials

$$\mathrm{P}_{sos}^* = \underset{\gamma, \sigma_i}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS_{2d}$$
$$\sigma_i(x) \in SOS_{2d_i}, \ i = 1, ..., m$$

Looks for coefficients of (SOS) polynomial of degree 2d

Moments Space

Approximate measures with finite moments

$$\mathrm{P}_{mom}^{*d} = \underset{y_\alpha, \alpha = 0, \ldots, 2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$
$$\mathbf{M}_{d - d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m.$$

Looks for moments up to order 2d of (nonnegative)measure

$$\mathrm{P}^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

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- Constrained Optimization $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}$

Equivalent

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Dual

Measure Space

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Relaxation

Relaxation

Space of SOS Polynomials

Approximate nonnegative functions with finite degree polynomials

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Dual

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$$\mathrm{P}_{mom}^{*d} = \underset{y_\alpha, \alpha = 0, \ldots, 2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$
$$\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m.$$

Looks for moments up to order 2d of (nonnegative)measure

Dual Optimization

Primal Optimization:

$$\mathrm{P}^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

$$\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}$$

- **KKT Optimality Condition:**

Lagrange multiplier

Lagrange function: $L(x, \mu) = p(x) - \sum_{i=1}^{m} \mu_i g_i(x)$

$\nabla_x L(x, \mu) = 0$	*Stationarity*
$g_i(x) \geq 0, \quad i = 1, ..., m$	*Primal Feasibility*
$\mu_i \geq 0$	*Dual Feasibility*
$\mu_i^* g_i(x^*) = 0, \quad i = 1, ..., m$	*Dual Complementary Slackness*

Primal Optimization:

$$P^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

$$\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, \ldots, m\}$$

- **KKT Optimality Condition:**

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$\mu_i \geq 0$	*Dual Feasibility*
$\mu_i^* g_i(x^*) = 0, \quad i = 1, \ldots, m$	*Dual Complementary Slackness*

Dual Optimization:

$$D^* = \underset{\mu \in \mathbb{R}^m}{\text{maximize}} \quad d(\mu)$$

$$\text{subject to} \qquad \mu_i \geq 0 \quad i = 1, \ldots, m$$

Lagrange Dual function: $d(\mu) = \min_{x \in \mathbb{R}^n} \ L(x, \mu)$

- Dual Optimization is convex.
- **Weak Duality:** provides lower bound $D^* \leq P^*$
  - For feasible x, μ

$$L(x, \mu) = p(x) - \sum_{i=1}^{m} \mu_i g_i(x) \leq p(x)$$

Primal Optimization:

$$\mathrm{P}^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

$$\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, \ldots, m\}$$

- **KKT Optimality Condition:**

Lagrange multiplier

Lagrange function: $L(x, \mu) = p(x) - \sum_{i=1}^{m} \mu_i g_i(x)$

$\nabla_x L(x, \mu) = 0$	*Stationarity*
$g_i(x) \geq 0, \quad i = 1, \ldots, m$	*Primal Feasibility*
$\mu_i \geq 0$	*Dual Feasibility*
$\mu_i^* g_i(x^*) = 0, \quad i = 1, \ldots, m$	*Dual Complementary Slackness*

Dual Optimization:

$$\mathrm{D}^* = \underset{\mu \in \mathbb{R}^m}{\text{maximize}} \quad d(\mu)$$

$$\text{subject to} \qquad \mu_i \geq 0 \quad i = 1, \ldots, m$$

Lagrange Dual function: $d(\mu) = \min_{x \in \mathbb{R}^n} L(x, \mu)$

- Dual Optimization is convex.
- **Weak Duality:** provides lower bound $\mathrm{D}^* \leq \mathrm{P}^*$
  - For feasible x, μ

$$L(x, \mu) = p(x) - \sum_{i=1}^{m} \mu_i g_i(x) \leq p(x)$$

- **Strong Duality:** $\mathrm{D}^* = \mathrm{P}^*$

  If **i)** primal problem is **convex**, **ii) strictly feasible** (Slater's Condition)

Primal Optimization:

$$P^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

$$\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, \ldots, m\}$$

- **KKT Optimality Condition:**

Lagrange multiplier

Lagrange function: $L(x, \mu) = p(x) - \sum_{i=1}^{m} \mu_i g_i(x)$

$\nabla_x L(x, \mu) = 0$	*Stationarity*
$g_i(x) \geq 0, \quad i = 1, \ldots, m$	*Primal Feasibility*
$\mu_i \geq 0$	*Dual Feasibility*
$\mu_i^* g_i(x^*) = 0, \quad i = 1, \ldots, m$	*Dual Complementary Slackness*

Dual Optimization:

$$D^* = \underset{\mu \in \mathbb{R}^m}{\text{maximize}} \quad d(\mu)$$

$$\text{subject to} \quad \mu_i \geq 0 \quad i = 1, \ldots, m$$

Lagrange Dual function: $d(\mu) = \min_{x \in \mathbb{R}^n} L(x, \mu)$

- Dual Optimization is convex.
- **Weak Duality:** provides lower bound $D^* \leq P^*$
  - For feasible x, μ

$$L(x, \mu) = p(x) - \sum_{i=1}^{m} \mu_i g_i(x) \leq p(x)$$

- **Strong Duality:** $D^* = P^*$
  If **i)** primal problem is **convex**, **ii) strictly feasible** (Slater's Condition)
- If Feasible x^* feasible and μ^* satisfies $d(\mu^*) = p(x^*)$
  Then, x^* is a global solution of the original nonlinear optimization problem.

$$P^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$
- Constrained Optimization $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ \ i = 1, \dots, m\}$

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$P_f^* = \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Dual

Convex and Strictly feasible problems
(Slater's Condition)

$$\boxed{P_f^* = P_\mu^*}$$

(nonnegative) Measure Space

$$P_\mu^* = \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad E_\mu[p(x)] = \int p(x)d\mu$$
$$\text{subject to} \quad \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Relaxation

Relaxation

Space of SOS Polynomials

$$P_{sos}^{*d} = \underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$
$$\sigma_i(x) \in SOS_{2d_i}, \ \ i = 1, \dots, m$$

Dual

Convex and Strictly feasible problems
(Slater's Condition)

$$\boxed{P_{sos}^{*d} = P_{mom}^{*d}}$$

Moments Space

$$P_{mom}^{*d} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$
$$\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \dots, m.$$

$$P^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$
- Constrained Optimization $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., m\}$

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$P_f^* = \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Dual

Convex and Strictly feasible problems
(Slater's Condition)

$$\boxed{P_f^* = P_\mu^*}$$

(nonnegative) Measure Space

$$P_\mu^* = \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad E_\mu[p(x)] = \int p(x)d\mu$$
$$\text{subject to} \quad \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Relaxation

Relaxation

Space of SOS Polynomials

$$P_{sos}^{*d} = \underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$
$$\sigma_i(x) \in SOS_{2d_i}, \ i = 1, ..., m$$

Dual

Convex and Strictly feasible problems
(Slater's Condition)

$$\boxed{P_{sos}^{*d} = P_{mom}^{*d}}$$

Moments Space

$$P_{mom}^{*d} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$
$$\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m.$$

Due to Duality:

i) if SDP relaxation in one space achieves the optimal solution of the original problem, the dual SDP should do the same,

ii) we can use the same algorithm for the SOS and Moment relaxations to solve nonlinear optimization.

SOS and Moment SDP for Unconstrained Optimization

Unconstrained Optimization

$$\mathrm{P}^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$

$$deg(p(x)) = 2d$$ (otherwise, (odd degree) $\mathbf{P}^* = -\infty$)

Unconstrained Optimization

$$\mathrm{P}^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$

$deg(p(x)) = 2d$ (otherwise (odd degree) $\mathbf{P}^* = -\infty$)

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$\begin{aligned}\mathrm{P}_f^* = &\underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma \\ &\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n\end{aligned}$$

Dual

$$\mathrm{P}_f^* = \mathrm{P}_\mu^*$$

(nonnegative) Measure Space

$$\begin{aligned}\mathrm{P}_\mu^* = &\underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad \mathrm{E}_\mu[p(x)] = \int p(x)d\mu \\ &\text{subject to} \quad \int_\Omega d\mu = 1\end{aligned}$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Unconstrained Optimization

$$P^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$

$deg(p(x)) = 2d$ (otherwise (odd degree) $\mathbf{P}^* = -\infty$)

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$\begin{aligned} P_f^* = & \underset{\gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma \\ & \text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n \end{aligned}$$

Dual

$$P_f^* = P_\mu^*$$

(nonnegative) Measure Space

$$\begin{aligned} P_\mu^* = & \underset{\mu \in \mathcal{M}(\Omega)}{\text{minimize}} \quad \mathrm{E}_\mu[p(x)] = \int p(x) d\mu \\ & \text{subject to} \quad \int_\Omega d\mu = 1 \end{aligned}$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Relaxation

Relaxation

Space of SOS Polynomials

$$\begin{aligned} P_{sos}^{*d} = & \underset{\gamma}{\text{maximize}} \quad \gamma \\ & \text{subject to} \quad p(x) - \gamma \in SOS_{2d} \end{aligned}$$

2d-degree Polynomial

Looks for coefficients of (SOS) polynomial of degree 2d

Dual

$$P_{sos}^{*d} = P_{mom}^{*d}$$

Moments Space

$$\begin{aligned} P_{mom}^{*d} = & \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha \\ & \text{subject to} \quad y_0 = 1 \\ & \qquad \mathbf{M}_d(y) \succcurlyeq 0 \end{aligned}$$

Looks for moments up to order 2d of (nonnegative)measure

- We just need to construct single SOS SDP with relaxation order of 2d.

  Hence, relaxation order of Moment SDP is also 2d.

Unconstrained Optimization

$$\mathrm{P}^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$

$deg(p(x)) = 2d$ (otherwise (odd degree) $\mathbf{P}^* = -\infty$)

Equivalent

Space Nonnegative Continuous functions

$$\mathrm{P}_f^* = \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n$$

Dual

$$\mathrm{P}_f^* = \mathrm{P}_\mu^*$$

Equivalent

(nonnegative) Measure Space

$$\mathrm{P}_\mu^* = \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad \mathrm{E}_\mu[p(x)] = \int p(x)d\mu$$
$$\text{subject to} \quad \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Relaxation

Space of SOS Polynomials

$$\mathrm{P}_{sos}^{*d} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

2d-degree Polynomial

Looks for coefficients of (SOS) polynomial of degree 2d

Dual

$$\mathrm{P}_{sos}^{*d} = \mathrm{P}_{mom}^{*d}$$

Relaxation

Moments Space

$$\mathrm{P}_{mom}^{*d} = \underset{y_\alpha,\alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$

Looks for moments up to order 2d of (nonnegative)measure

- **SOS and Moment SDP** $\xrightarrow{\text{relaxation}}$ $\mathrm{P}_{sos}^{*d} = \mathrm{P}_{mom}^{*d} \leq \mathrm{P}^*$

i) Instead of searching space of nonnegative polynomials, we search space of SOS polynomials. (SOS condition is sufficient condition for nonnegativity).

ii) We use finite number of moments to represent measures. Also moment condition in $\mathbb{R}^n$ is necessary condition.

Unconstrained Optimization

$$P^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$

$deg(p(x)) = 2d$ (otherwise (odd degree) $\mathbf{P}^* = -\infty$)

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$P^*_f = \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n$$

(nonnegative) Measure Space

$$P^*_\mu = \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad E_\mu[p(x)] = \int p(x)d\mu$$
$$\text{subject to} \quad \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Dual

$$P^*_f = P^*_\mu$$

Relaxation

Relaxation

Space of SOS Polynomials

$$P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

2d-degree Polynomial

Looks for coefficients of (SOS) polynomial of degree 2d

Moments Space

$$P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$

Looks for moments up to order 2d of (nonnegative)measure

Dual

$$P^{*d}_{sos} = P^{*d}_{mom}$$

- **SOS and Moment SDP** $\xrightarrow{\text{relaxation}}$ $P^{*d}_{sos} = P^{*d}_{mom} \leq P^*$
- **Condition for** $P^{*d}_{sos} = P^{*d}_{mom} = P^*$ **:**

Unconstrained Optimization

$$P^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$

$deg(p(x)) = 2d$ (otherwise (odd degree) $\mathbf{P}^* = -\infty$)

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$P_f^* = \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n$$

Dual

$$P_f^* = P_\mu^*$$

(nonnegative) Measure Space

$$P_\mu^* = \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad \mathrm{E}_\mu[p(x)] = \int p(x)d\mu$$
$$\text{subject to} \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Relaxation

Relaxation

Space of SOS Polynomials

$$P_{sos}^{*d} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

2d-degree Polynomial

Looks for coefficients of (SOS) polynomial of degree 2d

Dual

$$P_{sos}^{*d} = P_{mom}^{*d}$$

Moments Space

$$P_{mom}^{*d} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$

Looks for moments up to order 2d of (nonnegative)measure

- **SOS and Moment SDP** $\xrightarrow{\text{relaxation}}$ $P_{sos}^{*d} = P_{mom}^{*d} \leq P^*$

- **Condition for** $P_{sos}^{*d} = P_{mom}^{*d} = P^*$ **: Moment SDP:**
  - y^* is moment sequence of Dirac measures
  - If $r = \text{Rank } M_d(y^*) = \text{Rank } M_{d-1}(y^*)$

- *Proposition 5.2, Theorem 5.3.* Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.

Unconstrained Optimization

$$P^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$

$deg(p(x)) = 2d$ (otherwise (odd degree) $\mathbf{P}^* = -\infty$)

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$\begin{aligned} P_f^* = & \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma \\ & \text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n \end{aligned}$$

Dual

$$P_f^* = P_\mu^*$$

(nonnegative) Measure Space

$$\begin{aligned} P_\mu^* = & \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad E_\mu[p(x)] = \int p(x) d\mu \\ & \text{subject to} \quad \int_\Omega d\mu = 1 \end{aligned}$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Relaxation

Relaxation

Space of SOS Polynomials

$$\begin{aligned} P_{sos}^{*d} = & \underset{\gamma}{\text{maximize}} \quad \gamma \\ & \text{subject to} \quad p(x) - \gamma \in SOS_{2d} \end{aligned}$$

2d-degree Polynomial

Looks for coefficients of (SOS) polynomial of degree 2d

Dual

$$P_{sos}^{*d} = P_{mom}^{*d}$$

Moments Space

$$\begin{aligned} P_{mom}^{*d} = & \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha \\ & \text{subject to} \quad y_0 = 1 \\ & \qquad\qquad\quad \mathbf{M}_d(y) \succcurlyeq 0 \end{aligned}$$

Looks for moments up to order 2d of (nonnegative)measure

- **SOS and Moment SDP** $\xrightarrow{\text{relaxation}}$ $P_{sos}^{*d} = P_{mom}^{*d} \leq P^*$
- **Condition for** $P_{sos}^{*d} = P_{mom}^{*d} = P^*$ **:** **Moment SDP:** $\left\{ \begin{array}{l} y^* \text{is moment sequence of Dirac measures} \\ \text{If } r = \text{Rank } M_d(y^*) = \text{Rank } M_{d-1}(y^*) \end{array} \right.$ **SOS SDP:** $\left\{ \begin{array}{l} \text{Let } \gamma^* = P^* \\ p(x) - \gamma^* \in SOS_{2d} \end{array} \right.$

- *Proposition 5.2, Theorem 5.3.* Jean Bernard Lasserre, "Moments, Positive Polynomials and Their Applications" Imperial College Press Optimization Series, V. 1, 2009.

Unconstrained Optimization

$\Omega = \mathbb{R}^n$

$$P^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

$2d = deg(p(x))$

Moment Relaxation (SDP)

$$
\begin{aligned}
P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\
\text{subject to} \quad & y_0 = 1 \\
& \mathbf{M}_d(y) \succcurlyeq 0
\end{aligned}
$$

Dual

SOS Relaxation (SDP)

$$
\begin{aligned}
P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad & \gamma \\
\text{subject to} \quad & p(x) - \gamma \in SOS_{2d}
\end{aligned}
$$

Solve single Moment SDP:

- If $r = \text{Rank } M_d(y^*) = \text{Rank } M_{d-1}(y^*) \longrightarrow$ y^* is moment sequence of Dirac measures

  $i) P^{*d}_{mom} = P^*$

  $ii) x_i^*, i = 1, \ldots, r,$ global solutions can be extracted by linear algebra

- If $\text{Rank } M_d(y^*) \neq \text{Rank } M_{d-1}(y^*)$

  $P^{*d}_{mom} \leq P^*$

Solve single SOS SDP:

$P^{*d}_{sos} = P^*$

$P^{*d}_{sos} \leq P^*$

- *Algorithm 5.1,* Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.

SOS and Moment SDP for Constrained Optimization

Constrained Optimization

$$\mathrm{P}^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x) \qquad\qquad \Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, \ldots, m\}$$

Relaxation

Relaxation

Space of SOS Polynomials

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS_{2d}$$

$$\sigma_i(x) \in SOS_{2d_i},\ i = 1, \ldots, m$$

Looks for coefficients of (SOS) polynomial of degree 2d

- **d: relaxation order**

$2d \geq \max(deg(p(x)), deg(g_i(x)))$

Dual

$$\mathrm{P}^{*d}_{sos} = \mathrm{P}^{*d}_{mom}$$

Moments Space

$$\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$

$$\begin{aligned} \text{subject to} \quad & y_0 = 1 \\ & \mathbf{M}_d(y) \succcurlyeq 0 \\ & \mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, m. \end{aligned}$$

Looks for moments up to order 2d of (nonnegative)measure

- **d: relaxation order**

$2d \geq \max(deg(p(x)), deg(g_i(x)))$

In constrained problem:

i) Based on Putinar's condition, there exist a degree $2d$ for which positive polynomial takes the SOS representation on the given set Ω.

ii) Moment condition is necessary and sufficient condition.

Constrained Optimization

$$\mathrm{P}^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

$$\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}$$

Moment Relaxation (SDP)

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha = 0, \ldots, 2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\ \text{subject to} \quad & y_0 = 1 \\ & \mathbf{M}_d(y) \succcurlyeq 0 \\ & \mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, m. \end{aligned}$$

$d_{g_i} = \lceil \frac{deg(g_i(x))}{2} \rceil$

$2d \geq \max\left(deg(p(x)), deg(g_i(x))\right)$

Dual

SOS Relaxation (SDP)

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = \underset{\gamma, \sigma_i}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma - \sum_{i=1}^m \sigma_i(x) g_i(x) \in SOS_{2d} \\ & \sigma_i(x) \in SOS_{2d_i}, \ i = 1, \ldots, m \end{aligned}$$

- If $r = \text{Rank } M_d(y^*) = \text{Rank } M_{d-v}(y^*)$ → y^* is moment sequence of Dirac measures

  $v = \max\{d_{g_i}\}$

  $i) \mathrm{P}^{*d}_{mom} = \mathrm{P}^*$ ⟺ $\mathrm{P}^{*d}_{sos} = \mathrm{P}^*$

  $ii) x_i^*, i = 1, ..., r,$ global solutions can be extracted by linear algebra
- If $\text{Rank } M_d(y^*) \neq \text{Rank } M_{d-v}(y^*)$, increas d and solve new SDP
- Otherwise $\mathrm{P}^{*d}_{mom} \leq \mathrm{P}^*$ ⟺ $\mathrm{P}^{*d}_{sos} \leq \mathrm{P}^*$
- as $d \to \infty$ $\mathrm{P}^{*d}_{mom} \uparrow \mathrm{P}^*$ ⟺ as $d \to \infty$ $\mathrm{P}^{*d}_{sos} \uparrow \mathrm{P}^*$

  Finite Convergence: $\exists\, d^* \quad \mathrm{P}^{*d^*}_{mom} = \mathrm{P}^* \quad d \geq d^*$ ⟺ $\exists\, d^* \quad \mathrm{P}^{*d^*}_{sos} = \mathrm{P}^* \quad d \geq d^*$

- *Theorem 5.6, Chapter 7*: Jean Bernard Lasserre, "Moments, Positive Polynomials and Their Applications" Imperial College Press Optimization Series, V. 1, 2009.

Examples

Example 1: Unconstrained Optimization

https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_1_MOM.m
https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_1_SOS.m

$$\mathrm{P}^* = \underset{x \in \mathbb{R}}{\text{minimize}} \quad x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5$$

$$\begin{aligned} &\underset{\gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \in SOS \end{aligned}$$

$$\begin{aligned} \mathrm{P}^{*2}_{sos} = &\underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && \text{coefficients of } (x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma) = \text{coefficients of } B_2^T(x) Q B_2(x) \\ &&& Q \succcurlyeq 0 \end{aligned}$$

Looks for polynomial of order 4

Looks for moments up to order 4

$$\begin{aligned} \mathrm{P}^{*2}_{sos} = &\underset{Q \in \mathcal{S}^3, \gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && q_{22} = 1, \quad 2q_{12} = 4, \quad q_{11} + 2q_{02} = 6, \quad 2q_{01} = 4, \quad q_{00} = 5 - \gamma \\ &&& Q = \begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{01} & q_{11} & q_{12} \\ q_{02} & q_{12} & q_{22} \end{bmatrix} \succcurlyeq 0 \end{aligned}$$

Dual

$$\mathrm{P}^{*2}_{mom} = \underset{y_0, y_1, y_2, y_3, y_4}{\text{minimize}} \quad y_4 + 4y_3 + 6y_2 + 4y_1 + 5y_0$$

$$\text{subject to} \quad y_0 = 1 \qquad \mathbf{M}_2(y) = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succcurlyeq 0$$

`Yalmip``-Mosek`

$$\mathrm{P}^{*2}_{sos} = \gamma^* = 4$$

$$\mathrm{P}^{*2}_{sos} - \gamma^* \in SOS$$

$$\mathrm{P}^{*2}_{sos} - \gamma^* = B^T(x) Q B(x), \quad Q \succcurlyeq 0$$

$$Q = \begin{bmatrix} 1 & 2 & 0.9998 \\ 2 & 4.0004 & 2 \\ 0.9998 & 2 & 1 \end{bmatrix}$$

`GloptiPOly``-Mosek`

$$\mathrm{P}^{*2}_{mom} = 4 \qquad \mathrm{Rank}(M_2(y)) = \mathrm{Rank}(M_1(y)) = 1 \quad \Rightarrow \quad x^* = y_1 = -0.9998$$

```
[status,obj,m,dual]= msol(P)
```

$$\text{Dual} = \begin{bmatrix} 1 & 2 & 0.9996 \\ 2 & 4.0009 & 2 \\ 0.9996 & 2 & 1 \end{bmatrix}$$

Example 2: Unconstrained Optimization

https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_2_MOM.m
https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_2_SOS.m

$$\mathrm{P}^* = \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad 3+2x_1+2x_2+3x_1^2+2x_1x_2+3x_2^2+x_1^4+x_2^4$$

$$\begin{aligned} &\underset{\gamma\in\mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && 3+2x_1+2x_2+3x_1^2+2x_1x_2+3x_2^2+x_1^4+x_2^4-\gamma \in SOS \end{aligned}$$

Looks for polynomial of order 4

$$\mathrm{P}^{*2}_{sos} = \underset{Q\in\mathcal{S}^6,\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$

subject to

$$\text{coefficients of } (3+2x_1+2x_2+3x_1^2+2x_1x_2+3x_2^2+x_1^4+x_2^4-\gamma) = \text{coefficients of } B_2^T(x)QB_2(x)$$

$$Q \succcurlyeq 0$$

Dual

Looks for moments up to order 4

$$y = [y_{00}|y_{10},y_{01}|y_{20},y_{11},y_{02}|y_{30},y_{21},y_{12},y_{03}|y_{40},y_{31},y_{22},y_{13},y_{04}]$$

$$\mathrm{P}^{*2}_{mom} = \underset{y}{\text{minimize}} \quad 3y_{00}+2y_{10}+2y_{01}+3y_{20}+2y_{11}+3y_{02}+y_{40}+y_{04}$$

subject to

$$\mathrm{M}_2(y) = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{01} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succcurlyeq 0$$

$$y_{00} = 1$$

Yalmip-Mosek

$$\mathrm{P}^{*2}_{sos} = \gamma^* = 2.5074$$

$$Q = \begin{bmatrix} 0.4926 & 1.0000 & 1.0000 & -0.0533 & -0.0114 & -0.0533 \\ 1.0000 & 3.1066 & 1.0114 & -0.0000 & 0.1013 & -0.1013 \\ 1.0000 & 1.0114 & 3.1066 & -0.1013 & 0.1013 & 0.0000 \\ -0.0533 & -0.0000 & -0.1013 & 1.0000 & -0.0000 & -0.5140 \\ -0.0114 & 0.1013 & 0.1013 & -0.0000 & 1.0280 & -0.0000 \\ -0.0533 & -0.1013 & 0.0000 & -0.5140 & -0.0000 & 1.0000 \end{bmatrix}$$

GloptiPOly-Mosek

$$\mathrm{P}^{*2}_{mom} = 2.5074 \qquad \mathrm{Rank}(M_2(y)) = \mathrm{Rank}(M_1(y)) = 1 \Rightarrow \begin{aligned} x_1^* &= y_{10} = -0.2428 \\ x_2^* &= y_{01} = -0.2428 \end{aligned}$$

```
[status,obj,m,dual]= msol(P)
```

$$\text{Dual} = \begin{bmatrix} 0.4926 & 1.0000 & 1.0000 & -0.0369 & -0.0442 & -0.0369 \\ 1.0000 & 3.0737 & 1.0442 & -0.0000 & 0.0022 & -0.0022 \\ 1.0000 & 1.0442 & 3.0737 & -0.0022 & 0.0022 & 0.0000 \\ -0.0369 & -0.0000 & -0.0022 & 1.0000 & -0.0000 & -0.3843 \\ -0.0442 & 0.0022 & 0.0022 & -0.0000 & 0.7686 & -0.0000 \\ -0.0369 & -0.0022 & 0.0000 & -0.3843 & -0.0000 & 1.0000 \end{bmatrix}$$

Example 3: Unconstrained Optimization

https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_3_MOM.m
https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_3_SOS.m

$$\mathrm{P}^* = \underset{x \in \mathbb{R}^2}{\text{minimize}} \quad -x_1^2 x_2^2 + x_1^4 x_2^2 + x_1^2 x_2^4$$

$$\mathrm{P}^* = -1/27, \;\; x^* = [\pm\sqrt{3}/3, \pm\sqrt{3}/3]$$

$$\begin{aligned} \mathrm{P}^{*3}_{sos} = & \underset{Q \in \mathcal{S}^6, \gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma \\ & \text{subject to} \\ & \quad \text{coefficients of } (-x_1^2 x_2^2 + x_1^4 x_2^2 + x_1^2 x_2^4 - \gamma) = \text{coefficients of } B_3^T(x) Q B_3(x) \\ & \quad Q \succcurlyeq 0 \end{aligned}$$

Dual

$$\begin{aligned} \mathrm{P}^{*3}_{mom} = & \underset{y}{\text{minimize}} \quad -y_{22} + y_{42} + y_{24} \\ & \text{subject to} \\ & y_{00} = 1 \qquad \mathbf{M}_3(y) \succcurlyeq 0 \end{aligned}$$

Looks for polynomial of order 6

Looks for moments up to order 6

Yalmip-Mosek

Primal infeasible

$$-x_1^2 x_2^2 + x_1^4 x_2^2 + x_1^2 x_2^4 - \gamma^* \notin SOS$$

GloptiPOly-Mosek

Status=-1:
moment SDP could NOT be solved (unbounded SDP).

Example 4: Example 3 with Constraint

https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_4_MOM.m
https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_4_SOS.m

$$\mathrm{P}^* = \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad -x_1^2x_2^2 + x_1^4x_2^2 + x_1^2x_2^4$$
$$\text{subject to} \quad x \in \mathbf{K} = \{x \in \mathbb{R}^2 : 1 - x_1^2 - x_2^2 \geq 0\}$$

$$\mathrm{P}^* = -1/27, \ x^* = [\pm\sqrt{3}/3, \pm\sqrt{3}/3]$$

- $2d = \max(\deg(p(x)), \deg(g(x))) = 6$

$$\mathrm{P}^{*3}_{sos} = \underset{\gamma,\sigma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad \underbrace{-x_1^2x_2^2 + x_1^4x_2^2 + x_1^2x_2^4 - \gamma - \sigma_1 g(x)}_{\text{Degree 6}} \in SOS_{2d} \quad \sigma_1 \in SOS_{\underbrace{2d-deg(g)}_{4}}$$

- Looks for polynomial of order 6

$$\mathrm{P}^{*3}_{sos} = \underset{Q\in\mathcal{S}^6,\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to}$$
$$\text{coefficients of } (-x_1^2x_2^2+x_1^4x_2^2+x_1^2x_2^4-\gamma-\sigma g(x)) = \text{coefficients of } B_3^T(x)QB_3(x)$$
$$\text{coefficients of } (\sigma_1) = \text{coefficients of } B_2^T(x)Q_1B_2(x)$$
$$Q \succcurlyeq 0, \ Q_1 \succcurlyeq 0$$

Yalmip-Mosek

$$\mathrm{P}^{*3}_{sos} = \gamma = -0.0370$$

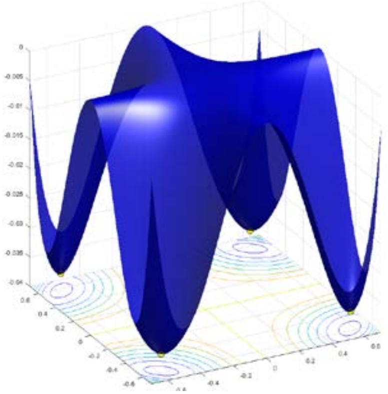
Dual

- $2d = \max(\deg(p(x)), \deg(g(x))) = 6$
- Looks for moments up to order 6

$$\mathrm{P}^{*3}_{mom} = \underset{y}{\text{minimize}} \quad -y_{22} + y_{42} + y_{24}$$
$$\text{subject to} \quad y_{00} = 1 \quad \mathbf{M}_3(y) \succcurlyeq 0 \quad \mathbf{M}_2(g(x)y) \succcurlyeq 0$$

$d - d_g = 3 - 1 = 2$

GloptiPOly-Mosek



$$\mathrm{Rank}(M_3(y)) = \mathrm{Rank}(M_2(y)) = 4$$

$$\Rightarrow \quad \mathrm{P}^{*3}_{mom} = -0.0370$$

$$(x_1^*, x_2^*) = (0.5773, 0.5773)$$
$$(x_1^*, x_2^*) = (0.5773, -0.5773)$$
$$(x_1^*, x_2^*) = (-0.5773, 0.5773)$$
$$(x_1^*, x_2) = (-0.5773, -0.5773)$$

Example 5: Example 3 with moment constraints

https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_5_MOM.m

$$\mathrm{P}^* = \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad -x_1^2x_2^2 + x_1^4x_2^2 + x_1^2x_2^4$$

$$\mathrm{P}^* = -1/27, \; x^* = [\pm\sqrt{3}/3, \pm\sqrt{3}/3]$$

$\mathbf{M}_d(y) \succcurlyeq 0, \;\; \forall d \qquad \exists\; c, a > 0 : \; |y_\alpha| \leq c a^\alpha$ (linear condition) $\Rightarrow$ y has a representing measure with support contained in $[-a, a]^n$

Proposition 3.5: Jean Bernard Lasserre, "Moments, Positive Polynomials and Their Applications" Imperial College Press Optimization Series, V. 1, 2009.

$$\mathrm{P}^{*3}_{mom} = \underset{y}{\text{minimize}} \quad -y_{22} + y_{42} + y_{24}$$
$$\text{subject to} \quad y_{00} = 1 \qquad \mathbf{M}_3(y) \succcurlyeq 0 \qquad |y| \leq 1$$

$\longrightarrow$ Moments of measure supported in $[-1,1]^2$

GloptiPOly-Mosek

Statues=-1

$$\mathrm{P}^{*4}_{mom} = \underset{y}{\text{minimize}} \quad -y_{22} + y_{42} + y_{24}$$
$$\text{subject to} \quad y_{00} = 1 \qquad \mathbf{M}_4(y) \succcurlyeq 0 \qquad |y| \leq 1$$

GloptiPOly-Mosek

Statues=0

$\mathrm{P}^{*4}_{mom} = -0.0370 \leq \mathrm{P}^*$

$$\mathrm{P}^{*5}_{mom} = \underset{y}{\text{minimize}} \quad -y_{22} + y_{42} + y_{24}$$
$$\text{subject to} \quad y_{00} = 1 \qquad \mathbf{M}_5(y) \succcurlyeq 0 \qquad |y| <= 1$$

GloptiPOly-Mosek

$\mathrm{P}^{*5}_{mom} = -0.0370$

$\mathrm{Rank}(M_3(y)) = \mathrm{Rank}(M_2(y)) = 4 \;\Rightarrow$

$(x_1^*, x_2^*) = (0.5773, 0.5773)$

$(x_1^*, x_2^*) = (0.5773, -0.5773)$

$(x_1^*, x_2^*) = (-0.5773, 0.5773)$

$(x_1^*, x_2) = (-0.5773, -0.5773)$

Example 6: Constrained Optimization

https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_6_MOM.m
https://github.com/jasour/rarnop19/blob/master/Lecture5_Duality_SOS-Moment/Example_6_SOS.m

$$\mathrm{P}^* = \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad -x_1$$

$$\text{subject to} \quad x \in \mathbf{K} = \{x \in \mathbb{R}^2 : 3 - 2x_2 - x_1^2 - x_2^2 \geq 0,\ -x_1 - x_2 - x_1x_2 \geq 0,\ 1 + x_1x_2 \geq 0\}$$

- $2d = \max(\deg(p(x)), \deg(g(x))) = 2$
- Looks for polynomial of order 2

$$\mathrm{P}^{*1}_{sos} = \underset{\gamma,\sigma}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad \underbrace{-x_1 - \gamma - \sigma_1 g_1(x) - \sigma_2 g_2(x) - \sigma_3 g_3(x)}_{\text{Degree 2}} \in SOS_{2d}$$

$$\sigma_1 \in SOS_{\underbrace{2d-deg(g_1)}_{0}} \quad \sigma_2 \in SOS_{\underbrace{2d-deg(g_2)}_{0}} \quad \sigma_3 \in SOS_{\underbrace{2d-deg(g_3)}_{0}}$$

Nonnegative scalar

Yalmip-Mosek $\mathrm{P}^{*1}_{sos} = \gamma = -1.7808$

- $2d = 4$

Yalmip-Mosek

$$\mathrm{P}^{*2}_{sos} = \gamma^* = -1.6180$$

Dual

- $2d = \max(\deg(p(x)), \deg(g(x))) = 2$
- Looks for moments up to order 2

$$\mathrm{P}^{*1}_{mom} = \underset{y}{\text{minimize}} \quad -y_{10}$$

$$\text{subject to} \quad y_{00} = 1 \quad \mathbf{M}_3(y) \succcurlyeq 0$$

$$\mathbf{M}_0(g_1(x)y) \succcurlyeq 0 \quad \mathbf{M}_0(g_2(x)y) \succcurlyeq 0 \quad \mathbf{M}_0(g_3(x)y) \succcurlyeq 0$$

$$d - d_{g_1} = 1 - 1 = 1 \quad d - d_{g_2} = 1 - 1 = 0 \quad d - d_{g_3} = 1 - 1 = 0$$

GloptiPOly-Mosek Statues=0 $\mathrm{P}^{*1}_{mom} = -1.7808 \leq \mathrm{P}^*$

- $2d = 4$

GloptiPOly-Mosek

$$\mathrm{P}^{*2}_{mom} = -1.6180 = \mathrm{P}^*$$

$$\mathrm{Rank}(M_2(y)) = \mathrm{Rank}(M_1(y)) = 1 \quad \Rightarrow \quad (x_1^*, x_2^*) = (1.6180, -0.6180)$$

How to use SOS and Moment Relaxations?

- Formulate the original problem as an (nonlinear,nonconvex) optimization problem.
  Then, use SOS or Moment relaxations to obtain SDP problem (convex).

- Obtain SOS or Measure conditions from the original problem.

Example: Stability of nonlinear systems

energy function: $V(x) \geq 0, -\dot{V}(x) \geq 0 \longrightarrow V(x) \in SOS, -\dot{V}(x) \in SOS$

How to use SOS and Moment Relaxations?

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SOS or Moment Relaxation ?

Moment relaxation:

1) to find x^*
2) Uncertain problems with probabilistic uncertainties

SOS relaxation:

1) Problems involving function or set approximation, e.g., Lyapunov function, Region of attraction set.
2) Uncertain problems with uncertainty set.

Duality of SOS and Moment Approaches

(Proofs)

Constrained Optimization

$$P^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$
- Constrained Optimization $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}$

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$P_f^* = \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Dual

Convex and Strictly feasible problems
(Slater's Condition)

$$\boxed{P_f^* = P_\mu^*}$$

(nonnegative) Measure Space

$$P_\mu^* = \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad E_\mu[p(x)] = \int p(x)d\mu$$

$$\text{subject to} \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

LP

Relaxation

Relaxation

Space of SOS Polynomials

$$P_{sos}^{*d} = \underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x)g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i}, \ i = 1, ..., m$$

Dual

Convex and Strictly feasible problems
(Slater's Condition)

$$\boxed{P_{sos}^{*d} = P_{mom}^{*d}}$$

Moments Space

$$P_{mom}^{*d} = \underset{y_\alpha,\alpha=0,\dots,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$

$$\text{subject to} \quad y_0 = 1$$

$$\mathbf{M}_d(y) \succcurlyeq 0$$

$$\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m.$$

SDP

Duality

Duality of Linear Programs:

Primal LP

$$
\begin{aligned}
\mathrm{P}^* = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\mathrm{D}^* = \underset{y, s \in \mathbb{R}^n}{\text{maximize}} \quad & b^T y \\
\text{subject to} \quad & c - A^T y = s \\
& s \geq 0.
\end{aligned}
$$

Duality

Duality of Linear Programs:

Primal LP

$$\begin{aligned} \mathrm{P}^* = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} \mathrm{D}^* = \underset{y, s \in \mathbb{R}^n}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & c - A^T y = s \\ & s \geq 0. \end{aligned}$$

Euclidean space: $x \in \mathbb{R}^n$

Duality

Duality of Linear Programs:

Primal LP

$$\begin{aligned} \mathrm{P}^* = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} \mathrm{D}^* = \underset{y, s \in \mathbb{R}^n}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & c - A^T y = s \\ & s \geq 0. \end{aligned}$$

Euclidean space: $x \in \mathbb{R}^n$

Extend to different spaces

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = \underset{x}{\text{minimize}} \quad & \langle c, x \rangle \\ \text{subject to} \quad & A(x) = b \\ & x \in K. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = \underset{y, s}{\text{maximize}} \quad & \langle b, y \rangle \\ \text{subject to} \quad & c - A^*(y) = s \\ & s \in K^*. \end{aligned}$$

e.g.,

- x ∈ *Euclidean space*
- x ∈ *Space of symmetric matrices*
- x ∈ *Space of continuous functions*

Duality

Duality of Linear Programs:

Primal LP

$$\begin{aligned} P^* = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} D^* = \underset{y,s \in \mathbb{R}^n}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & c - A^T y = s \\ & s \geq 0. \end{aligned}$$

Euclidean space: $x \in \mathbb{R}^n$

Extend to different spaces

Primal Conic Program

$$\begin{aligned} P^* = \underset{x}{\text{minimize}} \quad & \langle c, x \rangle \\ \text{subject to} \quad & A(x) = b \\ & x \in K. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} D^* = \underset{y,s}{\text{maximize}} \quad & \langle b, y \rangle \\ \text{subject to} \quad & c - A^*(y) = s \\ & s \in K^*. \end{aligned}$$

e.g.,

- x ∈ *Euclidean space*
- x ∈ *Space of symmetric matrices*
- x ∈ *Space of continuous functions*

Duality of Linear Programs → Duality of Conic Programs →
- Duality of LP in measure space
- Duality of SOS and Moment SDP

Duality of Linear Programs

Standard Linear Program

$$
\begin{aligned}
\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0.
\end{aligned}
$$

Standard Linear Program

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Lagrange multipliers

Lagrange function: $$L(x, y, s) = c^T x - y^T (Ax - b) - s^T x$$
$$= b^T y + (c - A^T y - s)^T x$$

KKT Condition: $$\nabla_x L(x, y, s) = c - A^T y - s = 0$$
$$Ax = b \quad x \geq 0 \qquad \boxed{s \geq 0} \quad s^T x = 0$$

Standard Linear Program

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Lagrange multipliers

Lagrange function:

$$\begin{aligned} L(x, y, s) &= c^T x - y^T (Ax - b) - s^T x \\ &= y^T b + (c - A^T y - s)^T x \end{aligned}$$

Dual Lagrange function

$$d(y, s) = \underset{x \in \mathbb{R}^n}{\text{minimize}}\ L(x, y, s)$$

Standard Linear Program

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Lagrange multipliers

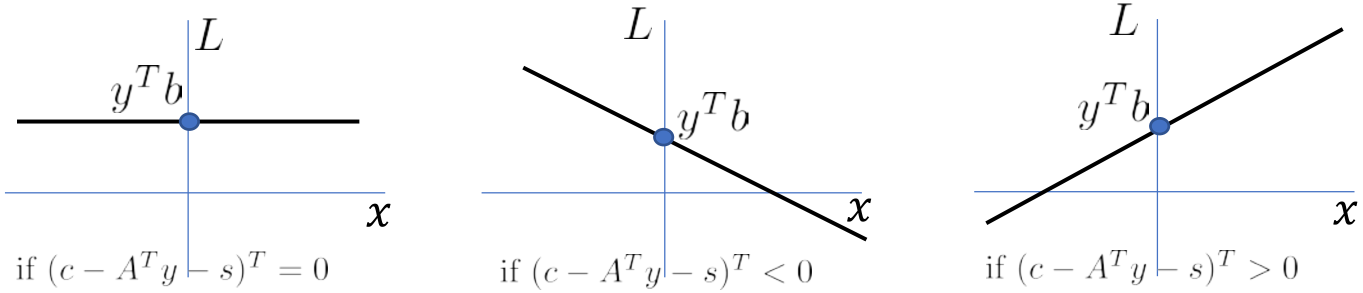
Lagrange function:

$$\begin{aligned} L(x, y, s) &= c^T x - y^T (Ax - b) - s^T x \\ &= y^T b + (c - A^T y - s)^T x \end{aligned}$$

Dual Lagrange function

$$d(y, s) = \underset{x \in \mathbb{R}^n}{\text{minimize}} \; L(x, y, s) \longrightarrow d(y, s) = \underset{x \in \mathbb{R}^n}{\text{minimize}} \; L(x, y, s) = \begin{cases} y^T b & c - A^T y - s = 0 \\ -\infty & \text{otherwise} \end{cases}$$

- We want to minimize Lagrange function with respect to x.
- $y^T b + (c - A^T y - s)^T x$: Lagrange function is a line in terms of x.



$$d(y, s) = \underset{x \in \mathbb{R}^n}{\text{minimize}} \; L(x, y, s) = y^T b \qquad d(y, s) = \underset{x \in \mathbb{R}^n}{\text{minimize}} \; L(x, y, s) = -\infty$$

- We are looking for the best minimum point (best lower bound of the objective function of the primal).

Standard Linear Program

$$
\begin{aligned}
\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0.
\end{aligned}
$$

Lagrange multipliers

Lagrange function:

$$
\begin{aligned}
L(x, y, s) &= c^T x - y^T (Ax - b) - s^T x \\
&= y^T b + (c - A^T y - s)^T x
\end{aligned}
$$

Dual Lagrange function

$$d(y, s) = \underset{x \in \mathbb{R}^n}{\text{minimize}}\; L(x, y, s) \longrightarrow d(y, s) = \underset{x \in \mathbb{R}^n}{\text{minimize}}\; L(x, y, s) = \begin{cases} y^T b & c - A^T y - s = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual LP: $\underset{y, s \geq 0}{\text{maximize}}\; d(y, s)$

$$
\boxed{
\begin{aligned}
\underset{y, s}{\text{maximize}} \quad & y^T b \\
\text{subject to} \quad & c - A^T y = s \\
& s \geq 0.
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
\underset{y}{\text{maximize}} \quad & y^T b \\
\text{subject to} \quad & c - A^T y \geq 0
\end{aligned}
}
$$

Primal LP

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \quad (1) \\ & x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & y^T b \\ \text{subject to} \quad & c - A^T y = s \quad (2) \\ & s \geq 0. \end{aligned}$$

Primal LP

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \quad (1) \\ & x \geq 0. \end{aligned}$$

Primal Space : $x \in \mathbb{R}^n$

$$\text{Cone}: \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

Dual LP

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & y^T b \\ \text{subject to} \quad & c - A^T y = s \quad (2) \\ & s \geq 0. \end{aligned}$$

Dual Space: $s \in \mathbb{R}^n$

$$\text{Dual Cone:} \quad K^* = \{s \in \mathbb{R}^n : s \geq 0\}$$

Primal LP

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c^T x$$
$$\text{subject to} \quad Ax = b \quad (1)$$
$$x \geq 0.$$

Primal Space : $x \in \mathbb{R}^n$

$$\text{Cone}: \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

Dual LP

$$\underset{y,s}{\text{maximize}} \quad y^T b$$
$$\text{subject to} \quad c - A^T y = s \quad (2)$$
$$s \geq 0.$$

Dual Space: $s \in \mathbb{R}^n$

$$\text{Dual Cone:} \quad K^* = \{s \in \mathbb{R}^n : s \geq 0\}$$

Weak Duality: $y^T b \leq c^T x \longrightarrow c^T x - y^T b \geq 0$

Primal LP

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \quad (1) \\ & x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & y^T b \\ \text{subject to} \quad & c - A^T y = s \quad (2) \\ & s \geq 0. \end{aligned}$$

Primal Space : $x \in \mathbb{R}^n$

$$\text{Cone}: \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

Dual Space: $s \in \mathbb{R}^n$

$$\text{Dual Cone:} \quad K^* = \{s \in \mathbb{R}^n : s \geq 0\}$$

Weak Duality: $y^T b \leq c^T x \longrightarrow c^T x - y^T b \geq 0$

$$c^T x - y^T b \geq 0 \Rightarrow (A^T y + s)^T x - y^T (Ax) \geq 0 \Rightarrow \underbrace{(A^T y)^T x}_{\mathbb{R}} + s^T x - \underbrace{y^T (Ax)}_{\mathbb{R}} \geq 0 \Rightarrow s^T x \geq 0$$

2 1

$$(A^T y)^T x = y^T (Ax)$$

Primal LP

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \quad \text{1} \\ & x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & y^T b \\ \text{subject to} \quad & c - A^T y = s \quad \text{2} \\ & s \geq 0. \end{aligned}$$

Primal Space : $x \in \mathbb{R}^n$

Cone : $K = \{x \in \mathbb{R}^n : x \geq 0\}$

Dual Space: $s \in \mathbb{R}^n$

Dual Cone: $K^* = \{s \in \mathbb{R}^n : s \geq 0\}$

Weak Duality: $y^T b \leq c^T x \longrightarrow c^T x - y^T b \geq 0$

$c^T x - y^T b \geq 0 \Rightarrow (A^T y + s)^T x - y^T (Ax) \geq 0 \Rightarrow \underbrace{(A^T y)^T x}_{\mathbb{R}} + s^T x - \underbrace{y^T (Ax)}_{\mathbb{R}} \geq 0 \Rightarrow \boxed{s^T x \geq 0}$

2 1

$(A^T y)^T x = y^T (Ax)$

Characterizes the dual cone

Dual Cone: $K^* = \{s \in \mathbb{R}^n : s^T x \geq 0 \ \forall x \in K\} = \{s \in \mathbb{R}^n : s \geq 0\}$

Linear functional in Euclidean space : $x \rightarrow \mathbb{R}$

(Dual space : space of all linear functionals of the vector space)

For more information see Appendix I

Primal LP

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b \quad (1) \\ & x \geq 0. \end{array}$$

Primal Space : $x \in \mathbb{R}^n$

$$\text{Cone : } \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

Dual LP

$$\begin{array}{ll} \underset{y,s}{\text{maximize}} & y^T b \\ \text{subject to} & c - A^T y = s \quad (2) \\ & s \geq 0. \end{array}$$

Dual Space: $s \in \mathbb{R}^n$

$$\begin{aligned} \text{Dual Cone: } \; K^* &= \{s \in \mathbb{R}^n : s^T x \geq 0 \ \forall x \in K\} \\ &= \{s \in \mathbb{R}^n : s \geq 0\} \end{aligned}$$

Primal LP

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & Ax = b \quad (1) \\ & x \geq 0. \end{array}$$

Primal Space : $x \in \mathbb{R}^n$

$$\text{Cone :} \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

Dual LP

$$\begin{array}{ll} \underset{y,s}{\text{maximize}} & y^T b \\ \text{subject to} & c - A^T y = s \quad (2) \\ & s \geq 0. \end{array}$$

Dual Space: $s \in \mathbb{R}^n$

$$\begin{aligned} \text{Dual Cone:} \quad K^* &= \{s \in \mathbb{R}^n : s^T x \geq 0 \; \forall x \in K\} \\ &= \{s \in \mathbb{R}^n : s \geq 0\} \end{aligned}$$

$\text{Linear Map } A(x) : \mathbb{R}^n \to \mathbb{R}^m$ $\qquad A(x) = Ax$

$\text{Dual Map } A^*(y) : \mathbb{R}^m \to \mathbb{R}^n$ $\qquad A^*(y) = A^T y$

Primal LP

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^Tx \\ \text{subject to} \quad & Ax = b \quad (1) \\ & x \geq 0. \end{aligned}$$

Primal Space : $x \in \mathbb{R}^n$

$$\text{Cone}: \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

Dual LP

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & y^Tb \\ \text{subject to} \quad & c - A^Ty = s \quad (2) \\ & s \geq 0. \end{aligned}$$

Dual Space: $s \in \mathbb{R}^n$

$$\text{Dual Cone:} \quad K^* = \{s \in \mathbb{R}^n : s \geq 0\}$$

Weak Duality: $y^Tb \leq c^Tx \longrightarrow c^Tx - y^Tb \geq 0$

Characterizes the dual cone

$$c^Tx - y^Tb \geq 0 \Rightarrow (A^Ty + s)^Tx - y^T(Ax) \geq 0 \Rightarrow \underbrace{(A^Ty)^Tx}_{\mathbb{R}} + s^Tx - \underbrace{y^T(Ax)}_{\mathbb{R}} \geq 0 \Rightarrow \boxed{s^Tx \geq 0}$$

2 1

$$\boxed{(A^Ty)^Tx = y^T(Ax)}$$

Characterizes the Dual Linear Map

Primal LP

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \qquad \text{1} \\ & x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & y^T b \\ \text{subject to} \quad & c - A^T y = s \qquad \text{2} \\ & s \geq 0. \end{aligned}$$

Primal Space : $x \in \mathbb{R}^n$

$$\text{Cone :} \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

Dual Space: $s \in \mathbb{R}^n$

$$\begin{aligned} \text{Dual Cone:} \quad K^* &= \{s \in \mathbb{R}^n : s^T x \geq 0 \ \forall x \in K\} \\ &= \{s \in \mathbb{R}^n : s \geq 0\} \end{aligned}$$

$$\text{Linear Map } A(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m \qquad A(x) = Ax$$

$$\text{Dual Map } A^*(y) : \mathbb{R}^m \rightarrow \mathbb{R}^n \qquad A^*(y) = A^T y$$

Linear Map and Dual Map satisfies $\langle A^*(y), x \rangle = \langle y, A(x) \rangle \longrightarrow (A^T y)^T x = y^T (Ax)$

Primal LP

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

1

Dual LP

$$\begin{aligned} \underset{y,s}{\text{maximize}} \quad & y^T b \\ \text{subject to} \quad & c - A^T y = s \\ & s \geq 0. \end{aligned}$$

2

Primal Space : $x \in \mathbb{R}^n$

Dual Space: $s \in \mathbb{R}^n$

$$\text{Cone :} \quad K = \{x \in \mathbb{R}^n : x \geq 0\}$$

$$\begin{aligned} \text{Dual Cone:} \quad K^* &= \{s \in \mathbb{R}^n : s^T x \geq 0 \ \forall x \in K\} \\ &= \{s \in \mathbb{R}^n : s \geq 0\} \end{aligned}$$

$$\text{Linear Map } A(x) : \mathbb{R}^n \to \mathbb{R}^m \qquad A(x) = Ax$$

$$\text{Dual Map } A^*(y) : \mathbb{R}^m \to \mathbb{R}^n \qquad A^*(y) = A^T y$$

Linear Map and Dual Map satisfies $\langle A^*(y), x\rangle = \langle y, A(x)\rangle \longrightarrow (A^T y)^T x = y^T (Ax)$

$$\begin{aligned} \mathrm{P}^* = \underset{x}{\text{minimize}} \quad & \langle c, x\rangle \\ \text{subject to} \quad & A(x) = b \\ & x \in K. \end{aligned}$$

Dual

$$\begin{aligned} \mathrm{D}^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b\rangle \\ \text{subject to} \quad & c - A^*(y) = s \\ & s \in K^*. \end{aligned}$$

Primal Conic Program

$$
\begin{aligned}
\mathrm{P}^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle \\
& \text{subject to} && A(x) = b \\
& && x \in K.
\end{aligned}
$$

Dual

Dual Conic Program

$$
\begin{aligned}
\mathrm{D}^* = & \underset{y,s}{\text{maximize}} && \langle y, b\rangle \\
& \text{subject to} && c - A^*(y) = s \\
& && s \in K^*.
\end{aligned}
$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle \\ &\text{subject to} && A(x) = b \\ &&& x \in K. \end{aligned}$$

Dual

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle \\ &\text{subject to} && c - A^*(y) = s \\ &&& s \in K^*. \end{aligned}$$

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle \\ &\text{subject to} && A^*(x) = b \\ &&& x \in K^*. \end{aligned}$$

Dual

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle \\ &\text{subject to} && c - A(y) = s \\ &&& s \in K. \end{aligned}$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle \\ &\text{subject to} && A(x) = b \\ & && x \in K. \end{aligned}$$

Dual

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle \\ &\text{subject to} && c - A^*(y) = s \\ & && s \in K^*. \end{aligned}$$

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle \\ &\text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned}$$

Dual

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle \\ &\text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

We can reformulate different class of Optimization Problems as a conic program.

Use the duality of conic program, to obtain dual optimization.

Primal Conic Program

Dual Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle \\ &\text{subject to} && A(x) = b \\ & && x \in K. \end{aligned} \quad \xrightarrow{\textbf{Dual}} \quad \begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle \\ &\text{subject to} && c - A^*(y) = s \\ & && s \in K^*. \end{aligned}$$

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle \\ &\text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned} \quad \xrightarrow{\textbf{Dual}} \quad \begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle \\ &\text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

- SDP solvers like SeDuMi, MOSEK,… are conic program solvers.
  - MOEK: https://docs.mosek.com/slides/2017/aau/conic-opt.pdf
  - SeDuMi: http://sedumi.ie.lehigh.edu/sedumi/files/sedumi-downloads/SeDuMi_Guide_11.pdf

H. D. Mittelmann, “The State-of-the-Art in Conic Optimization Software”, http://www.optimization-online.org/DB_FILE/2010/08/2694.pdf

Primal Convex Conic

$$
\begin{aligned}
\mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x \rangle \\
&\text{subject to} && A(x) = b \\
& && x \in K.
\end{aligned}
$$

Dual Convex Conic

$$
\begin{aligned}
\mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b \rangle \\
&\text{subject to} && c - A^*(y) = s \\
& && s \in K^*.
\end{aligned}
$$

Primal SDP

$$
\begin{aligned}
&\underset{X}{\text{minimize}} && C \bullet X \\
&\text{subject to} && A_i \bullet X = b_i \quad i = 1, \ldots, m. \\
& && X \succcurlyeq 0.
\end{aligned}
$$

$K = \{X \in \mathbb{R}^{n \times n} : X \succcurlyeq 0\}$

Dual SDP

$$
\begin{aligned}
&\underset{S,y}{\text{maximize}} && y^T b \\
&\text{subject to} && C - \textstyle\sum_{i=1}^m y_i A_i = S \\
& && S \succcurlyeq 0.
\end{aligned}
$$

$K^* = \{S \in \mathbb{R}^{n \times n} : S \succcurlyeq 0\}$

- See Appendix I

Primal Convex Conic

$$\begin{aligned} \mathrm{P}^* = & \underset{x}{\text{minimize}} && \langle c, x \rangle \\ & \text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned}$$

Dual Convex Conic

$$\begin{aligned} \mathrm{D}^* = & \underset{y,s}{\text{maximize}} && \langle y, b \rangle \\ & \text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

LP in measure

$$\begin{aligned} \mathrm{P}^*_\mu = & \underset{\mu}{\text{minimize}} && \int p(x) d\mu \\ & \text{subject to} && \int d\mu = 1 \\ & \mu \in \text{space of (nonnegative) measures on } \Omega \end{aligned}$$

Dual

$$\begin{aligned} & \underset{y}{\text{maximize}} && y \\ & \text{subject to} && p(x) - y \in K \end{aligned}$$

$K^* = \{\text{(nonegative) measures } \mu : supp(\mu) \subset \Omega\}$

$K = \{\text{nonnegative continuous functions on } \Omega\}$

- See Appendix I

Constrained Optimization

$$P^* = \underset{x \in \Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$
- Constrained Optimization $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}$

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$P_f^* = \underset{\gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Dual

(nonnegative) Measure Space

$$P_\mu^* = \underset{\mu \in \mathcal{M}(\Omega)}{\text{minimize}} \ E_\mu[p(x)] = \int p(x) d\mu$$

$$\text{subject to} \ \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Convex and Strictly feasible problems
(Slater's Condition)

$$\boxed{P_f^* = P_\mu^*}$$

Duality of SOS and Moment SDPs

Semidefinite Conde and Semidefinite Program:

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succeq 0. \end{aligned}$$

Dual ⟺

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^m y_i A_i = S \\ & S \succeq 0. \end{aligned}$$

Semidefinite Conde and Semidefinite Program:

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^{m} y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

Unconstrained Optimization

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in SOS_{2d} \end{aligned}$$

Dual

moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\ \text{subject to} \quad & y_0 = 1 \\ & \mathbf{M}_d(y) \succcurlyeq 0 \end{aligned}$$

Semidefinite Conde and Semidefinite Program:

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^{m} y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

Unconstrained Optimization

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in SOS_{2d} \end{aligned}$$

Dual

moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\ \text{subject to} \quad & y_0 = 1 \\ & \mathbf{M}_d(y) \succcurlyeq 0 \end{aligned}$$

Constrained Optimization

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = \underset{\gamma, \sigma_i}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS_{2d} \\ & \sigma_i(x) \in SOS_{2d_i}, \; i = 1, \ldots, m \end{aligned}$$

Dual

moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_{\alpha} p_\alpha y_\alpha \\ \text{subject to} \quad & y_0 = 1 \\ & \mathbf{M}_d(y) \succcurlyeq 0 \\ & \mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, m. \end{aligned}$$

Unconstrained Optimization

SOS Optimization

$$
\begin{aligned}
P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad & \gamma \\
\text{subject to} \quad & p(x) - \gamma \in SOS_{2d}
\end{aligned}
$$

Moment SDP

$$
\begin{aligned}
P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\
\text{subject to} \quad & y_0 = 1 \\
& \mathbf{M}_d(y) \succcurlyeq 0
\end{aligned}
$$

Duality

Rewrite SOS Representation (Appendix II)

Rewrite Moment Matrix (Appendix II)

Standard SDP

$$
\begin{aligned}
\underset{X}{\text{minimize}} \quad & C \bullet X \\
\text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\
& X \succcurlyeq 0.
\end{aligned}
$$

Duality

Standard SDP

$$
\begin{aligned}
\underset{S,y}{\text{maximize}} \quad & b^T y \\
\text{subject to} \quad & C - \textstyle\sum_{i=1}^m y_i A_i = S \\
& S \succcurlyeq 0.
\end{aligned}
$$

Unconstrained Optimization

SOS Optimization

$$P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

Duality

Moment SDP

$$P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad \sum_{\alpha=0}^{2d} p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$

Rewrite SOS Representation (Appendix II)

$$p(x) = B^T(x) Q B(x) \xrightarrow{\text{Coefficients}} p_\alpha = \langle Q, \mathbf{B}_\alpha \rangle$$

Constant matrix

Rewrite Moment Matrix (Appendix II)

$$\mathbf{M}_d(y) = \sum_\alpha \mathbf{B}_\alpha y_\alpha$$

Constant matrix

Standard SDP

$$\underset{X}{\text{minimize}} \quad C \bullet X$$
$$\text{subject to} \quad A_i \bullet X = b_i \quad i = 1, \ldots, m.$$
$$X \succcurlyeq 0.$$

Duality

Standard SDP

$$\underset{S,y}{\text{maximize}} \quad b^T y$$
$$\text{subject to} \quad C - \textstyle\sum_{i=1}^m y_i A_i = S$$
$$S \succcurlyeq 0.$$

Constrained Optimization

SOS Optimization

$$
\begin{aligned}
\mathrm{P}^{*d}_{sos} = &\underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma \\
&\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS_{2d} \\
&\qquad\qquad \sigma_i(x) \in SOS_{2d_i}, \ \ i = 1, \ldots, m
\end{aligned}
$$

Duality

Moment SDP

$$
\begin{aligned}
\mathrm{P}^{*d}_{mom} = &\underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad \sum_{\alpha} p_\alpha y_\alpha \\
&\text{subject to} \quad y_0 = 1 \\
&\qquad \mathbf{M}_d(y) \succcurlyeq 0 \\
&\qquad \mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

Rewrite SOS Representation (Appendix II)

Rewrite Moment and Localizing Matrix (Appendix II)

Standard SDP

$$
\begin{aligned}
&\underset{X}{\text{minimize}} \quad C \bullet X \\
&\text{subject to} \quad A_i \bullet X = b_i \quad i = 1, \ldots, m. \\
&\qquad X \succcurlyeq 0.
\end{aligned}
$$

Duality

Standard SDP

$$
\begin{aligned}
&\underset{S,y}{\text{maximize}} \quad b^T y \\
&\text{subject to} \quad C - \textstyle\sum_{i=1}^{m} y_i A_i = S \\
&\qquad S \succcurlyeq 0.
\end{aligned}
$$

Constrained Optimization

SOS Optimization

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS_{2d}$$

$$\sigma_i(x) \in SOS_{2d_i}, \quad i = 1, \ldots, m$$

Duality

Moment SDP

$$\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad \sum_{\alpha} p_\alpha y_\alpha$$

$$\text{subject to} \quad y_0 = 1$$

$$\mathbf{M}_d(y) \succcurlyeq 0$$

$$\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, m.$$

Rewrite SOS Representation (Appendix II)

$$p(x) = B^T(x) Q B(x) \xrightarrow{\text{Coefficients}} p_\alpha = \langle Q, \mathbf{B}_\alpha \rangle$$

Constant matrix

$$g(x)\sigma(x) \longrightarrow \text{coefficients}(g(x)\sigma(x)) = \langle Q, \mathbf{B}_{g\alpha} \rangle$$

Rewrite Moment and Localizing Matrix (Appendix II)

$$\mathbf{M}_d(y) = \sum_\alpha \mathbf{B}_\alpha y_\alpha \qquad \mathbf{M}_d(gy) = \sum_\alpha \mathbf{B}_{g\alpha} y_\alpha$$

Constant matrix

Standard SDP

$$\underset{X}{\text{minimize}} \quad C \bullet X$$

$$\text{subject to} \quad A_i \bullet X = b_i \quad i = 1, \ldots, m.$$

$$X \succcurlyeq 0.$$

Duality

Standard SDP

$$\underset{S,y}{\text{maximize}} \quad b^T y$$

$$\text{subject to} \quad C - \sum_{i=1}^{m} y_i A_i = S$$

$$S \succcurlyeq 0.$$

Unconstrained Optimization

SOS Optimization

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in SOS_{2d} \end{aligned}$$

Moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\ \text{subject to} \quad & y_0 = 1 \\ & \mathbf{M}_d(y) \succcurlyeq 0 \end{aligned}$$

Unconstrained Optimization

SOS Optimization

$$P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

$$P^{*d}_{sos} = \underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma$$

subject to

$$\text{coefficients of } p(x) - \gamma = \text{coefficients of } B^T(x) Q B(x)$$
$$Q \succcurlyeq 0$$

Moment SDP

$$P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad \sum_{\alpha=0}^{2d} p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$

Unconstrained Optimization

SOS Optimization

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

$$\mathrm{P}^{*d}_{sos} = \underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to}$$
$$\text{coefficients of } p(x) - \gamma = \text{coefficients of } B^T(x) Q B(x)$$
$$Q \succcurlyeq 0$$

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = &\underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma \\ &\text{subject to} \quad p_0 - \gamma = \langle Q, \mathbf{B}_0 \rangle, \\ &\qquad\qquad\quad p_\alpha = \langle Q, \mathbf{B}_\alpha \rangle, \alpha \neq 0 \\ &\qquad\qquad\quad Q \succcurlyeq 0 \end{aligned}$$

Moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = &\underset{y_\alpha, \alpha = 0, \ldots, 2d}{\text{minimize}} \quad \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\ &\text{subject to} \quad y_0 = 1 \\ &\qquad\qquad\quad \mathbf{M}_d(y) \succcurlyeq 0 \end{aligned}$$

Unconstrained Optimization

SOS Optimization

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

$$\mathrm{P}^{*d}_{sos} = \underset{Q\in\mathcal{S}^n,\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to}$$
$$\text{coefficients of } p(x) - \gamma = \text{coefficients of } B^T(x)QB(x)$$
$$Q \succcurlyeq 0$$

$$\begin{aligned}\mathrm{P}^{*d}_{sos} = \underset{Q\in\mathcal{S}^n,\gamma\in\mathbb{R}}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p_0 - \gamma = \langle Q, \mathbf{B}_0\rangle, \\ & p_\alpha = \langle Q, \mathbf{B}_\alpha\rangle, \alpha \neq 0 \\ & Q \succcurlyeq 0\end{aligned}$$

$$\begin{aligned}\mathrm{P}^{*d}_{sos} = \underset{Q\in\mathcal{S}^n}{\text{maximize}} \quad & p_0 - \langle Q, \mathbf{B}_0\rangle \\ \text{subject to} \quad & \langle Q, \mathbf{B}_\alpha\rangle = p_\alpha, \ \ \alpha \neq 0 \\ & Q \succcurlyeq 0\end{aligned}$$

Moment SDP

$$\begin{aligned}\mathrm{P}^{*d}_{mom} = \underset{y_\alpha,\alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\ \text{subject to} \quad & y_0 = 1 \\ & \mathbf{M}_d(y) \succcurlyeq 0\end{aligned}$$

Unconstrained Optimization

SOS Optimization

$$P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n,\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to}$$
$$\text{coefficients of } p(x) - \gamma = \text{coefficients of } B^T(x)QB(x)$$
$$Q \succcurlyeq 0$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n,\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p_0 - \gamma = \langle Q, \mathbf{B}_0\rangle,$$
$$p_\alpha = \langle Q, \mathbf{B}_\alpha\rangle, \alpha \neq 0$$
$$Q \succcurlyeq 0$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n}{\text{maximize}} \quad p_0 - \langle Q, \mathbf{B}_0\rangle$$
$$\text{subject to} \quad \langle Q, \mathbf{B}_\alpha\rangle = p_\alpha, \quad \alpha \neq 0$$
$$Q \succcurlyeq 0$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n}{\text{maximize}} \quad -\langle Q, \mathbf{B}_0\rangle$$
$$\text{subject to} \quad \langle Q, \mathbf{B}_\alpha\rangle = p_\alpha, \quad \alpha \neq 0$$
$$Q \succcurlyeq 0$$

SOS SDP

Moment SDP

$$P^{*d}_{mom} = \underset{y_\alpha,\alpha=0,\dots,2d}{\text{minimize}} \quad \sum_{\alpha=0}^{2d} p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$

Unconstrained Optimization

SOS Optimization

$$P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n, \gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to}$$
$$\text{coefficients of } p(x) - \gamma = \text{coefficients of } B^T(x)QB(x)$$
$$Q \succcurlyeq 0$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n, \gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p_0 - \gamma = \langle Q, \mathbf{B}_0 \rangle,$$
$$p_\alpha = \langle Q, \mathbf{B}_\alpha \rangle, \alpha \neq 0$$
$$Q \succcurlyeq 0$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n}{\text{maximize}} \quad p_0 - \langle Q, \mathbf{B}_0 \rangle$$
$$\text{subject to} \quad \langle Q, \mathbf{B}_\alpha \rangle = p_\alpha, \quad \alpha \neq 0$$
$$Q \succcurlyeq 0$$

$$P^{*d}_{sos} = \underset{Q\in\mathcal{S}^n}{\text{maximize}} \quad -\langle Q, \mathbf{B}_0 \rangle$$
$$\text{subject to} \quad \langle Q, \mathbf{B}_\alpha \rangle = p_\alpha, \quad \alpha \neq 0$$
$$Q \succcurlyeq 0$$

SOS SDP

Moment SDP

$$P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_{\alpha=0}^{2d} p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\mathbf{M}_d(y) \succcurlyeq 0$$

$$P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad \sum_{\alpha=0}^{2d} p_\alpha y_\alpha$$
$$\text{subject to} \quad y_0 = 1$$
$$\sum_\alpha y_\alpha \mathbf{B}_\alpha \succcurlyeq 0$$

Unconstrained Optimization

SOS Optimization

$$P^{*d}_{sos} = \underset{\gamma}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma \in SOS_{2d}$$

$$P^{*d}_{sos} = \underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} \quad \gamma$$
$$\text{subject to}$$
$$\text{coefficients of } p(x) - \gamma = \text{coefficients of } B^T(x) Q B(x)$$
$$Q \succcurlyeq 0$$

$$\begin{aligned} P^{*d}_{sos} = &\underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && p_0 - \gamma = \langle Q, \mathbf{B}_0 \rangle, \\ &&& p_\alpha = \langle Q, \mathbf{B}_\alpha \rangle, \alpha \neq 0 \\ &&& Q \succcurlyeq 0 \end{aligned}$$

$$\begin{aligned} P^{*d}_{sos} = &\underset{Q \in \mathcal{S}^n}{\text{maximize}} && p_0 - \langle Q, \mathbf{B}_0 \rangle \\ &\text{subject to} && \langle Q, \mathbf{B}_\alpha \rangle = p_\alpha, \ \alpha \neq 0 \\ &&& Q \succcurlyeq 0 \end{aligned}$$

$$\begin{aligned} P^{*d}_{sos} = &\underset{Q \in \mathcal{S}^n}{\text{maximize}} && -\langle Q, \mathbf{B}_0 \rangle \\ &\text{subject to} && \langle Q, \mathbf{B}_\alpha \rangle = p_\alpha, \ \alpha \neq 0 \\ &&& Q \succcurlyeq 0 \end{aligned}$$

SOS SDP

Moment SDP

$$\begin{aligned} P^{*d}_{mom} = &\underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} && \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\ &\text{subject to} && y_0 = 1 \\ &&& \mathbf{M}_d(y) \succcurlyeq 0 \end{aligned}$$

$$\begin{aligned} P^{*d}_{mom} = &\underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} && \sum_{\alpha=0}^{2d} p_\alpha y_\alpha \\ &\text{subject to} && y_0 = 1 \\ &&& \sum_{\alpha} y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \end{aligned}$$

$$\begin{aligned} P^{*d}_{mom} = &\underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} && \sum_{\alpha} p_\alpha y_\alpha \\ &\text{subject to} && \mathbf{B}_0 + \sum_{\alpha \neq 0} y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \end{aligned}$$

Moment SDP

Unconstrained Optimization

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^{m} y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

Dual

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = & \underset{Q \in \mathcal{S}^n}{\text{maximize}} \quad -\langle Q, \mathbf{B}_0 \rangle \\ & \text{subject to} \quad \langle Q, \mathbf{B}_\alpha \rangle = p_\alpha, \;\; \alpha = 1, \ldots, 2d \\ & \qquad\qquad\quad Q \succcurlyeq 0 \end{aligned}$$

Moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = & \underset{y_\alpha, \alpha = 0, \ldots, 2d}{\text{minimize}} \quad \sum_{\alpha} p_\alpha y_\alpha \\ & \text{subject to} \quad \mathbf{B}_0 + \sum_{\alpha=1}^{2d} y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \end{aligned}$$

- Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.
- Jean Bernard Lasserre, “Global Optimization with Polynomials and the Problem of Moments”, SIAM J. Optim., 11(3), 796–817, 2001.

Unconstrained Optimization

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^m y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

Dual

$b \rightarrow -[p_\alpha], A_i \rightarrow -\mathbf{B}_i, C = \mathbf{B}_0, m = 2d$

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \mathbf{B}_0 \bullet Q \\ \text{subject to} \quad & -\mathbf{B}_i \bullet Q = -p_i \quad i = 1, \ldots, 2d. \\ & Q \succcurlyeq 0. \end{aligned}$$

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & -[p_\alpha]^T y \\ \text{subject to} \quad & \mathbf{B}_0 + \textstyle\sum_{i=1}^{2d} y_i \mathbf{B}_i \succcurlyeq 0 \end{aligned}$$

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = \underset{Q \in \mathcal{S}^n}{\text{maximize}} \quad & -\langle Q, \mathbf{B}_0 \rangle \\ \text{subject to} \quad & \langle Q, \mathbf{B}_\alpha \rangle = p_\alpha, \ \alpha = 1, \ldots, 2d \\ & Q \succcurlyeq 0 \end{aligned}$$

Moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha = 0, \ldots, 2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\ \text{subject to} \quad & \mathbf{B}_0 + \sum_{\alpha=1}^{2d} y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \end{aligned}$$

- Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.
- Jean Bernard Lasserre, “Global Optimization with Polynomials and the Problem of Moments”, SIAM J. Optim., 11(3), 796–817, 2001.

Constrained Optimization

SOS SDP

$$
\begin{aligned}
P_{sos}^{*d} = &\underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma \\
&\text{subject to} \quad p(x) - \gamma - \textstyle\sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS \\
&\qquad\qquad\quad \sigma_i(x) \in SOS_{2d_i}, \ i = 1, ..., m
\end{aligned}
$$

Constrained Optimization

SOS SDP

$$
\begin{aligned}
\mathrm{P}^{*d}_{sos} = & \underset{\gamma,\sigma_i}{\text{maximize}} && \gamma \\
& \text{subject to} && p(x) - \gamma - \textstyle\sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS \\
& && \sigma_i(x) \in SOS_{2d_i},\ i = 1, \dots, m
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}^{*d}_{sos} = & \underset{\gamma, Q_i|_{i=0}^{m}}{\text{maximize}} && \gamma \\
& && \text{coefficients of } \left(B^T(x) Q_0 B(x) - \gamma + \textstyle\sum_{i=1}^{m} \sigma_i g_i(x) \right) = \text{coefficients of } (p(x) - \gamma) \\
& \text{subject to} && \sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x),\quad i = 1, \dots, m
\end{aligned}
$$

Constrained Optimization

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = &\underset{\gamma,\sigma_i}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x) - \gamma - \textstyle\sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS \\ &&& \sigma_i(x) \in SOS_{2d_i},\ i = 1, \dots, m \end{aligned}$$

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = &\underset{\gamma, Q_i|_{i=0}^{m}}{\text{maximize}} && \gamma \\ &&& \text{coefficients of } \left(B^T(x) Q_0 B(x) - \gamma + \textstyle\sum_{i=1}^{m} \sigma_i g_i(x) \right) = \text{coefficients of } (p(x) - \gamma) \\ &\text{subject to} && \sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, \dots, m \end{aligned}$$

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = &\underset{Q_i \in \mathcal{S}^n}{\text{maximize}} && \gamma \\ &\text{subject to} && \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha \neq 0 \qquad \langle Q_0, \mathbf{B}_0 \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i 0} \rangle = p_\alpha - \gamma \\ &&& Q_i \succcurlyeq 0, \quad i = 0, \dots, m \end{aligned}$$

Constrained Optimization

SOS SDP

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma - \textstyle\sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$
$$\sigma_i(x) \in SOS_{2d_i}, \;\; i = 1, \dots, m$$

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma, Q_i|_{i=0}^{m}}{\text{maximize}} \quad \gamma$$
$$\text{coefficients of } \left(B^T(x) Q_0 B(x) - \gamma + \textstyle\sum_{i=1}^{m} \sigma_i g_i(x) \right) = \text{coefficients of } (p(x) - \gamma)$$
$$\text{subject to} \quad \sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, \dots, m$$

$$\mathrm{P}^{*d}_{sos} = \underset{Q_i \in \mathcal{S}^n}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha \neq 0 \qquad \langle Q_0, \mathbf{B}_0 \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i 0} \rangle = p_\alpha - \gamma$$
$$Q_i \succcurlyeq 0, \quad i = 0, \dots, m$$

$$\mathrm{P}^{*d}_{sos} = \underset{Q_i \in \mathcal{S}^n}{\text{maximize}} \quad p_0 - \langle Q_0, \mathbf{B}_0 \rangle - \sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i 0} \rangle$$
$$\text{subject to} \quad \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha \neq 0$$
$$Q_i \succcurlyeq 0, \quad i = 0, \dots, m$$

Constrained Optimization

SOS SDP

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma, \sigma_i}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad p(x) - \gamma - \textstyle\sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$
$$\sigma_i(x) \in SOS_{2d_i}, \;\; i = 1, \dots, m$$

$$\mathrm{P}^{*d}_{sos} = \underset{\gamma, Q_i|_{i=0}^{m}}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad \text{coefficients of } \left(B^T(x) Q_0 B(x) - \gamma + \textstyle\sum_{i=1}^{m} \sigma_i g_i(x) \right) = \text{coefficients of } (p(x) - \gamma)$$
$$\sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, \dots, m$$

$$\mathrm{P}^{*d}_{sos} = \underset{Q_i \in \mathcal{S}^n}{\text{maximize}} \quad \gamma$$
$$\text{subject to} \quad \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha \neq 0 \qquad \langle Q_0, \mathbf{B}_0 \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i 0} \rangle = p_\alpha - \gamma$$
$$Q_i \succcurlyeq 0, \quad i = 0, \dots, m$$

$$\mathrm{P}^{*d}_{sos} = \underset{Q_i \in \mathcal{S}^n}{\text{maximize}} \quad p_0 - \langle Q_0, \mathbf{B}_0 \rangle - \sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i 0} \rangle$$
$$\text{subject to} \quad \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha \neq 0$$
$$Q_i \succcurlyeq 0, \quad i = 0, \dots, m$$

$$\mathrm{P}^{*d}_{sos} = \underset{Q_i \in \mathcal{S}^n}{\text{maximize}} \quad -\langle Q_0, \mathbf{B}_0 \rangle - \sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i 0} \rangle$$
$$\text{subject to} \quad \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha \neq 0$$
$$Q_i \succcurlyeq 0, \quad i = 0, \dots, m$$

SOS SDP

Moment SDP

$$
\begin{aligned}
\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\
\text{subject to} \quad & y_0 = 1 \\
& \mathbf{M}_d(y) \succcurlyeq 0 \\
& \mathbf{M}_d(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

Constrained Optimization

Moment SDP

$$
\begin{aligned}
\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\
\text{subject to} \quad & y_0 = 1 \\
& \mathbf{M}_d(y) \succcurlyeq 0 \\
& \mathbf{M}_d(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, m.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\ldots,2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\
\text{subject to} \quad & y_0 = 1 \\
& \sum_\alpha y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \\
& \sum_\alpha y_\alpha \mathbf{B}_{g_i \alpha} \succcurlyeq 0, i = 1, \ldots, m
\end{aligned}
$$

Constrained Optimization

Moment SDP

$$
\begin{aligned}
\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\
\text{subject to} \quad & y_0 = 1 \\
& \mathbf{M}_d(y) \succcurlyeq 0 \\
& \mathbf{M}_d(g_i y) \succcurlyeq 0, \quad i = 1, \dots, m.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\
\text{subject to} \quad & y_0 = 1 \\
& \sum_\alpha y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \\
& \sum_\alpha y_\alpha \mathbf{B}_{g_i\alpha} \succcurlyeq 0, i = 1, \dots, m
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}^{*d}_{mom} = \underset{y_\alpha, \alpha=0,\dots,2d}{\text{minimize}} \quad & \sum_\alpha p_\alpha y_\alpha \\
\text{subject to} \quad & \mathbf{B}_0 + \sum_{\alpha \neq 0} y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \\
& \mathbf{B}_{g_i 0} + \sum_{\alpha \neq 0} y_\alpha \mathbf{B}_{g_i\alpha} \succcurlyeq 0, i = 1, \dots, m
\end{aligned}
$$

Moment SDP

Constrained Optimization

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, \hat{m}. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^{\hat{m}} y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = & \underset{Q_i|_{i=0}^{m} \in \mathcal{S}^n}{\text{maximize}} \quad - \langle Q_0, \mathbf{B}_0 \rangle - \sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i 0} \rangle \\ & \text{subject to} \quad \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^{m} \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha = 1, \ldots, 2d \\ & \qquad\qquad\quad Q_i \succcurlyeq 0, \quad i = 0, \ldots, m \end{aligned}$$

Moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = & \underset{y_\alpha, \alpha = 0, \ldots, 2d}{\text{minimize}} \quad \sum_{\alpha} p_\alpha y_\alpha \\ & \text{subject to} \quad \mathbf{B}_0 + \sum_{\alpha=1}^{2d} y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \\ & \qquad\qquad\quad \mathbf{B}_{g_i 0} + \sum_{\alpha=1}^{2d} y_\alpha \mathbf{B}_{g_i \alpha} \succcurlyeq 0, i = 1, \ldots, m \end{aligned}$$

- Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.
- Jean Bernard Lasserre, “Global Optimization with Polynomials and the Problem of Moments”, SIAM J. Optim., 11(3), 796–817, 2001.

Constrained Optimization

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \dots, \hat{m}. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \sum_{i=1}^{\hat{m}} y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

Dual

$$X = blkdiag([Q_0, Q_1, ..., Q_m])$$

$$C = blkdiag([\mathbf{B}_0, \mathbf{B}_{g_1 0}, \dots, \mathbf{B}_{g_m 0}])$$

$$A_i = -blkdiag([\mathbf{B}_i, \mathbf{B}_{g_1 i}, \dots, \mathbf{B}_{g_m i}]), i = 1, ..., 2d$$

$$b \to -[p_\alpha] \qquad \hat{m} \to 2d$$

SOS SDP

$$\begin{aligned} \mathrm{P}^{*d}_{sos} = & \underset{Q_i|_{i=0}^m \in \mathcal{S}^n}{\text{maximize}} \quad - \langle Q_0, \mathbf{B}_0 \rangle - \sum_{i=1}^m \langle Q_i, \mathbf{B}_{g_i 0} \rangle \\ & \text{subject to} \quad \langle Q_0, \mathbf{B}_\alpha \rangle + \textstyle\sum_{i=1}^m \langle Q_i, \mathbf{B}_{g_i \alpha} \rangle = p_\alpha, \quad \alpha = 1, ..., 2d \\ & \qquad Q_i \succcurlyeq 0, \quad i = 0, ..., m \end{aligned}$$

Dual

Moment SDP

$$\begin{aligned} \mathrm{P}^{*d}_{mom} = & \underset{y_\alpha, \alpha = 0, \dots, 2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha \\ & \text{subject to} \quad \mathbf{B}_0 + \sum_{\alpha=1}^{2d} y_\alpha \mathbf{B}_\alpha \succcurlyeq 0 \\ & \qquad \mathbf{B}_{g_i 0} + \sum_{\alpha=1}^{2d} y_\alpha \mathbf{B}_{g_i \alpha} \succcurlyeq 0, i = 1, ..., m \end{aligned}$$

- Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.
- Jean Bernard Lasserre, “Global Optimization with Polynomials and the Problem of Moments”, SIAM J. Optim., 11(3), 796–817, 2001.

$$P^* = \underset{x\in\Omega}{\text{minimize}} \quad p(x)$$

- Unconstrained Optimization $\Omega = \mathbb{R}^n$
- Constrained Optimization $\Omega = \mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \; i = 1, ..., m\}$

Equivalent

Equivalent

Space Nonnegative Continuous functions

$$P^*_f = \underset{\gamma\in\mathbb{R}}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma \geq 0, \quad \forall x \in \Omega$$

Dual

(nonnegative) Measure Space

$$P^*_\mu = \underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} \quad \mathrm{E}_\mu[p(x)] = \int p(x) d\mu$$

$$\text{subject to} \quad \int_\Omega d\mu = 1$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

K : cone of nonnegative continuous functions on $\mathbf{K}$

K^* : cone of nonnegative measures supported on $\mathbf{K}$

Space of SOS Polynomials

$$P^*_{sos} \underset{\gamma,\sigma_i}{\text{maximize}} \quad \gamma$$

$$\text{subject to} \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i}, \; i = 1, ..., m$$

Dual

$$P^{*d}_{sos} = P^{*d}_{mom}$$

Moments Space

$$P^{*d}_{mom} = \underset{y_\alpha, \alpha=0,...,2d}{\text{minimize}} \quad \sum_\alpha p_\alpha y_\alpha$$

$$\text{subject to} \quad y_0 = 1$$

$$\mathbf{M}_d(y) \succcurlyeq 0$$

$$\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, ..., m.$$

K : cone of SOS poynomials up to order 2d

K^* : cone of truncated moments up to order 2d of (nonnegative) measures

- Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.
- Jean Bernard Lasserre, “Global Optimization with Polynomials and the Problem of Moments”, SIAM J. Optim., 11(3), 796–817, 2001.

Primal-Dual Interior Point Method

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^m y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^m y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

From the **Duality** a triple (X, S, y) solves the primal-dual SDP if and only if:

$$\left\{ \begin{array}{lll} A_i \bullet X = b_i \quad i = 1, \ldots, m & X \succcurlyeq 0 & \text{Primal feasibility} \\ C - \sum_{i=1}^m y_i A_i = S & S \succcurlyeq 0 & \text{Dual feasibility} \\ SX = 0 & & \text{Dual Complementary slackness} \\ (\text{Strong Duality:} \quad C \bullet X - y^T b = 0 \rightarrow S \bullet X = 0(\text{psd matrices}) \rightarrow SX = 0) & & \end{array} \right.$$

Primal SDP

$$\begin{array}{ll} \underset{X}{\text{minimize}} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i \quad i = 1, \dots, m. \\ & X \succcurlyeq 0. \end{array}$$

Dual SDP

$$\begin{array}{ll} \underset{S,y}{\text{maximize}} & b^T y \\ \text{subject to} & C - \sum_{i=1}^m y_i A_i = S \\ & S \succcurlyeq 0. \end{array}$$

From the **Duality** a triple (X, S, y) solves the primal-dual SDP if and only if:

$$\begin{cases} A_i \bullet X = b_i \quad i = 1, \dots, m & X \succcurlyeq 0 & \text{Primal feasibility} \\ C - \sum_{i=1}^m y_i A_i = S & S \succcurlyeq 0 & \text{Dual feasibility} \\ SX = 0 & & \text{Dual Complementary slackness} \end{cases}$$

- Primal-dual interior point methods solves the system of nonlinear systems.
- From Lecture 2, interior point method i) turns constrained optimization with inequality constraints into constraint optimization with only equality constraints, ii) Applies Newton's method to solve optimality conditions.

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \dots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^m y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

From the **Duality** a triple (X, S, y) solves the primal-dual SDP if and only if:

$$\begin{cases} A_i \bullet X = b_i \quad i = 1, \dots, m & X \succcurlyeq 0 \quad \text{Primal feasibility} \\ C - \sum_{i=1}^m y_i A_i = S & S \succcurlyeq 0 \quad \text{Dual feasibility} \\ SX = 0 & \text{Dual Complementary slackness} \end{cases}$$

- Primal-dual interior point methods solves the system of nonlinear systems.
- From Lecture 2, interior point method i) turns constrained optimization with inequality constraints into constraint optimization with only equality constraints, ii) Applies Newton's method to solve optimality conditions.

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \dots, m. \\ & X \succcurlyeq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X + \frac{1}{t} f(X) \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \dots, m \end{aligned}$$

$$\text{where, barrier function } f(X) = log(det(X))$$

Optimality condition $\Rightarrow$

$$\begin{cases} A_i \bullet X = b_i \quad i = 1, \dots, m \\ C - \sum_{i=1}^m y_i A_i = S \\ SX = tI_n \end{cases}$$

System of nonlinear equations

Primal SDP

$$\begin{array}{ll} \underset{X}{\text{minimize}} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{array}$$

Dual SDP

$$\begin{array}{ll} \underset{S,y}{\text{maximize}} & b^T y \\ \text{subject to} & C - \sum_{i=1}^m y_i A_i = S \\ & S \succcurlyeq 0. \end{array}$$

- Hence, Primal-dual SDP solvers (e.g., SeDuMi, MOSEK,….), have access to the dual and primal variables.

- By solving moment SDP, we can also obtain SOS polynomial. (see examples 1 and 2)

Appendix I: Conic Duality

Primal and Dual Spaces

- Primal spaces V : vector space

  e.g., Euclidean space, space of symmetric matrices, space of continuous functions

- Dual spaces V^* : vector space of real-valued linear functionals.

- $\langle .,. \rangle_V : V \times V^* \to \mathbb{R}$: "Duality Pairing" between an element of vector space and an element of dual space

Examples:

Euclidean spaces (self-dual): $V = \mathbb{R}^n, \; V^* = \mathbb{R}^n, \; \langle x, y \rangle_V = x^T y$

Space of symmetric matrices (self-dual): $V = \mathcal{S}^n, \; V^* = \mathcal{S}^n, \; \langle X, Y \rangle_V = X \bullet Y = trace(X^T Y)$

Space of continuous functions and spaces of measures: $V = \mathcal{C}, \; V^* = \mathcal{M}, \; \langle f, \mu \rangle_V = \int f d\mu$

- Chapter IV.3: Alexander Barvinok "A Course in Convexity", American Mathematical Society, Graduate Studies in Mathematics, Volume 54

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A(x) = b \\ & && x \in K. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ &\text{subject to} && c - A^*(y) = s \\ & && s \in K^*. \end{aligned}$$

- Proper cone:[1] $K \subset V_1$

$x \in V_1 \quad c \in V_1^*$

- Dual cone: $K^* \subset V_1^*$

$b \in V_2 \quad y \in V_2^*$

- Linear map $A(x) : V_1 \to V_2$
- Dual Linear map $A^*(y) : V_2^* \to V_1^*$

satisfies → $< \underbrace{A^*(y)}_{V_1^*}, \underbrace{x}_{V_1} >_{V_1} = < \underbrace{y}_{V_2^*}, \underbrace{A(x)}_{V_2} >_{V_2}$ ← satisfies

- Chapter IV.3: Alexander Barvinok "A Course in Convexity", American Mathematical Society, Graduate Studies in Mathematics, Volume 54

1: proper cone: convex, closed, nonempty interior, pointed.

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A(x) = b \\ &&& x \in K. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ &\text{subject to} && c - A^*(y) = s \\ &&& s \in K^*. \end{aligned}$$

- Proper cone:[1] $K \subset V_1$

$x \in V_1 \quad c \in V_1^*$

- Dual cone: $K^* \subset V_1^*$

$b \in V_2 \quad y \in V_2^*$

- Linear map $A(x) : V_1 \to V_2$
- Dual Linear map $A^*(y) : V_2^* \to V_1^*$

satisfies $\quad < A^*(y), x >_{V_1} = < y, A(x) >_{V_2} \quad$ satisfies

$$\underbrace{A^*(y)}_{V_1^*},\ \underbrace{x}_{V_1},\ \underbrace{y}_{V_2^*},\ \underbrace{A(x)}_{V_2}$$

To obtain the dual optimization:
i) define the primal cone and associated dual cone, ii) identify linear map and find dual map iii) Use dual conic program to construct the dual optimization

1: proper cone: convex, closed, nonempty interior, pointed.

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = & \underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ & \text{subject to} && A(x) = b \\ & && x \in K. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = & \underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ & \text{subject to} && c - A^*(y) = s \\ & && s \in K^*. \end{aligned}$$

- Proper cone: $K = \{x \in \mathbb{R}^n : x \geq 0\} \subset V_1 = \mathbb{R}^n$
- Dual cone: $K^* = \{s \in \mathbb{R}^n : s^T x \geq 0 \ \forall x \in K\}$ $= \{s \in \mathbb{R}^n : s \geq 0\} \subset V_1^* = \mathbb{R}^n$

$x \in V_1 = \mathbb{R}^n \quad c \in V_1^* = \mathbb{R}^n \qquad b \in V_2 = \mathbb{R}^m \quad y \in V_2^* = \mathbb{R}^m$

- Linear map $A(x) = Ax : V_1 = \mathbb{R}^n \to V_2 = \mathbb{R}^m$
- Dual Linear map $A^*(y) = A^T y : V_2^* = \mathbb{R}^m \to V_1^* = \mathbb{R}^n$

$$\langle \underbrace{A^*(y)}_{V_1^*}, \underbrace{x}_{V_1} \rangle_{V_1} = \langle \underbrace{y}_{V_2^*}, \underbrace{A(x)}_{V_2} \rangle_{V_2} \rightarrow \langle A^T y, x \rangle_{V_1} = \langle y, Ax \rangle_{V_2} \rightarrow (A^T y)^T x = (y)^T A x \rightarrow y^T A x = y^T A x$$

Primal LP

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && c^T x \\ & \text{subject to} && Ax = b \\ & && x \geq 0. \end{aligned}$$

Dual LP

$$\begin{aligned} & \underset{y,s}{\text{maximize}} && y^T b \\ & \text{subject to} && c - A^T y = s \\ & && s \geq 0. \end{aligned}$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = \underset{x}{\text{minimize}} \quad & \langle c, x \rangle_{V_1} \\ \text{subject to} \quad & A(x) = b \\ & x \in K. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = \underset{y,s}{\text{maximize}} \quad & \langle y, b \rangle_{V_2} \\ \text{subject to} \quad & c - A^*(y) = s \\ & s \in K^*. \end{aligned}$$

Primal SDP

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \underset{S,y}{\text{maximize}} \quad & y^T b \\ \text{subject to} \quad & C - \textstyle\sum_{i=1}^m y_i A_i = S \\ & S \succcurlyeq 0. \end{aligned}$$

$$K = \{X \in \mathbb{R}^{n \times n} : X \succcurlyeq 0\} \subset V_1 = \mathcal{S}^n$$

$$\begin{aligned} K^* &= \{S \in \mathbb{R}^{n \times n} : S \bullet X \geq 0 \ \forall X \in K\} \\ &= \{S \in \mathbb{R}^{n \times n} : S \succcurlyeq 0\} \subset V_1^* = \mathcal{S}^n \end{aligned}$$

$$A(X) = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \bullet X : V_1 = \mathcal{S}^n \to V_2 = \mathbb{R}^m \qquad b \in V_2 = \mathbb{R}^m \qquad y \in V_2^* = \mathbb{R}^m$$

$$< A^*(y), X >_{V_1} = < y, A(X) >_{V_2} \to A^*(y) \bullet X = y^T A(X) \to A^*(y) \bullet X = y^T \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \bullet X \to A^*(y) = y^T \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$$

$$\to A^*(y) = \sum_{i=1}^m A_i y_i : V_2^* = \mathbb{R}^m \to V_1^* = \mathcal{S}^n$$

Primal Conic Program

$$\begin{aligned} \mathrm{P}^* = &\underset{x}{\text{minimize}} && \langle c, x\rangle_{V_1} \\ &\text{subject to} && A^*(x) = b \\ & && x \in K^*. \end{aligned}$$

Dual Conic Program

$$\begin{aligned} \mathrm{D}^* = &\underset{y,s}{\text{maximize}} && \langle y, b\rangle_{V_2} \\ &\text{subject to} && c - A(y) = s \\ & && s \in K. \end{aligned}$$

LP in Measure

$$\begin{aligned} \mathrm{P}^*_\mu = &\underset{\mu\in\mathcal{M}(\Omega)}{\text{minimize}} && \mathrm{E}_\mu[p(x)] = \int p(x)d\mu \\ &\text{subject to} && \int_\Omega d\mu = 1 \end{aligned}$$

$\mathcal{M}(\Omega)$: space of measures supported on Ω

Dual

$$\begin{aligned} &\underset{y}{\text{maximize}} && y \\ &\text{subject to} && p(x) - y \in K \end{aligned}$$

$$\begin{aligned} K^* &= \{\mu \in \mathcal{M} : \textstyle\int f d\mu \geq 0 \ \forall f \in K\} \\ &= \{(\text{nonegative) measures } \mu : supp(\mu) \subset \Omega\} \end{aligned}$$

$$\begin{aligned} K &= \{f \in \mathcal{C} : f \geq 0 \text{ on } \Omega\} \\ &= \{\text{nonnegative continuous functions on } \Omega\} \end{aligned}$$

Linear Map $A^*(\mu) : V_2^* = \mathcal{M} \to V_1^* = \mathbb{R} \longrightarrow A(\mu) = \int d\mu$

Dual Linear Map $A(y) : V_1 = \mathbb{R} \to V_2 = \mathcal{C}$

$< A^*(x), y >_{V_1} = < x, A(y) >_{V_2}$ $\qquad V_1 = \mathbb{R}, \quad V_1^* = \mathbb{R} \qquad V_2 = \mathcal{C}, \quad V_2^* = \mathcal{M}$

$$< \underbrace{\textstyle\int d\mu}_{\mathbb{R}}, \underbrace{y}_{\mathbb{R}} > = < \underbrace{\mu}_{\mathcal{M}}, \underbrace{A(y)}_{\mathcal{C}} > \longrightarrow (\textstyle\int d\mu)^T y = \int A(y) d\mu \longrightarrow \int y d\mu = \int A(y) d\mu \longrightarrow A(y) = y$$

Appendix II: Alternative Representations

Moment Matrix

Moment Matrix associated with a sequence of moments up to order $2d$:

Moment Matrix of order d: $$\mathbf{M}_d(y) = \mathrm{E}_\mu[B_d(x)B_d^T(x)] = \mathrm{E}_\mu[\sum_\alpha \mathbf{B}_\alpha x^\alpha] = \sum_\alpha \mathbf{B}_\alpha y_\alpha$$

Constant matrix

- $B_d\,(x)$: vector of monomial up to order d

Moment Matrix

$$\mathbf{M}_d(y) = \sum_\alpha \mathbf{B}_\alpha y_\alpha$$

$$\mathbf{M}_2(y) = \mathrm{E}[B_2(x)B_2^T(x)] = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y_0 + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} y_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} y_3 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} y_4$$

Localizing Matrix

Localizing matrix associated with a sequence of moments $[y_\alpha, \quad \alpha \in \mathbb{N}^n]$ and polynomial $g(x) \in \mathbb{R}[x]$

Localizing Matrix $$\mathbf{M}_d(gy) = \mathrm{E}[gB_d(x)B_d^T(x)] = \mathrm{E}_\mu[\textstyle\sum_\alpha \mathbf{B}_{g\alpha} x^\alpha] = \sum_\alpha \mathbf{B}_{g\alpha} y_\alpha$$

- $B_d(x)$: vector of monomial up to order d

Constant matrix

Localizing Matrix $\mathbf{M}_d(gy) = \sum_\alpha \mathbf{B}_{g\alpha} y_\alpha$

Sequence of moments up to order 4 and $g(x) = a - x_1^2 - x_2^2$

$$\mathbf{M}_1(gy) = \mathrm{E}[g(x)B_2(x)B_2^T(x)] = \mathrm{E}[g(x)\begin{bmatrix}1\\ x_1\\ x_2\end{bmatrix}\begin{bmatrix}1 & x_1 & x_2\end{bmatrix}] = \mathrm{E}[g(x)\begin{bmatrix}1 & x_1 & x_2\\ x_1 & x_1^2 & x_1x_2\\ x_2 & x_1x_2 & x_2^2\end{bmatrix}] = \mathrm{E}[\begin{bmatrix}g(x) & g(x)x_1 & g(x)x_2\\ g(x)x_1 & g(x)x_1^2 & g(x)x_1x_2\\ g(x)x_2 & g(x)x_1x_2 & g(x)x_2^2\end{bmatrix}]$$

$$= \mathrm{E}[\begin{bmatrix}a - x_1^2 - x_2^2 & ax_1 - x_1^3 - x_1x_2^2 & ax_2 - x_2x_1^2 - x_2^3\\ ax_1 - x_1^3 - x_1x_2^2 & ax_1^2 - x_1^4 - x_1^2x_2^2 & ax_1x_2 - x_1^3x_2 - x_1x_2^3\\ ax_2 - x_2x_1^2 - x_2^3 & ax_1x_2 - x_1^3x_2 - x_1x_2^3 & ax_2^2 - x_2^2x_1^2 - x_2^4\end{bmatrix}] = \begin{bmatrix}ay_{00} - y_{20} - y_{02} & ay_{10} - y_{30} - y_{12} & ay_{01} - y_{21} - y_{03}\\ ay_{10} - y_{30} - y_{12} & ay_{20} - y_{40} - y_{22} & ay_{11} - y_{31} - y_{13}\\ ay_{01} - y_{21} - y_{03} & ay_{11} - y_{31} - y_{13} & ay_{02} - y_{22} - y_{04}\end{bmatrix}$$

Localizing Matrix $\mathbf{M}_d(gy) = \sum_\alpha \mathbf{B}_{g\alpha} y_\alpha$

Sequence of moments up to order 4 and $g(x) = a - x_1^2 - x_2^2$

$$\mathbf{M}_1(gy) = \mathrm{E}[g(x)B_2(x)B_2^T(x)] = \mathrm{E}[g(x)\begin{bmatrix}1\\x_1\\x_2\end{bmatrix}\begin{bmatrix}1 & x_1 & x_2\end{bmatrix}] = \mathrm{E}[g(x)\begin{bmatrix}1 & x_1 & x_2\\x_1 & x_1^2 & x_1x_2\\x_2 & x_1x_2 & x_2^2\end{bmatrix}] = \mathrm{E}[\begin{bmatrix}g(x) & g(x)x_1 & g(x)x_2\\g(x)x_1 & g(x)x_1^2 & g(x)x_1x_2\\g(x)x_2 & g(x)x_1x_2 & g(x)x_2^2\end{bmatrix}]$$

$$= \mathrm{E}[\begin{bmatrix}a - x_1^2 - x_2^2 & ax_1 - x_1^3 - x_1x_2^2 & ax_2 - x_2x_1^2 - x_2^3\\ax_1 - x_1^3 - x_1x_2^2 & ax_1^2 - x_1^4 - x_1^2x_2^2 & ax_1x_2 - x_1^3x_2 - x_1x_2^3\\ax_2 - x_2x_1^2 - x_2^3 & ax_1x_2 - x_1^3x_2 - x_1x_2^3 & ax_2^2 - x_2^2x_1^2 - x_2^4\end{bmatrix}] = \begin{bmatrix}ay_{00} - y_{20} - y_{02} & ay_{10} - y_{30} - y_{12} & ay_{01} - y_{21} - y_{03}\\ay_{10} - y_{30} - y_{12} & ay_{20} - y_{40} - y_{22} & ay_{11} - y_{31} - y_{13}\\ay_{01} - y_{21} - y_{03} & ay_{11} - y_{31} - y_{13} & ay_{02} - y_{22} - y_{04}\end{bmatrix}$$

$$\mathbf{M}_1(gy) = \sum_\alpha y_\alpha \mathbf{B}_{g\alpha} =$$

$$= \begin{bmatrix}a & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} y_{00} + \begin{bmatrix}-1 & 0 & 0\\0 & a & 0\\0 & 0 & 0\end{bmatrix} y_{20} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & a\\0 & a & 0\end{bmatrix} y_{11} + \begin{bmatrix}-1 & 0 & 0\\0 & 0 & 0\\0 & 0 & a\end{bmatrix} y_{02} + \begin{bmatrix}0 & -1 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{bmatrix} y_{30} + \begin{bmatrix}0 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 0\end{bmatrix} y_{21}$$

$$+ \begin{bmatrix}0 & -1 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{bmatrix} y_{12} + \begin{bmatrix}0 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 0\end{bmatrix} y_{03} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix} y_{31} + \begin{bmatrix}0 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix} y_{22} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix} y_{13} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{bmatrix} y_{04}$$

SOS Polynomials

- $p(x) = B^T(x)QB(x) = \langle Q, B(x)B^T(x)\rangle$

$B(x)B^T(x) = \sum_\alpha \mathbf{B}_\alpha x^\alpha$

$$p(x) = \langle Q, \sum_\alpha \mathbf{B}_\alpha x^\alpha \rangle = \sum_\alpha \underbrace{\langle Q, \mathbf{B}_\alpha \rangle}_{p_\alpha\text{: Coefficient}} x^\alpha = \sum_\alpha p_\alpha x^\alpha \longrightarrow p_\alpha = \langle Q, \mathbf{B}_\alpha \rangle$$

SOS Polynomials

Example: $p(x) = x_1^2 - x_1 x_2^2 + x_2^4 + 1$

$$p(x) = \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}^T \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}$$

$$p(x) = B^T(x) Q B(x)$$

SOS Polynomials

Example: $p(x) = x_1^2 - x_1 x_2^2 + x_2^4 + 1$

$$p(x) = \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}^T \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}$$

$$p(x) = B^T(x) Q B(x)$$

$$B(x)B^T(x) = \begin{bmatrix} 1 & x_2 & x_2^2 & x_1 \\ x_2 & x_2^2 & x_2^3 & x_1x_2 \\ x_2^2 & x_2^3 & x_2^4 & x_1x_2^2 \\ x_1 & x_1x_2 & x_1x_2^2 & x_1^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{00}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{10}} x_1 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{B}_{20}} x_1^2 + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{01}} x_2 + \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{02}} x_2^2 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{03}} x_2^3 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{04}} x_2^4 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{11}} x_1x_2 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{B}_{12}} x_1x_2^2$$

SOS Polynomials

Example: $p(x) = x_1^2 - x_1 x_2^2 + x_2^4 + 1$

$$p(x) = \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}^T \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}$$

$$p(x) = B^T(x) Q B(x)$$

$$B(x)B^T(x) = \begin{bmatrix} 1 & x_2 & x_2^2 & x_1 \\ x_2 & x_2^2 & x_2^3 & x_1x_2 \\ x_2^2 & x_2^3 & x_2^4 & x_1x_2^2 \\ x_1 & x_1x_2 & x_1x_2^2 & x_1^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{00}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{10}} x_1 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{B}_{20}} x_1^2 + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{01}} x_2 + \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{02}} x_2^2 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{03}} x_2^3 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{04}} x_2^4 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{11}} x_1x_2 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{B}_{12}} x_1x_2^2$$

$$p_\alpha = \langle Q, \mathbf{B}_\alpha \rangle$$

$$p_{00} = \langle Q, \mathbf{B}_{00} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rangle = 1$$

$$p_{10} = \langle Q, \mathbf{B}_{10} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \rangle = 0$$

$$p_{20} = \langle Q, \mathbf{B}_{20} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rangle = 1$$

$$p_{01} = \langle Q, \mathbf{B}_{01} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rangle = 0$$

$$p_{02} = \langle Q, \mathbf{B}_{02} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rangle = 0$$

$$p_{03} = \langle Q, \mathbf{B}_{03} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rangle = 0$$

$$p_{04} = \langle Q, \mathbf{B}_{04} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rangle = 1$$

$$p_{11} = \langle Q, \mathbf{B}_{11} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rangle = 0$$

$$p_{12} = \langle Q, \mathbf{B}_{12} \rangle = \langle \frac{1}{6} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rangle = -1$$

SOS Polynomials

$\sigma(x) \in SOS \qquad g(x)$: polynomial

$$g(x)\sigma(x) = g(x)B(x)^T Q B(x) = \langle Q, gB(x)^T B(x)\rangle$$

$$g(x).B(x)B^T(x) = \sum_\alpha \mathbf{B}_{g\alpha} x^\alpha$$

$$g(x)\sigma(x) = \langle Q, \sum_\alpha \mathbf{B}_{g\alpha} x^\alpha\rangle = \sum_\alpha \underbrace{\langle Q, \mathbf{B}_{g\alpha}\rangle}_{\text{Coefficient of polynomial } g(x)\sigma(x)} x^\alpha \longrightarrow \text{coefficients}(g(x)\sigma(x)) = \langle Q, \mathbf{B}_{g\alpha}\rangle$$

SOS Polynomials

Example: $\sigma(x) \in SOS$ $\quad \sigma(x) = x_1^2 - x_1x_2^2 + x_2^4 + 1 = \frac{1}{6}\begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}^T \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}$

$g(x) = x_1$

$$g(x)\sigma(x) = x_1^3 - x_1^2x_2^2 + x_1x_2^4 + x_1$$

$$g(x).B(x)B^T(x) = \begin{bmatrix} x_1 & x_1x_2 & x_1x_2^2 & x_1^2 \\ x_1x_2 & x_1x_2^2 & x_1x_2^3 & x_1^2x_2 \\ x_1x_2^2 & x_1x_2^3 & x_1x_2^4 & x_1^2x_2^2 \\ x_1^2 & x_1^2x_2 & x_1^2x_2^2 & x_1^3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}}_{\mathbf{B}_{10}} x_1 + \underbrace{\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}}_{\mathbf{B}_{12}} x_1x_2 + \underbrace{\begin{bmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}}_{\mathbf{B}_{12}} x_1x_2^2 + \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}}_{\mathbf{B}_{13}} x_1x_2^3 + \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}}_{\mathbf{B}_{14}} x_1x_2^4 + \underbrace{\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}}_{\mathbf{B}_{21}} x_1^2x_2 + \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix}}_{\mathbf{B}_{21}} x_1^2x_2 + \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}}_{\mathbf{B}_{22}} x_1^2x_2^2 + \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}}_{\mathbf{B}_{30}} x_1^3$$

$$\underbrace{\text{coefficients}(g(x)\sigma(x)) = \langle Q, \mathbf{B}_{g\alpha} \rangle}$$

$$\langle Q, \mathbf{B}_{10} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \rangle = 1$$

$$\langle Q, \mathbf{B}_{20} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} \rangle = 0$$

$$\langle Q, \mathbf{B}_{30} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix} \rangle = 1$$

$$\langle Q, \mathbf{B}_{11} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \rangle = 0$$

$$\langle Q, \mathbf{B}_{12} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix} \rangle = 0$$

$$\langle Q, \mathbf{B}_{13} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} \rangle = 0$$

$$\langle Q, \mathbf{B}_{14} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix} \rangle = 1$$

$$\langle Q, \mathbf{B}_{21} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix} \rangle = 0$$

$$\langle Q, \mathbf{B}_{22} \rangle = \langle \frac{1}{6}\begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix} \rangle = -1$$

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