Lecture 3

Sum Of Squares For Nonlinear Optimization

MIT 16.S498: Risk Aware and Robust Nonlinear Planning
Fall 2019

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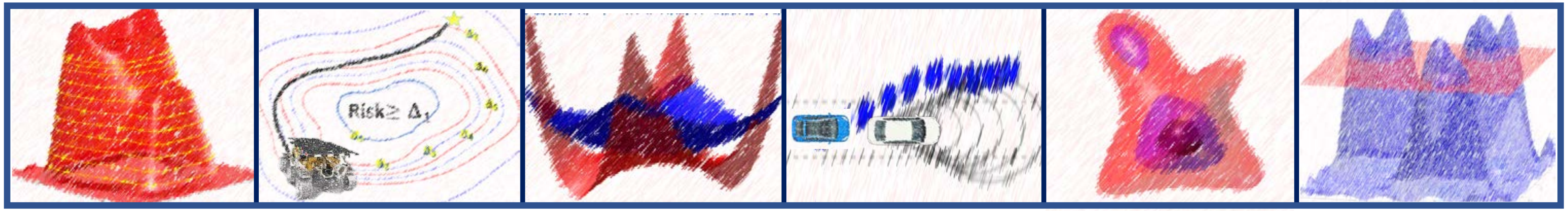

Nonlinear (nonconvex) Optimization

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \;\; i = 1, \ldots, m \end{array}$$

Objective function and constraints are polynomial functions.

Goal: Find Convex Relaxations of Nonlinear Optimization

Tools:

i) Nonnegative Polynomials   ii) Semidefinite Programs

Nonlinear (nonconvex) Optimization

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Tools: i) Nonnegative Polynomials ii) Semidefinite Programs

Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

Step 2:

Represent **Nonnegative Polynomials** with **Positive Semidefinite** Matrices (PSD)

Reformulate Nonlinear Optimization as **Semidefinite Program**

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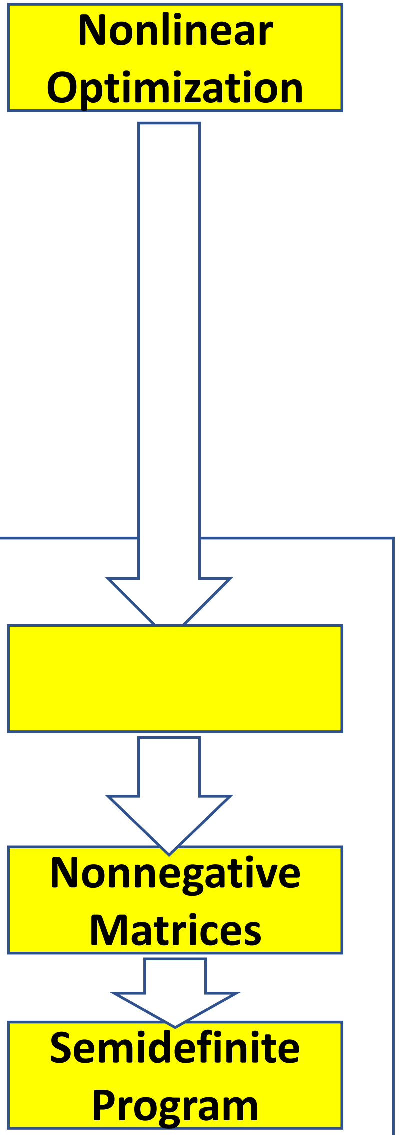

Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

Step 2:

Represent **Nonnegative Polynomials** with **Positive Semidefinite** Matrices (PSD)

$\Downarrow$

Reformulate Nonlinear Optimization as **Semidefinite Program**

Nonnegative Polynomials

- Monomials
- Polynomials
- Nonnegative Polynomials

Polynomials

- **Monomials:** product of powers of variables

variables x: $x = [x_1, ..., x_n]^T$

n-tuple: $\alpha = (\alpha_1, ..., \alpha_n), \quad \alpha_i \in \mathbb{N}$

- Monomial (powers of variables):

$$x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$$

- Degree of monomial: $\sum_{i=1}^{n} \alpha_i$

Polynomials

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- **Polynomials:** finite linear combination of *monomials*.
  - Polynomial: $p(x) : \mathbb{R}^n \rightarrow \mathbb{R}$

$$p(x) = \sum_\alpha p_\alpha x^\alpha$$

coefficients monomials

(univariate) Polynomial of order 3 in x_1

$$p(x_1) = 1 + 0.5x_1^2 + 0.75x_1^3$$

(multivariate) Polynomial of order 5 in x_1 and x_2

$$p(x_1, x_2) = 0.56 + 0.5x_1 + 2x_2^2 + 0.75x_1^3x_2^2$$

  - Degree of polynomial: Maximum degree of monomial in the polynomial

Polynomials

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- **Vector representation:** $p(x) = C^T B(x)$

Vector of coefficients Vector of monomials

$$p(x_1) = 1 + 0.5x_1^2 + 0.75x_1^3 = \begin{bmatrix} 1 \\ 0.5 \\ 0.75 \end{bmatrix}^T \begin{bmatrix} 1 \\ x_1^2 \\ x_1^3 \end{bmatrix}$$

$$p(x_1, x_2) = 0.56 + 0.5x_1 + 2x_2^2 + 0.75x_1^3x_2^2 = \begin{bmatrix} 0.56 \\ 0.5 \\ 2 \\ 0.75 \end{bmatrix}^T \begin{bmatrix} 1 \\ x_1 \\ x_2^2 \\ x_1^3x_2^2 \end{bmatrix}$$

Polynomials

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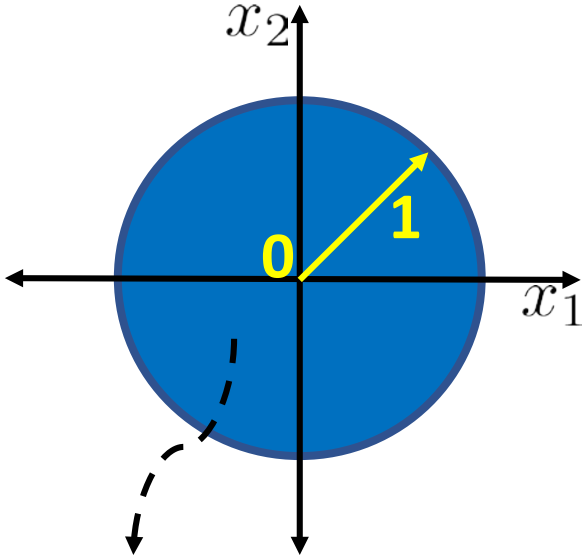
$\mathbb{R}[x]$ Set (ring)of real polynomial in the variables $x \in \mathbb{R}^n$

$\mathbb{R}_d[x] \subset \mathbb{R}[x]$ Set of polynomials of degree at most d

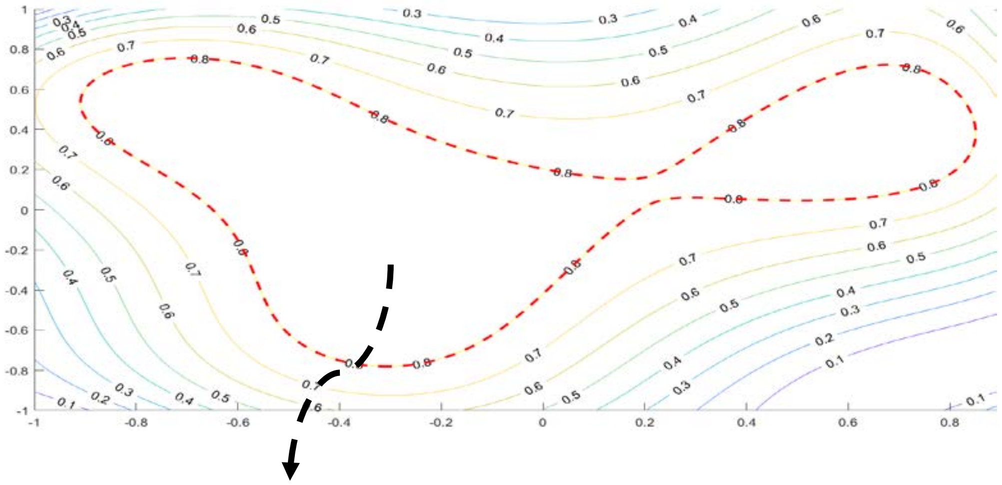
Level Set of Polynomials

Semialgebraic Set: Set described by level sets of polynomials

$$\{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, ..., n, \quad h_i(x) = 0, i = 1, ...m\}$$

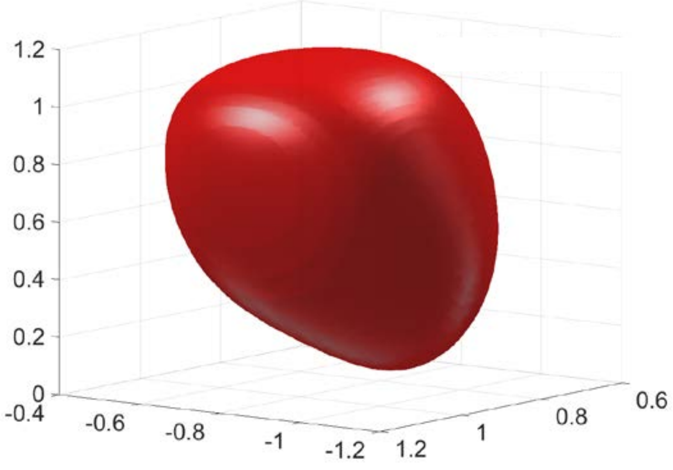


$\{x \in \mathbb{R}^2 : 1 - x_1^2 - x_2^2 \geq 0\}$



$\{x \in \mathbb{R}^2 : g(x_1, x_2) \geq 0.8\}$

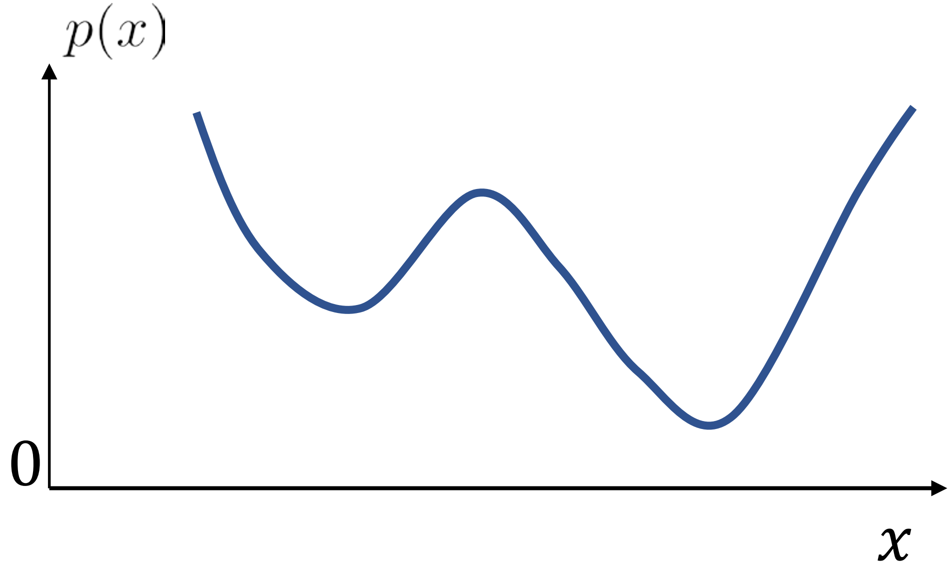
$-0.42x_1^5 - 1.2x_1^4x_2 - 0.48x_1^4 + 0.3x_1^3x_2^2 - 0.57x_1^3x_2 + 0.61x_1^3 - 0.66x_1^2x_2^3 + 0.17x_1^2x_2^2 + 1.9x_1^2x_2 + 0.066x_1^2 +$
$0.69x_1x_2^4 - 0.14x_1x_2^3 - 0.85x_1x_2^2 + 0.6x_1x_2 - 0.22x_1 + 0.011x_2^5 - 0.068x_2^4 - 0.07x_2^3 - 0.42x_2^2 - 0.084x_2 + 0.84$



$\{x \in \mathbb{R}^3 : g(x_1, x_2, x_3) \geq 1\}$

Nonnegative Polynomials

$$p(x) \in \mathbb{R}[x] \xrightarrow{\text{Nonnegative Polynomials}} p(x) \geq 0 \quad \forall x \in \mathbb{R}^n$$



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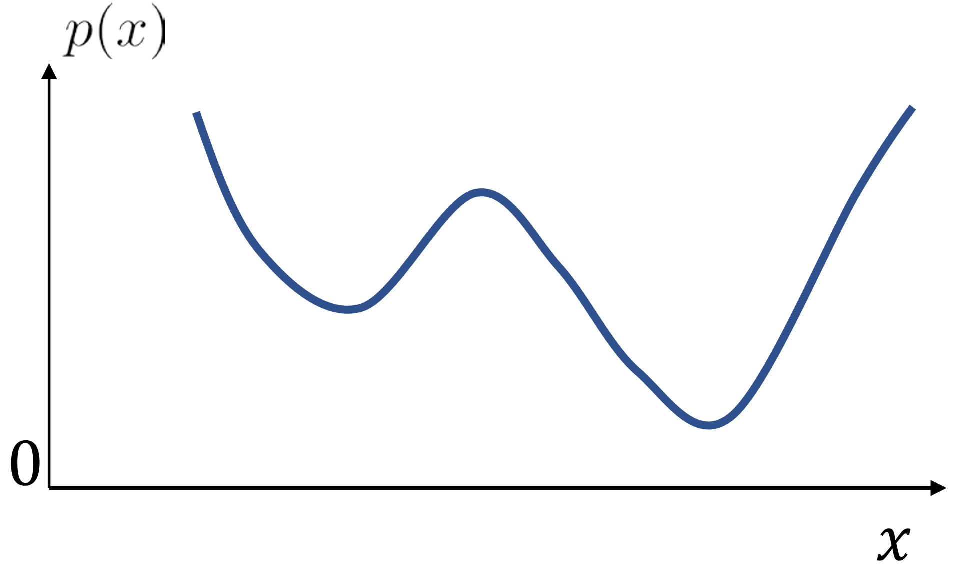
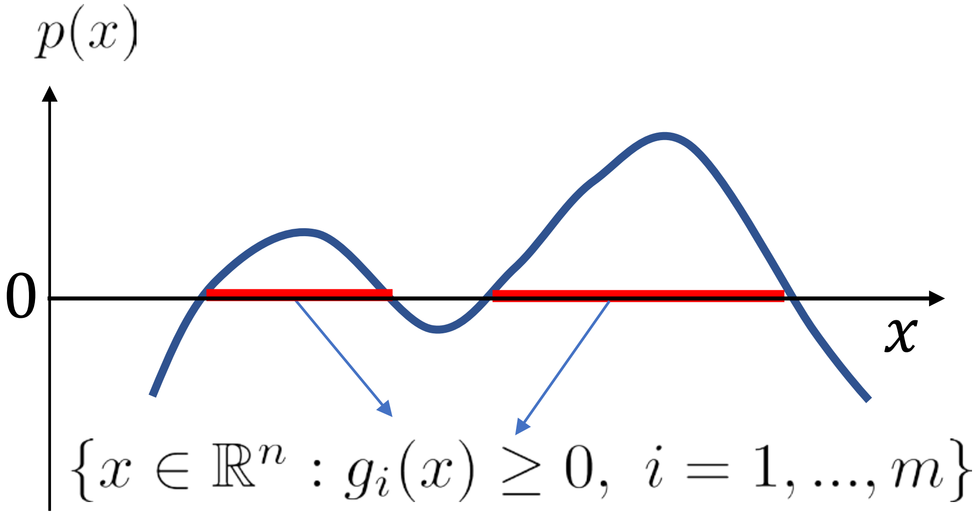


Nonnegative Polynomial on the Set

$$p(x) \geq 0 \quad \forall x \in \underbrace{\{x \in \mathbb{R}^n : g_i(x) \geq 0, \; i = 1, \ldots, m\}}_{\text{Set}}$$



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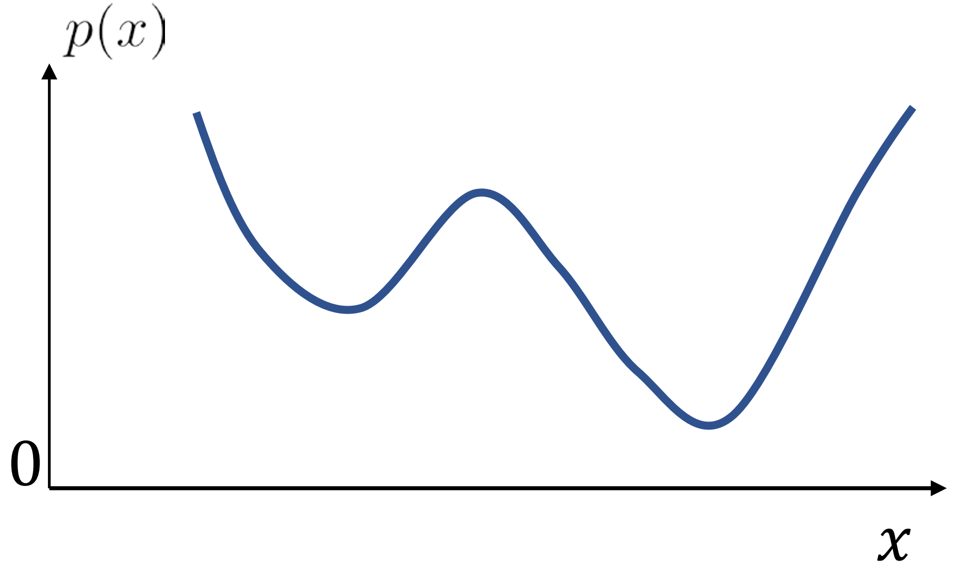
For Unconstrained Optimization



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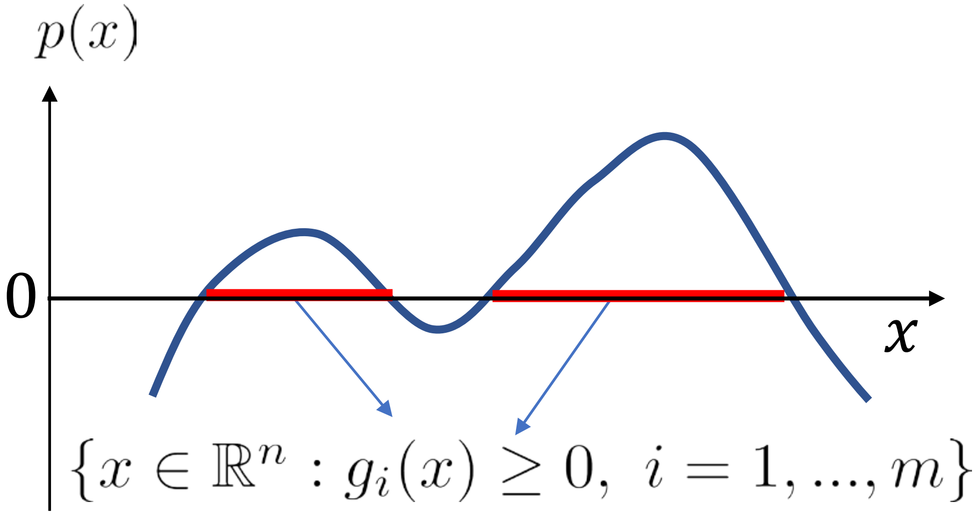
For Constrained Optimization



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Tools: i) Nonnegative Polynomials ii) Semidefinite Programs

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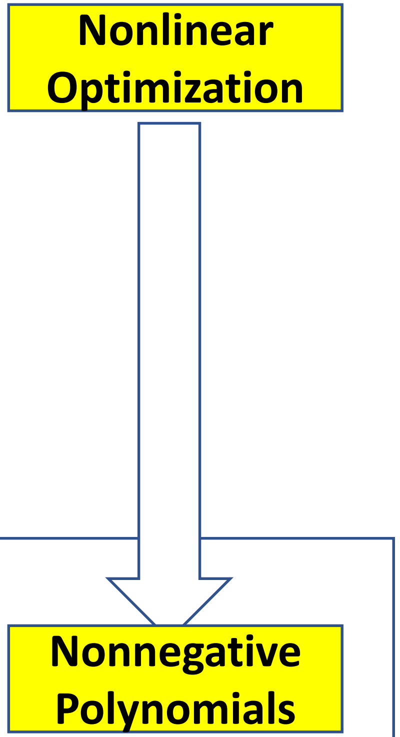
Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

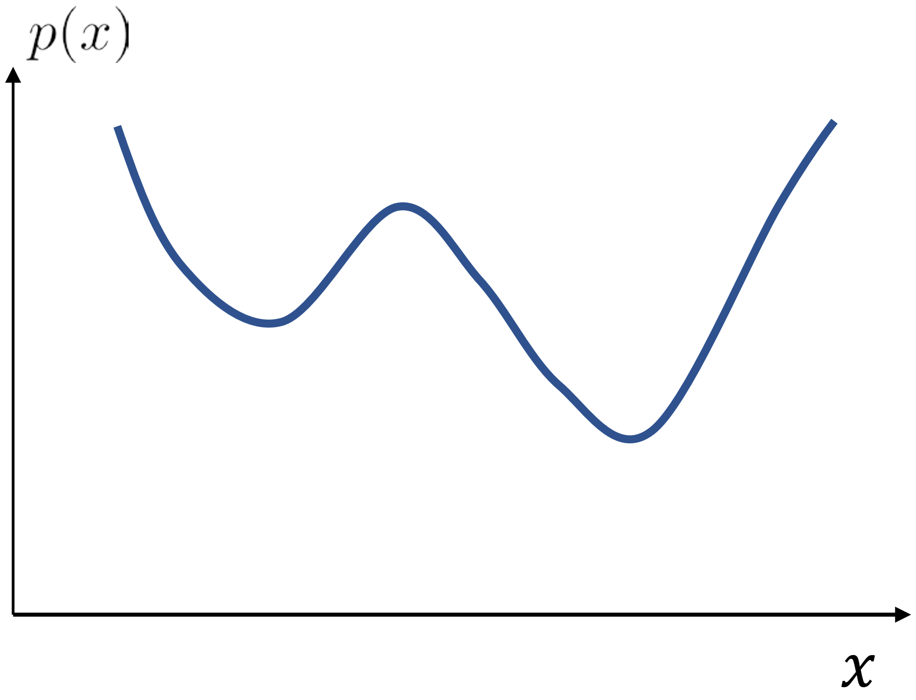


Nonlinear Optimization and Nonnegative Polynomials

Unconstrained Optimization and Nonnegative polynomials

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$

$p(x) \in \mathbb{R}[x]$



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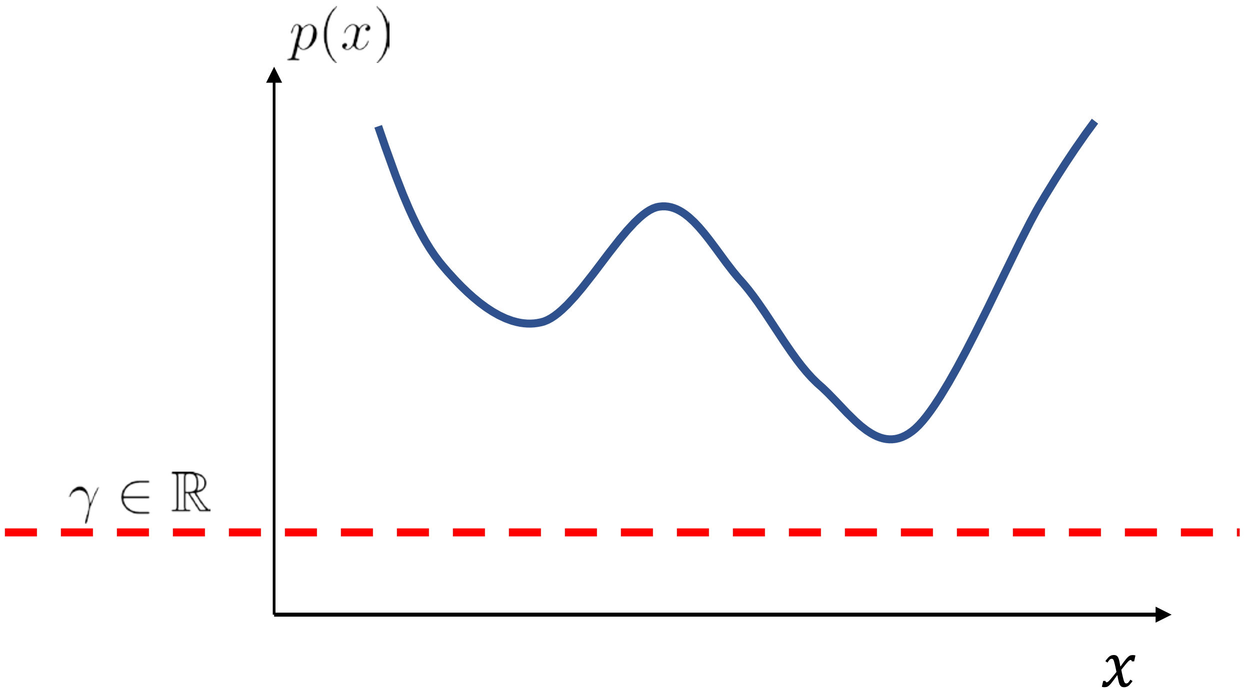

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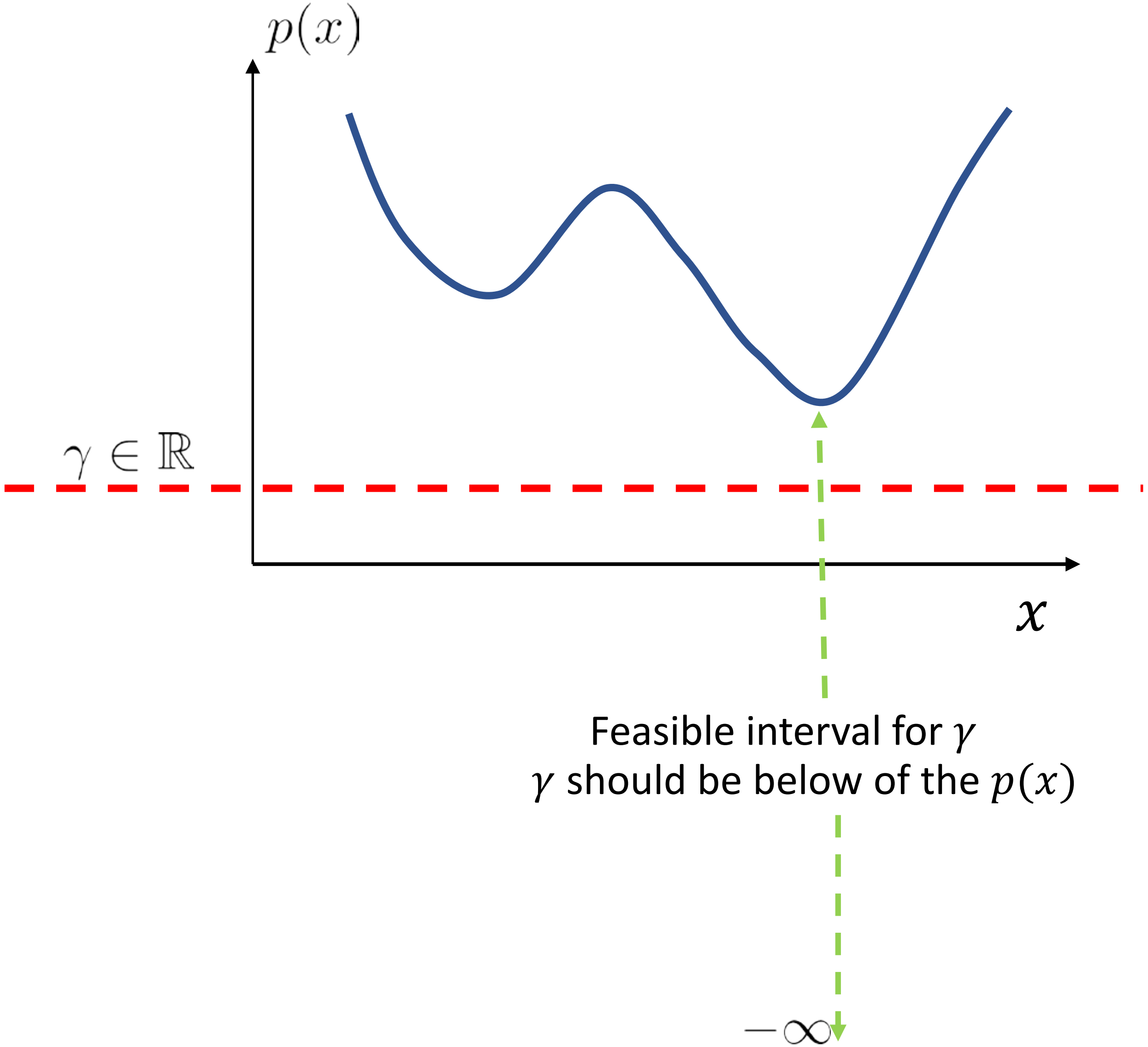

Feasible interval for γ
γ should be below of the $p(x)$

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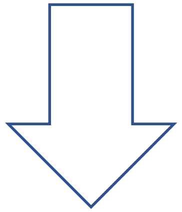

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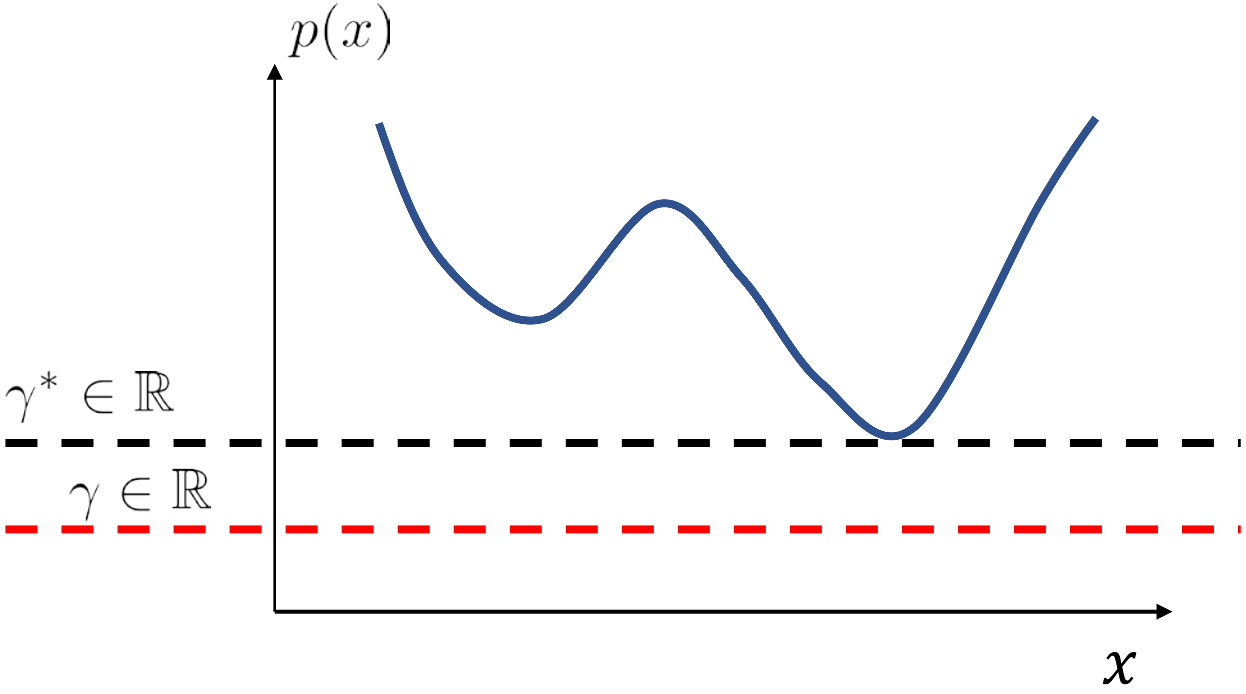

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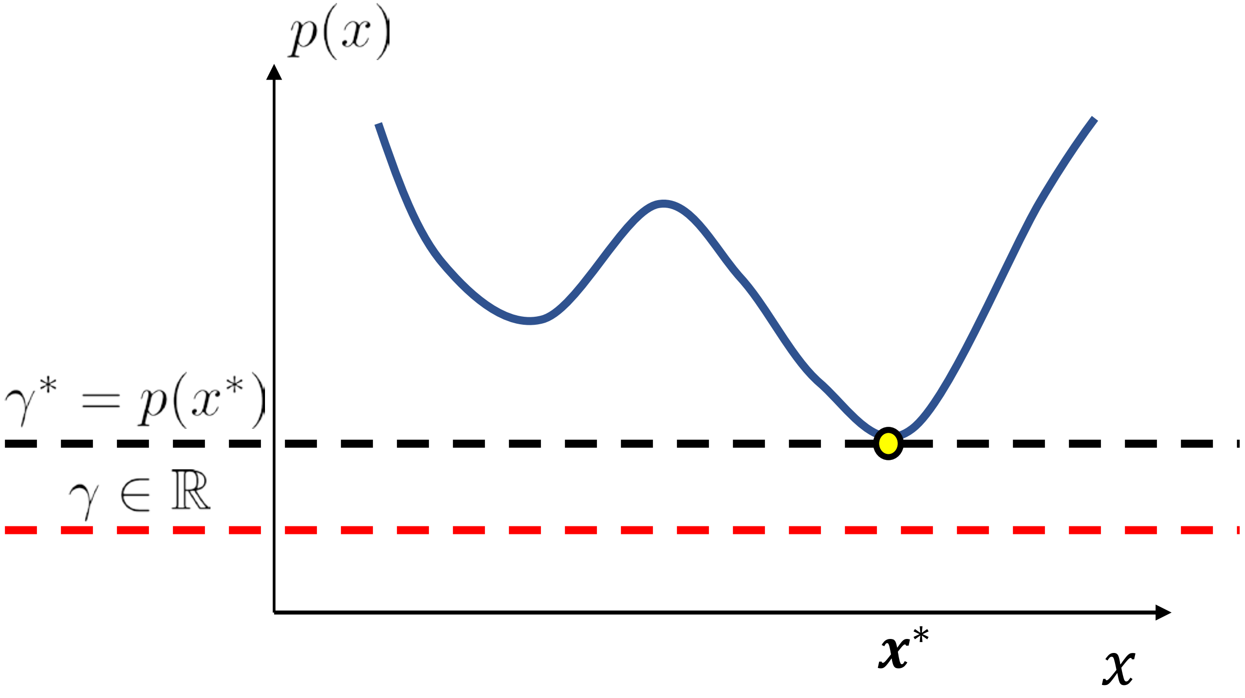

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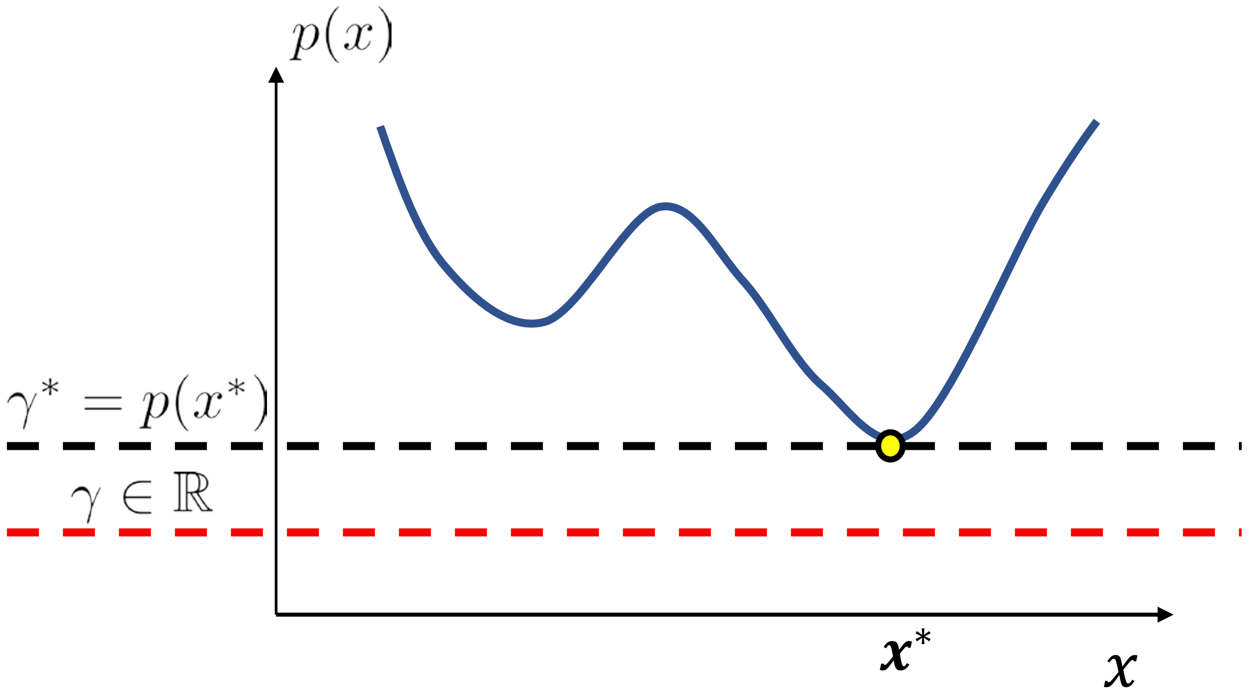
Linear Constraint

Polynomial Nonnegativity Constraint

We are looking for γ such that $p(x) - \gamma$ be a nonnegative polynomial.



Constrained Optimization and Nonnegative polynomials

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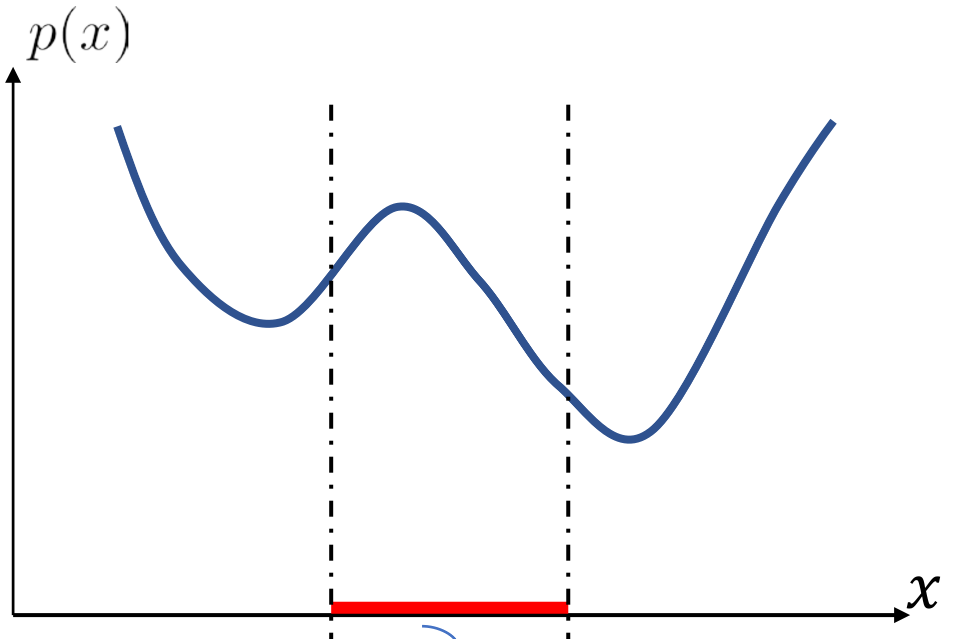
$$p(x), g_i(x) \in \mathbb{R}[x], \;\; i = 1, \dots, m$$



$$\{x \in \mathbb{R}^n : g_i(x) \geq 0, \; i = 1, \dots, m\}$$

Feasible Set

Constrained Optimization and Nonnegative polynomials

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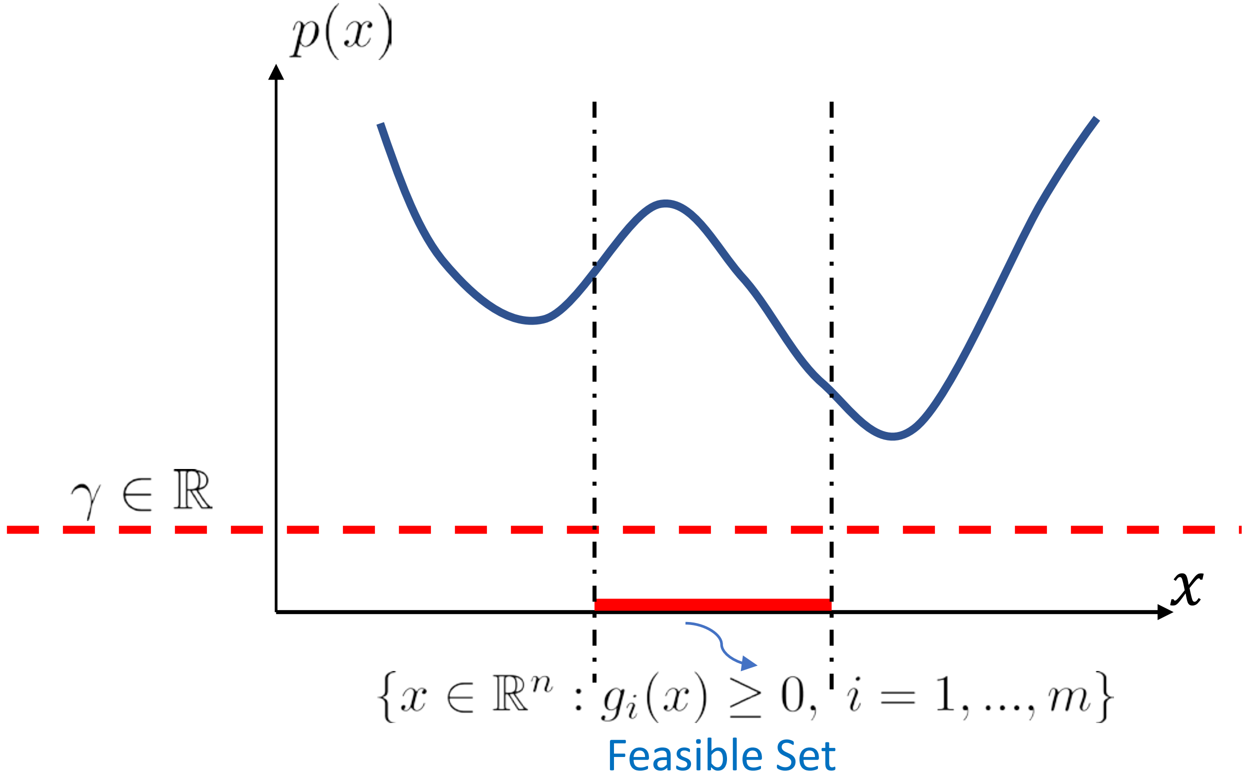

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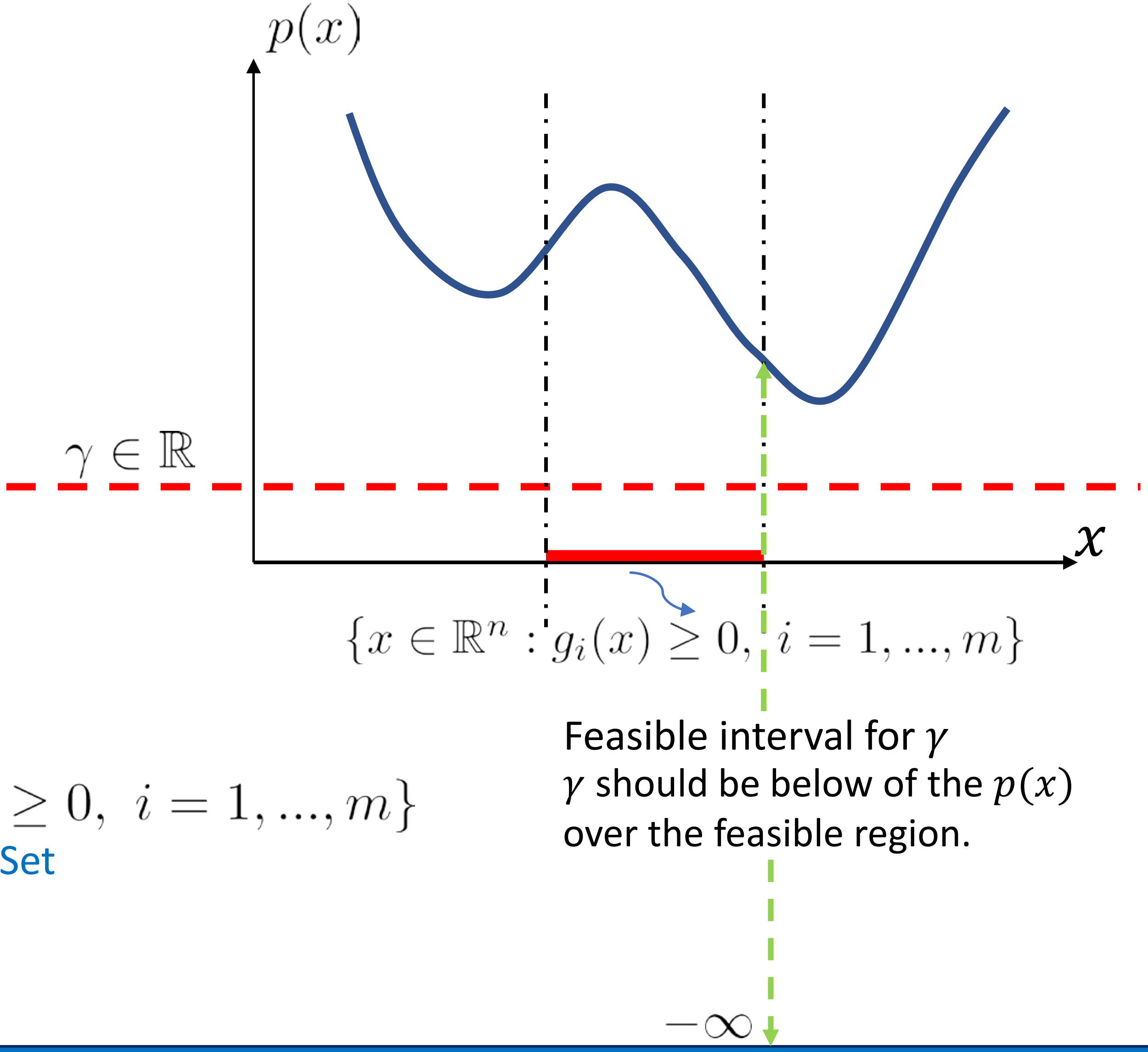
Feasible Set



Feasible interval for γ
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Constrained Optimization and Nonnegative polynomials

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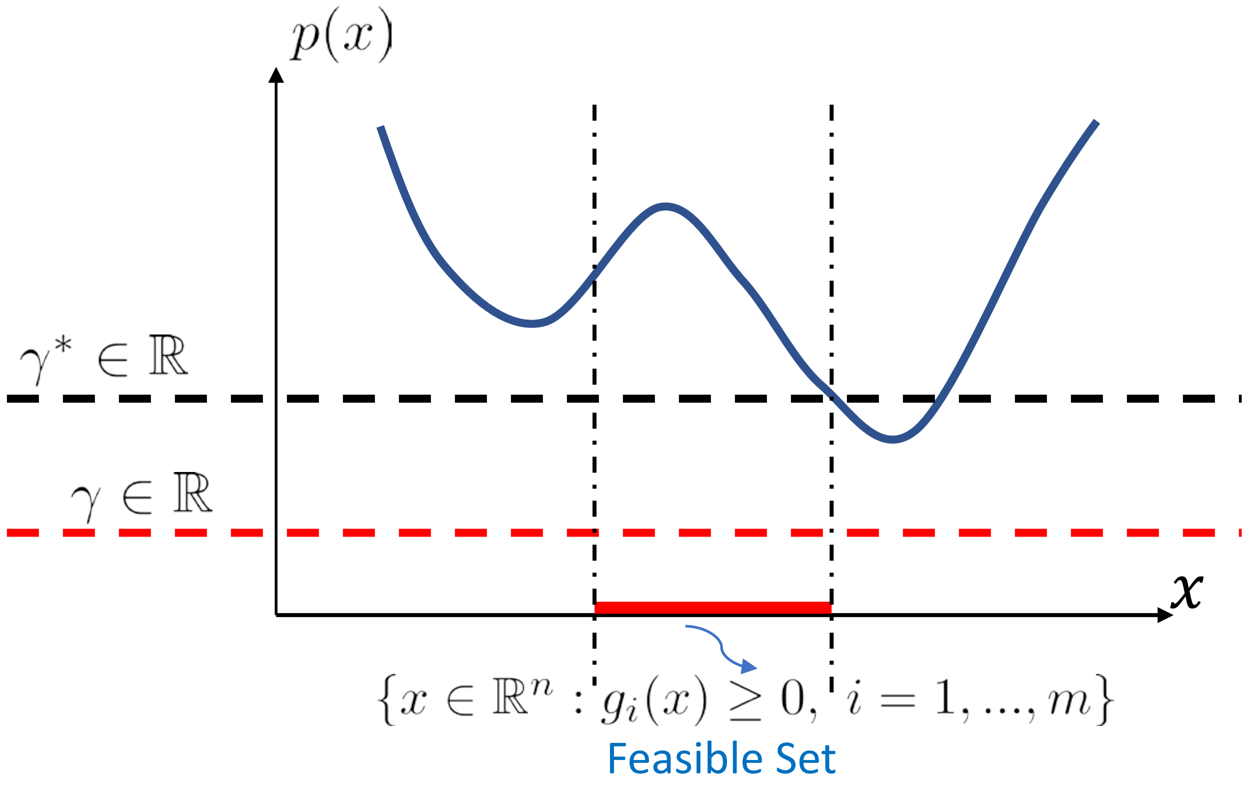

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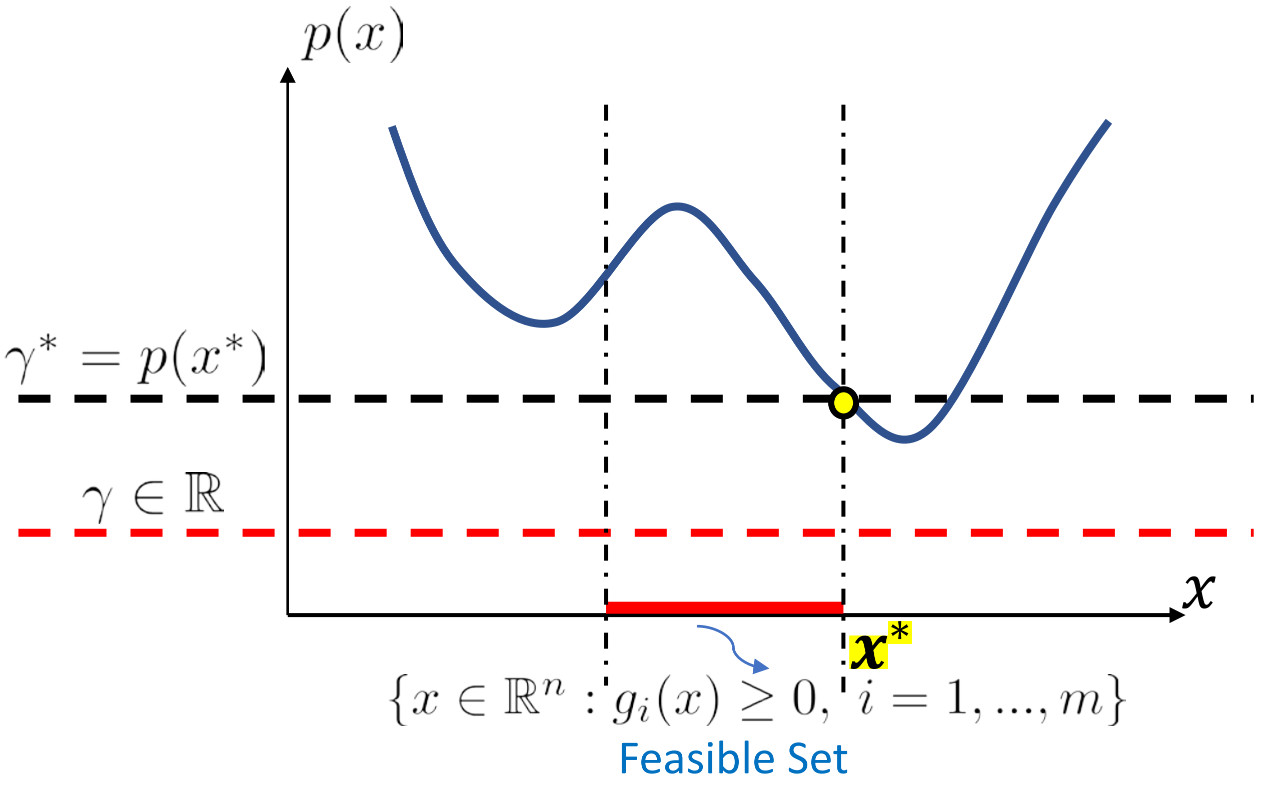

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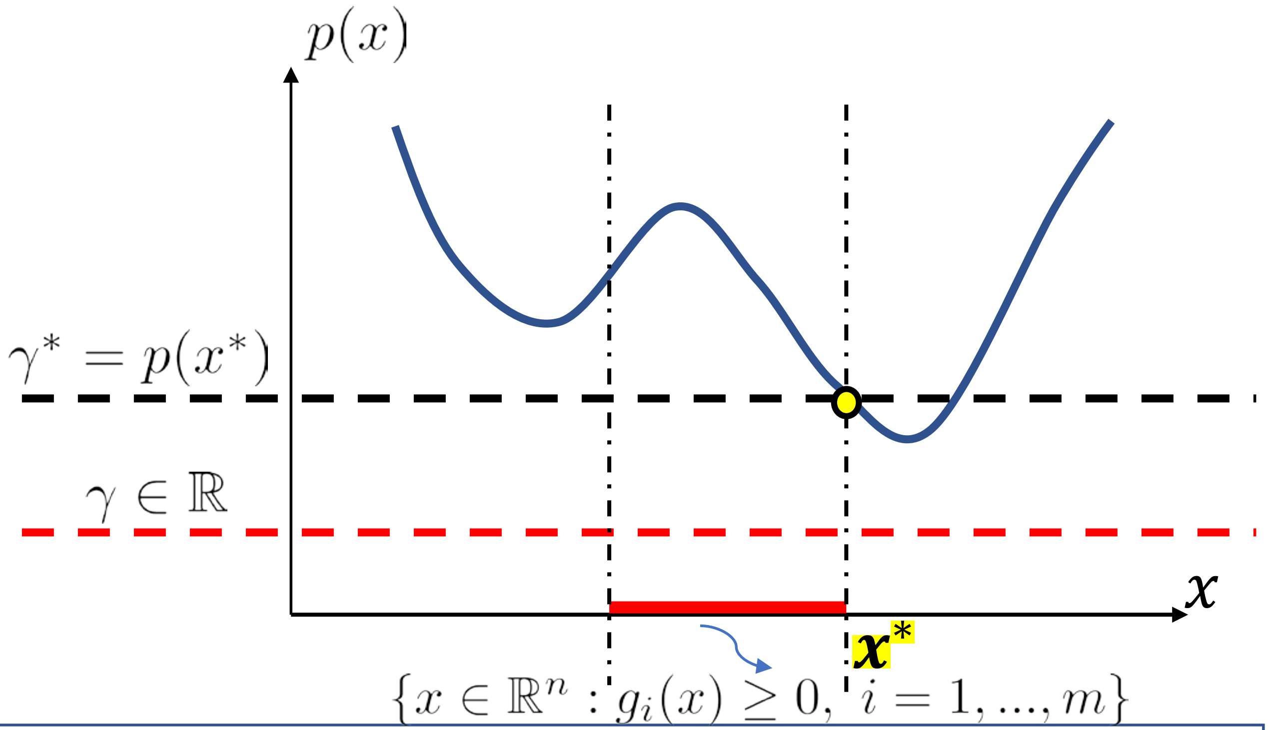

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→ Linear constraint

→ Polynomial Nonnegativity constraint

We are looking for γ such that $p(x) - \gamma$ be a nonnegative polynomial on the set $\{x \in \mathbb{R}^n : g_i(x) \geq 0, \;\; i = 1, \ldots, m\}$

Nonlinear Optimization and Nonnegative polynomials

Unconstrained Optimization:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x) \qquad \Longrightarrow \qquad \begin{aligned} &\underset{\gamma\in\mathbb{R}}{\text{maximize}} && \gamma \xrightarrow{\hspace{2em}} \text{linear function} \\ &\text{subject to} && \boxed{p(x)-\gamma \geq 0, \quad \forall x \in \mathbb{R}^n} \end{aligned}$$

Polynomial Nonnegativity Constraint

$p(x) \in \mathbb{R}[x]$

Constrained Optimization:

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linear function

Polynomial Nonnegativity Constraint

$\Downarrow$

Replace with convex constraints

Convex optimization

$$p(x)\in\mathbb{R}[x]$$

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linear function

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- Reformulation of Nonlinear Optimization in terms of nonnegative polynomials.

- **We can also reformulate different problems (in different domains)** in terms of nonnegative polynomials.

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Example: Stability of Nonlinear Systems

Given a nonlinear dynamical system $\dot{x} = f(x), \quad x(0) = x_0$

We want to show that solutions $x(t)$ converge to zero for all initial conditions (**stability**).

- To prove this, we need to find an energy function $V(x)$ with following properties

Lyapunov function: $V(x) > 0 \quad on \quad x \neq 0 \qquad -\dot{V}(x) > 0$

- Reformulation of Nonlinear Optimization in terms of nonnegative polynomials.

- **We can also reformulate different problems (in different domains)** in terms of nonnegative polynomials.

Example: Stability of Nonlinear Systems

Given a nonlinear dynamical system $\dot{x} = f(x), \qquad x(0) = x_0$

We want to show that solutions $x(t)$ converge to zero for all initial conditions (**stability**).

- To prove this, we need to find an energy function $V(x)$ with following properties

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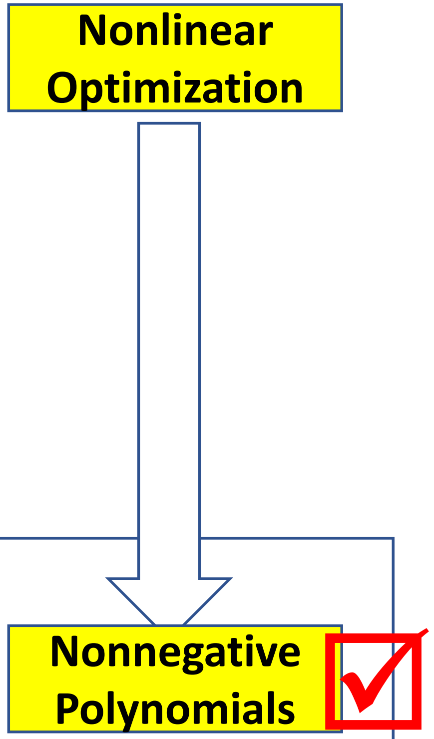
Example: MAX CUT Problem in Graph Theory

Nonlinear (nonconvex) Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \ i = 1, ..., m \end{aligned}$$

Tools: i) Nonnegative Polynomials ii) Semidefinite Programs



Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

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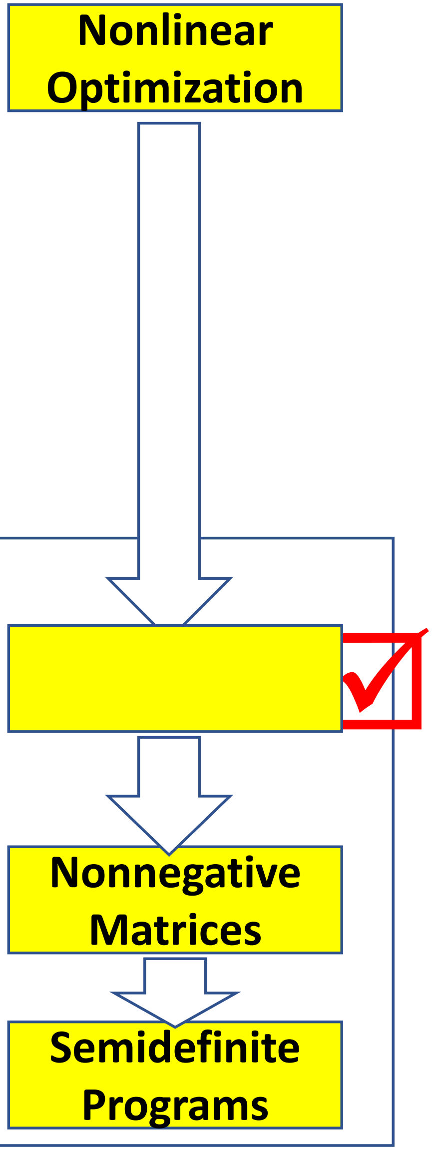
Step 2:

Represent **Nonnegative Polynomials** with **Positive Semidefinite** Matrices (PSD)

Reformulate Nonlinear Optimization as **Semidefinite Programs**



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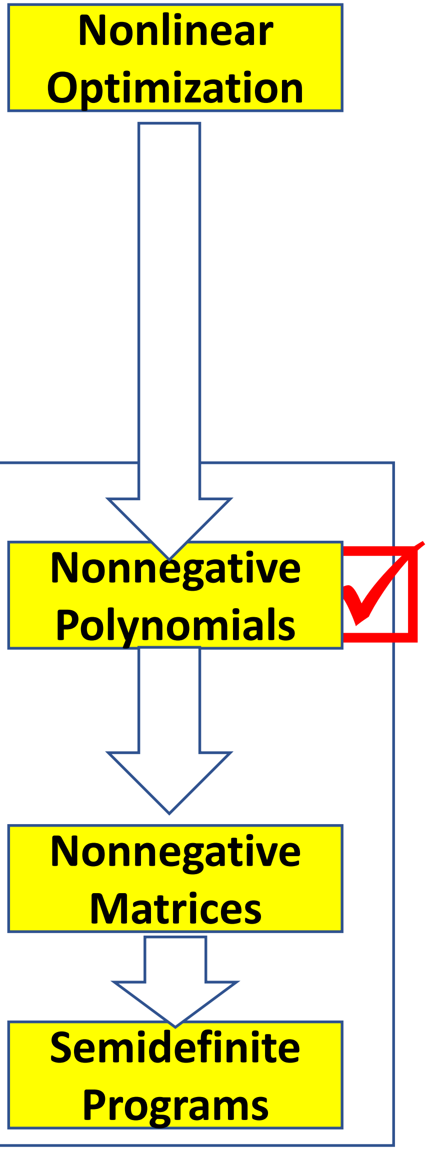

Nonlinear Optimization

Nonnegative Polynomials ☑

Nonnegative Matrices

Semidefinite Programs

Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

Step 2:

- **2.1 Replace Nonnegative Polynomials with Sum of Squares (SOS) Polynomials**
- **2.2** Represent **SOS Polynomials** with **Positive Semidefinite** Matrices (PSD)

⇓

Reformulate Nonlinear Optimization as **Semidefinite Programs**

Sum of Squares (SOS) Polynomials

Sum of Squares (SOS) Polynomials

Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if :

it can be written as a finite sum of squares of other polynomials.

$$p(x) \in \mathbb{R}[x] \xrightarrow{\text{SOS}} \boxed{p(x) = \sum_{i=1}^{\ell} h_i^2(x) \qquad h_i(x) \in \mathbb{R}[x], \quad i = 1, \dots, \ell}$$

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Example:

$$p(x) = x_1^2 - x_1 x_2^2 + x_2^4 + 1 \longrightarrow p(x) = \underbrace{\left(\frac{\sqrt{3}}{2}(x_1 - x_2^2)\right)}_{h_1}{}^2 + \underbrace{\left(\frac{1}{2}(x_1 + x_2^2)\right)}_{h_2}{}^2 + \underbrace{1}_{h_3}{}^2 \Rightarrow$$

$p(x)$ is SOS

$\Downarrow$

Nonnegative polynomial

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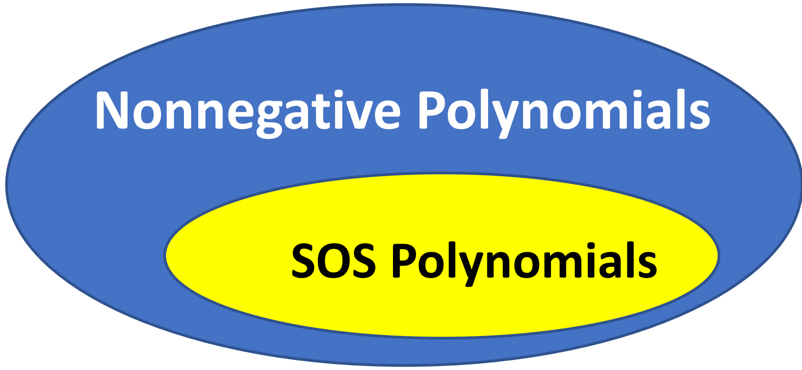

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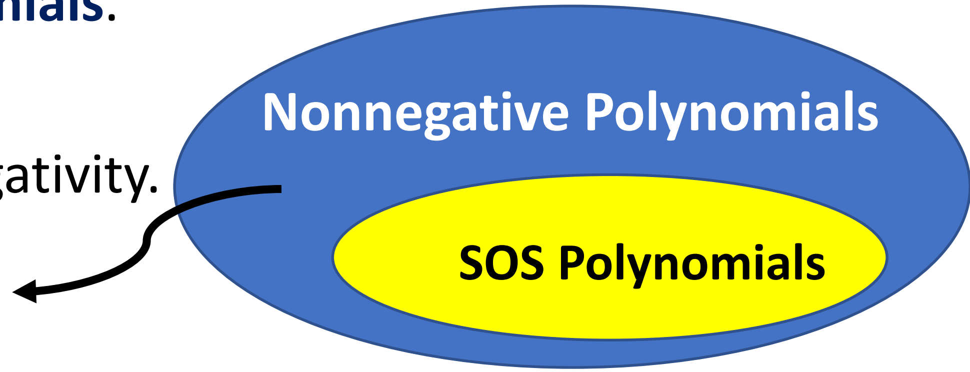
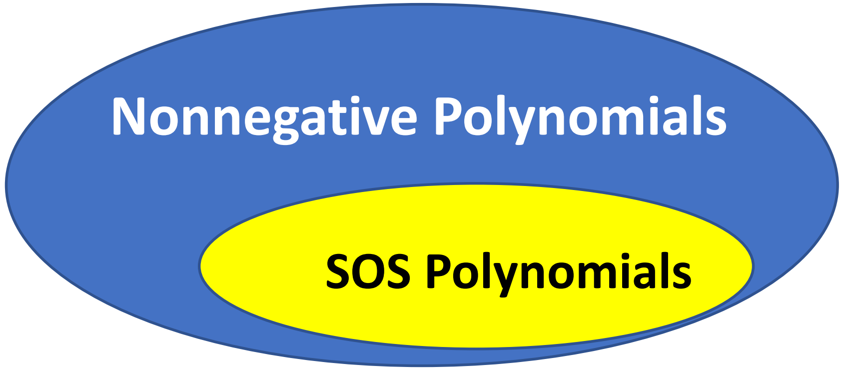


Example: Motzkin polynomial $\quad p(x_1, x_2) = x_1^4 x_2^2 + x_1^2 x_2^4 + 1 - 3x_1^2 x_2^2$

$p(x_1, x_2) \geq 0 \qquad p(x_1, x_2) \notin SOS$

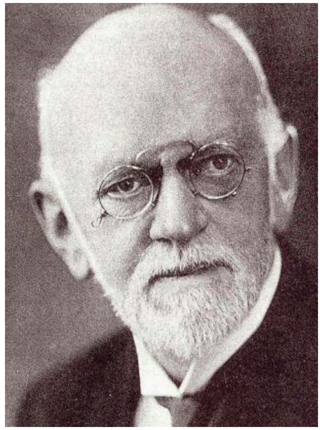
Sum of Squares (SOS) Polynomials



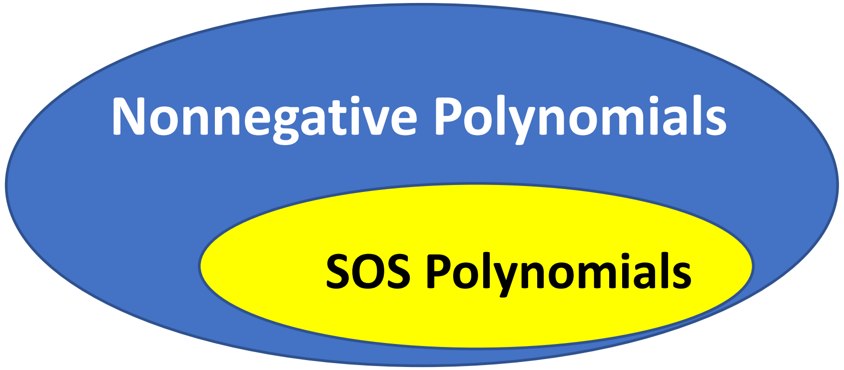
- ➢ SOS condition is a **sufficient** test for polynomial nonnegativity.

- The investigation of the relation between **nonnegativity** and **SOS** began in the paper of Hilbert from 1888.

D. Hilbert,“ Uber die Darstellung Definiter Formen als Summe von Formenquadraten“ , Math. Ann., 32 (1888)


David Hilbert

Sum of Squares (SOS) Polynomials



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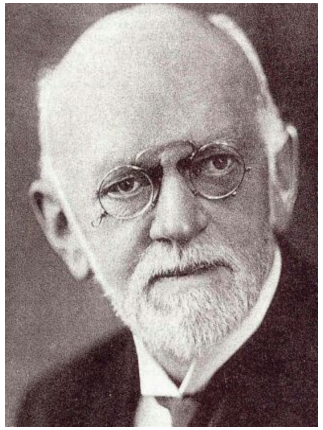
- Hilbert showed that every **nonnegative polynomial** is **SOS** only in the following three cases:

$$\underbrace{\textbf{i)}\ \text{Univariate Polynomials , }\textbf{ii)}\ \text{Quadratic Polynomials (d=2), }\textbf{iii)}\ \text{Bivariate polynomial of degree 4 (n=2,d=4)}}$$

Nonnegativity Condition = **SOS Condition**



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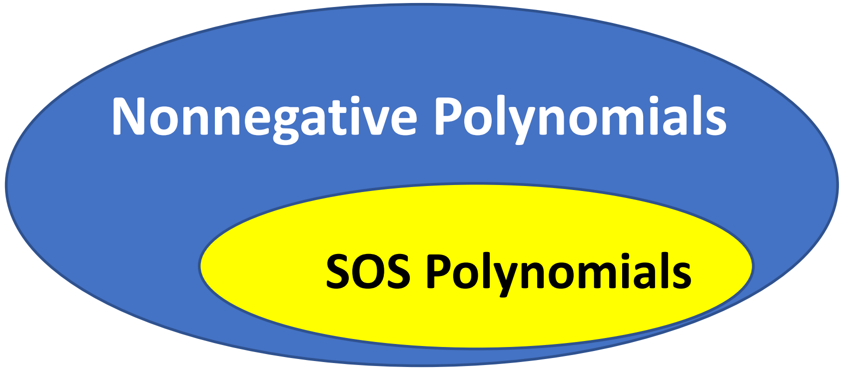

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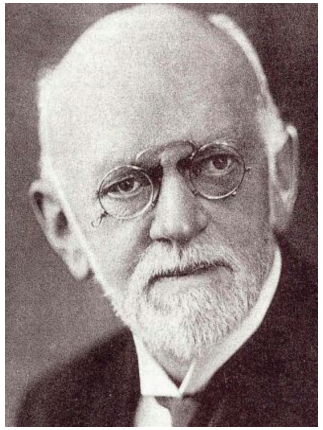

David Hilbert

Hilbert’s 17th problem asked whether this is true in general:

Hilbert’s 17th problem (1900):
Given a nonnegative polynomial, can it be represented as a sum of squares of rational functions?

Hilbert, David "Mathematical Problems". Bulletin of the American Mathematical Society. 8 (10): 437–479.

Sum of Squares (SOS) Polynomials

1) Nonnegativity Condition of $p(x) \in \mathbb{R}[x]$

Nonnegative polynomial

$$p(x) \geq 0, \ \forall x \in \mathbb{R}^n$$

$\xrightarrow{\text{SOS}}$

SOS Condition

$$p(x) \in SOS$$

$\Rightarrow$

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \ldots, \ell$$

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2) Nonnegativity Condition Of $p(x) \in \mathbb{R}[x]$ On The Set

Nonnegative polynomial

$$p(x) \geq 0, \quad \forall x \in \underbrace{\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}}_{\text{set}}$$

$$\xrightarrow{\text{SOS}}$$

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Putinar's Certificate (Positivstellensatze):[1]

Let the semialgebraic set $\mathbf{K}$ be a compact set.[2] If Polynomial $p(x)$ is nonnegative on the set $\mathbf{K}$ then,

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2: *Archimedean set.*

Sum of Squares (SOS) Polynomials

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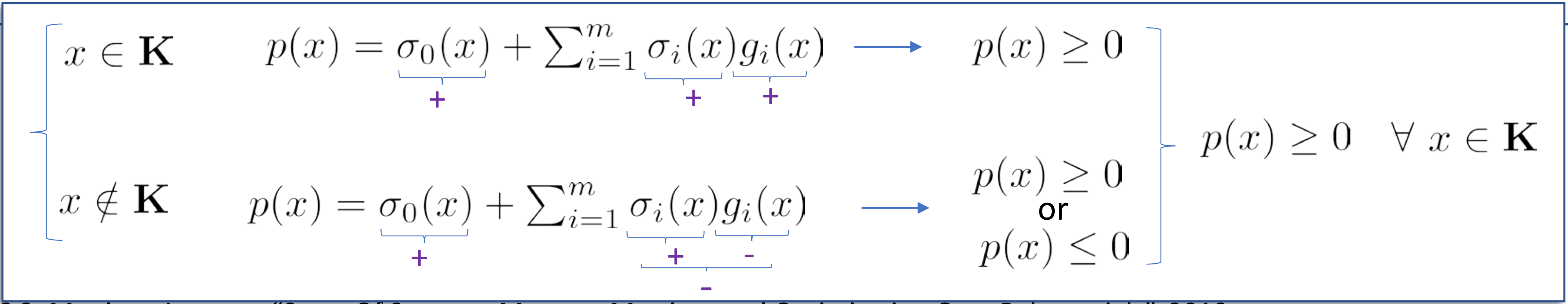

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- Archimedean property is not a geometric property of the set $\mathbf{K}$ but rather an algebraic property related to the representation of the set by its defining polynomials.[4]

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Archimedean : Set $\mathbf{K}$ is ***Archimedean*** **if** there exist a $u(x) \in \mathbf{K}$ of the form $u(x) = \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i(x), \sigma_i \in SOS$ such that set $\{x: u(x) \geq 0\}$ is compact.[1,2]

- Archimedean condition is bot very restrictive. Archimedean condition is satisfied in the following cases:
  - All the polynomials of the set $\mathbf{K}$ are affine and the set is a polytope .[1,3]
  - The set $\{x: g_i(x) \geq 0\}$ is compact for some $g_i(x) \in \mathbf{K}$.[1]
- If the set $\mathbf{K}$ is not Archimedean, we can add the (redundant) polynomial $g_{m+1}\,(x) = M - ||x||$^2 where $M \geq 0$ such that the set $\{x: g_{m+1}\,(x) \geq 0\}$ contains the set $\mathbf{K}$. Adding such polynomial to the set, does not change the geometry of the set.[1]
- Archimedean property is not a geometric property of the set $\mathbf{K}$ but rather an algebraic property related to the representation of the set by its defining polynomials.[4]
- If the set is *Archimedean* then necessarily is compact but the reverse is not true.

[1] *: Section 2.5*: Jean Bernard Lasserre, "Moments, Positive Polynomials and Their Applications" Imperial College Press Optimization Series, V. 1, 2009.
[2] M. Putinar, "Positive polynomials on compact semi-algebraic sets", Indiana University Mathematics Journal, 42, pp. 969-984, 1993.
[3] Theorem 7.1.3, M. Marshall. "Positive Polynomials and Sums of Squares". American Mathematical Society, Providence, Rhode Island, 2008.
[4] A. Jasour, N. S. Aybat, C. Lagoa "Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets", SIAM Journal on Optimization, 25(3), 1411–1440, 2015.

Sum of Squares (SOS) Polynomials

- In **Putinar's Certificate**, set $\mathbf{K}$ should be ***Archimedean*** (slightly stronger than compactness).

$$\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ \ i = 1, ..., m\} \Rightarrow \textit{Archimedean}$$

- In the presence Archimedean assumption, the number of the terms in the SOS representation, i.e., $p(x) = \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i(x)$, is **linear** in the number of polynomials that defines $\mathbf{K}$

- In the absence of Archimedean assumption, the number of terms in SOS representation is **exponential** in the number of polynomials that defines $\mathbf{K}$

$$p(x) = \sigma_0 + \sum_i \sigma_i g_i(x) + \sum_{i,j} \sigma_{ij} g_i(x) g_j(x) + \sum_{i,j,k} \sigma_{ij} g_i(x) g_j(x) g_k(x) + \ldots$$

- *Section 2.5*: Jean Bernard Lasserre, "Moments, Positive Polynomials and Their Applications" Imperial College Press Optimization Series, V. 1, 2009.
- M. Putinar, "Positive polynomials on compact semi-algebraic sets", Indiana University Mathematics Journal, 42, pp. 969-984, 1993.

2) Nonnegativity Condition of $p(x) \in \mathbb{R}[x]$ on the set

Nonnegative polynomial

$$p(x) \geq 0, \quad \forall x \in \underbrace{\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}}_{\text{set}}$$

SOS

Putinar's Certificate:

$$p(x) = \sigma_0(x) + \sum_{i=1}^{m} \sigma_i(x) g_i(x) \qquad \sigma_i(x) \in SOS, \ i = 0, ..., m$$

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SOS

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$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) = \sigma_0(x)$$

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS \qquad \sigma_i(x) \in SOS, \ i = 1, ..., m$$

Sum of Squares (SOS) Polynomials

1) Nonnegativity Condition of $p(x) \in \mathbb{R}[x]$

Nonnegative polynomial

$$p(x) \geq 0, \;\; \forall x \in \mathbb{R}^n$$

SOS

SOS Condition

$$p(x) \in SOS$$

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \ldots, \ell$$

2) Nonnegativity Condition of $p(x) \in \mathbb{R}[x]$ on the set

Nonnegative polynomial

$$p(x) \geq 0, \quad \forall x \in \mathbf{K} = \underbrace{\{x \in \mathbb{R}^n : g_i(x) \geq 0, \;\; i = 1, \ldots, m\}}_{\text{set}}$$

SOS

SOS Condition

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS, \;\; i = 1, \ldots, m$$

Sum of Squares (SOS) Polynomials

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SOS

SOS Condition

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

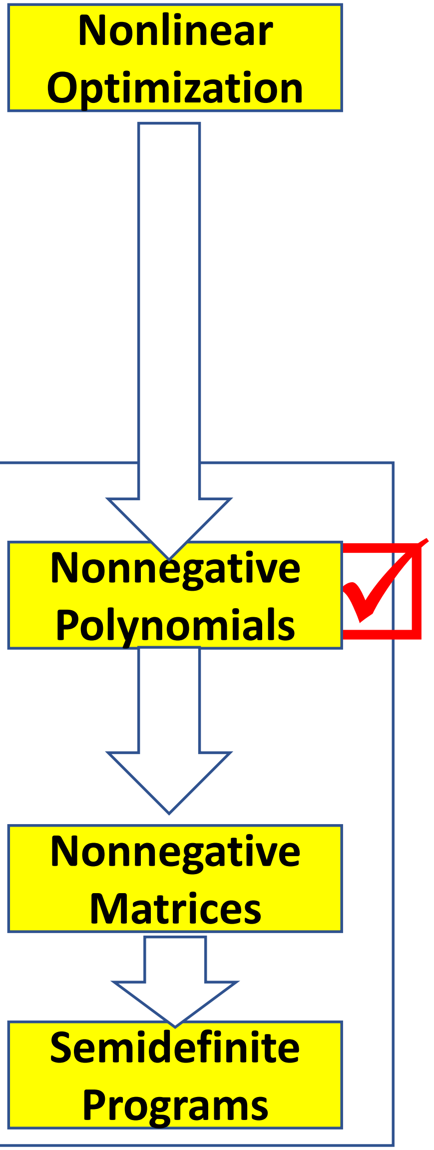
$$\sigma_i(x) \in SOS_{2d_i}, \;\; i = 1, ..., m$$

$$\boxed{deg(\sigma_i(x)) = 2d_i}$$

Nonlinear (nonconvex) Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \dots, m \end{aligned}$$

Tools: i) Nonnegative Polynomials ii) Semidefinite Programs



Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

Step 2:

2.1 Replace Nonnegative Polynomials with Sum of Squares (SOS) Polynomials ☑

2.2 Represent **SOS Polynomials** with **Positive Semidefinite** Matrices (PSD)

⇩

Reformulate Nonlinear Optimization as **Semidefinite Programs**

Nonlinear (nonconvex) Optimization

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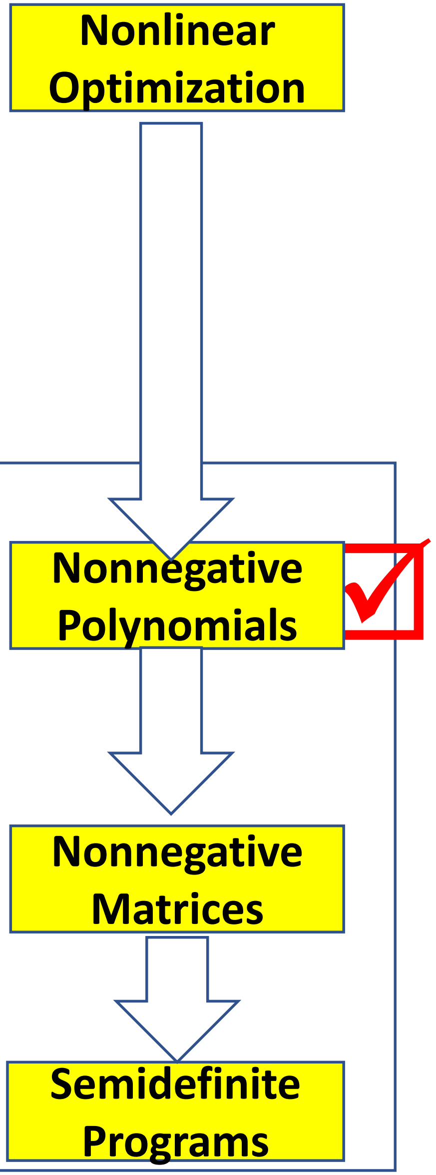

Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

Step 2:

2.1 Replace Nonnegative Polynomials with Sum of Squares (SOS) Polynomials ☑

SOS Programming using YALMIP

2.2 Represent **SOS Polynomials** with **Positive Semidefinite** Matrices (PSD)

⇓

Reformulate Nonlinear Optimization as **Semidefinite Programs**

SOS Programming

Problems with SOS Conditions

- Verification Problems
- Design Problems
- Optimization

YALMIP: J. Lofberg,"YALMIP : A Toolbox for Modeling and Optimization in MATLAB", In Proceedings of the CACSD Conference, 2004 https://yalmip.github.io/

SOSTOOLS: MATLAB toolbox for formulating and solving sums of squares (SOS) optimization programs https://www.cds.caltech.edu/sostools/

Input: SOS Program

- Generates Semidefinite Program (SDP) from SOS Program
- Solves the SDP using SDP solvers

SDP solvers: e.g.,

MOSEK https://www.mosek.com
SEDUMI http://sedumi.ie.lehigh.edu
SDPT3 http://www.math.nus.edu.sg/~mattohkc/sdpt3.html

Rely on **interior point** methods

SOS Programming

1) Nonnegativity Verification:

Given, $p(x) \in \mathbb{R}[x]$

Check if $p(x) \geq 0$

SOS

SOS Condition

$$p(x) \in SOS$$

Given, $p(x) \in \mathbb{R}[x]$ and the sset $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., m\}$

Check if $p(x) \geq 0 \quad \forall x \in \mathbf{K}$

SOS

SOS condition

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i},\ i = 1, ..., m$$

SOS Programming

1) Nonnegativity Verification:

Example: Check the nonnegativity of polynomial $p(x)$

$$p(x) = x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 \xrightarrow{\textbf{SOS}} p(x) \in SOS$$

SOS Programming

1) Nonnegativity Verification:

Example: Check the nonnegativity of polynomial $p(x)$

$$p(x) = x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 \xrightarrow{\textbf{SOS}} p(x) \in SOS$$

YALMIP

`x = sdpvar(1);` $\longrightarrow$ variables x

`p = x(1)^4+4*x(1)^3+6*x(1)^2+4x(1)+5;` $\longrightarrow$ $p(x)$

`F = sos(p);` $\longrightarrow$ $p(x) \in SOS$

`ops = sdpsettings('solver','mosek');` $\longrightarrow$ SDP solver

`[sol,v,Q]=solvesos(F);` $\longrightarrow$ solve SOS programming

`h=sosd(F); sdisplay(h’*h);` $\longrightarrow$ $h(x)$vector in $p(x) = \sum_{i=1}^{\ell} h_i^2(x)$

SOS Programming

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`F = sos(p);`	$p(x) \in SOS$
`ops = sdpsettings('solver','mosek');`	SDP solver
`[sol,v,Q]=solvesos(F);`	solve SOS programming
`h=sosd(F); sdisplay(h’*h);`	$h(x)$vector in $p(x) = \sum_{i=1}^{\ell} h_i^2(x)$

SOS Decomposition

$$p(x) = (-1.54 - 2.25x_1 - 0.65x^2)^2 + (1.61 - 0.92x - 0.63x^2)^2 + (0.066 - 0.163x + 0.405x^2)^2$$

p(x) is nonnegative

SOS Programming

1) Nonnegativity Verification:

Example: Check the nonnegativity of polynomial $p(x)$

$$p(x) = x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 \xrightarrow{\text{SOS}} p(x) \in SOS$$

YALMIP

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`p = x(1)^4+4*x(1)^3+6*x(1)^2+4x(1)+5;`	$p(x)$
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p(x) is nonnegative

- If $p(x)$ does not have SOS representation: Yalmip output: Problem status: The problem is primal infeasible

SOS Programming

1) Nonnegativity Verification:

Example: Check the nonnegativity of polynomial $p(x)$ on the set $\mathbf{K}$

$$p(x) = x_1^3 - 4x_1^2 + 2x_1x_2 - x_2^2 + x_2^3 \qquad \mathbf{K} = \{x : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 - 1 \geq 0\}$$

SOS Programming

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Example: Check the nonnegativity of polynomial $p(x)$ on the set $\mathbf{K}$

$$p(x) = x_1^3 - 4x_1^2 + 2x_1x_2 - x_2^2 + x_2^3 \qquad \mathbf{K} = \{x : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 - 1 \geq 0\}$$

SOS

SOS Condition

$$p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3(x_1 + x_2 - 1) \in SOS$$

where, $\sigma_i(x) \in SOS_2,\ i = 1, 2, 3$

SOS Condition

$$p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3 (x_1 + x_2 - 1) \in SOS$$

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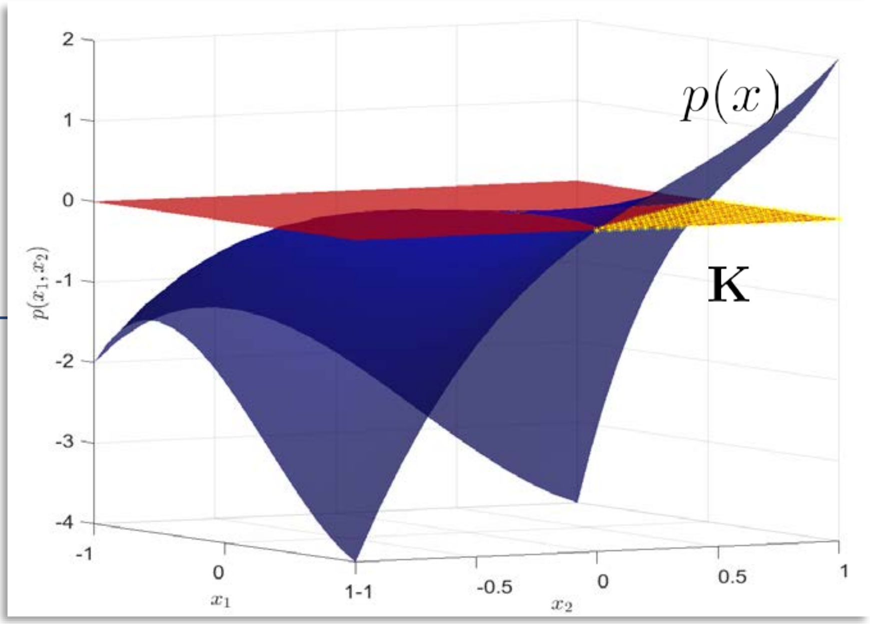


`sdpvar x1 x2` → variables x_1, x_2

`p = x1^3-x1^2+2*x1*x2-x2^2+x2^3;` → $p(x)$

`g = [x1;x2;x1+x2-1]` → $\mathbf{K}$

`d=2;` → order of σ_i

`[s1,c1] = polynomial([x1 x2],d);` → σ_1 with coefficients c_1

`[s2,c2] = polynomial([x1 x2],d);` → σ_2 with coefficients c_2

`[s3,c3] = polynomial([x1 x2],d);` → σ_3 with coefficients c_3

`ops = sdpsettings('solver','mosek');` → SDP solver

`F = [sos(p-[s1 s2 s3]*g), sos(s1), sos(s2), sos(s3)];` → $p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3 (x_1 + x_2 - 1) \in SOS$ $\sigma_i(x) \in SOS_2, \ i = 1,2,3$

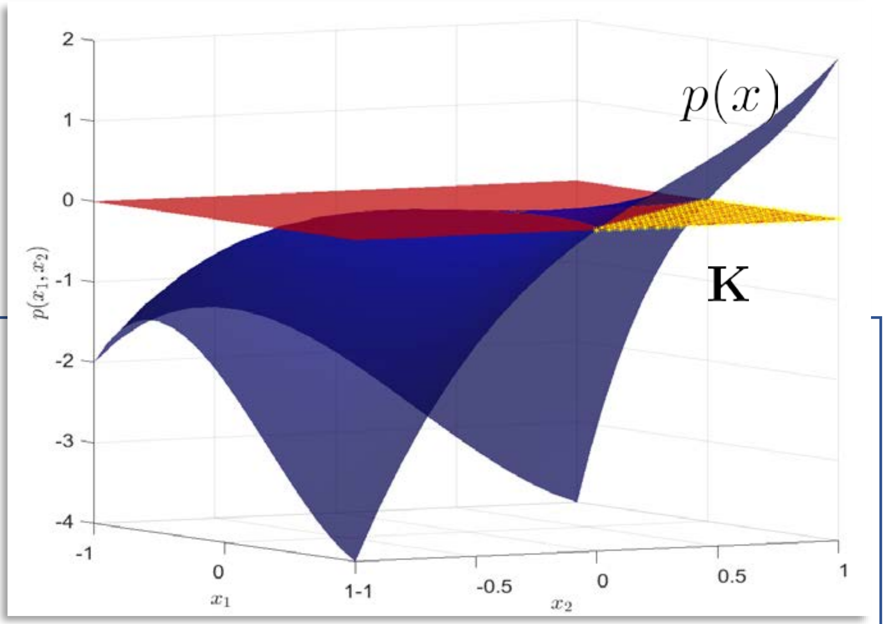
`[sol,v,Q]=solvesos(F,[],ops,[c0;c1;c2;c3]);` → solve SOS programming

https://github.com/jasour/rarnop19/blob/master/Lecture3_SOS_NonlinearOptimization/SOS_Decomposition/Example_3.m

SOS Condition

$$p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3(x_1 + x_2 - 1) \in SOS$$

where, $\sigma_i(x) \in SOS_2, \; i = 1, 2, 3$



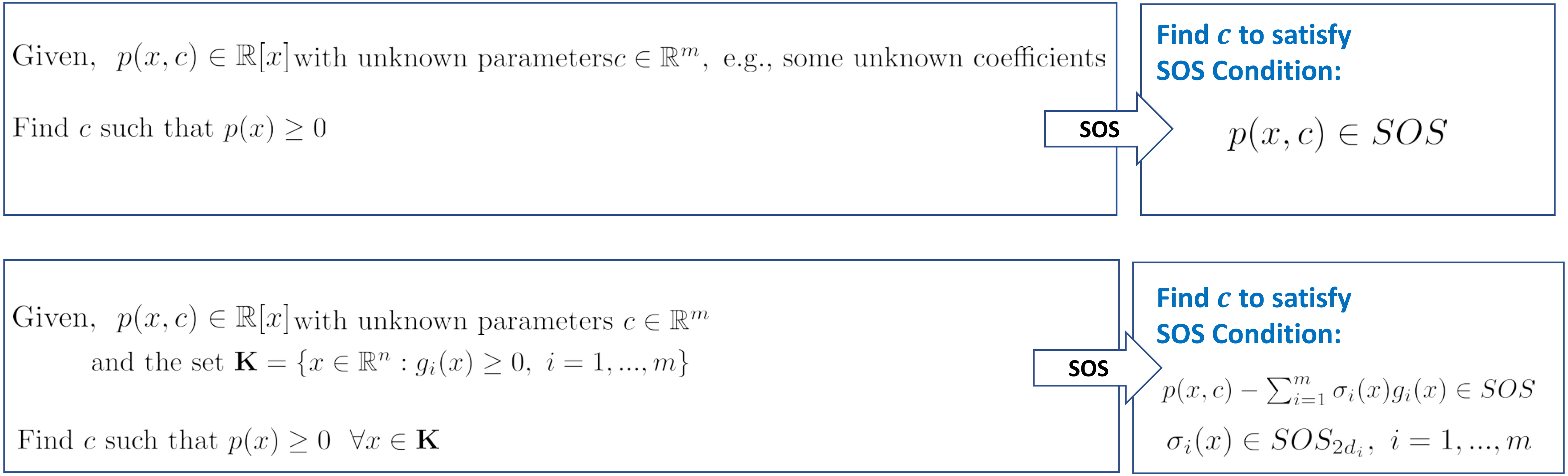
`sdpvar x1 x2`	variables x_1, x_2
`p = x1^3-x1^2+2*x1*x2-x2^2+x2^3;`	$p(x)$
`g = [x1;x2;x1+x2-1]`	**K**
`d=2;`	order of σ_i
`[s1,c1] = polynomial([x1 x2],d);`	$\sigma_1$with coefficients c_1
`[s2,c2] = polynomial([x1 x2],d);`	$\sigma_2$with coefficients c_2
`[s3,c3] = polynomial([x1 x2],d);`	$\sigma_3$with coefficients c_3
`ops = sdpsettings('solver','mosek');`	SDP solver
`F = [sos(p-[s1 s2 s3]*g), sos(s1), sos(s2), sos(s3)];`	$p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3(x_1 + x_2 - 1) \in SOS$ $\sigma_i(x) \in SOS_2, \; i = 1, 2, 3$
`[sol,v,Q]=solvesos(F,[],ops,[c0;c1;c2;c3]);`	solve SOS programming
`sdisplay(sosd(F(1))'*sosd(F(1)))`	$p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3(x_1 + x_2 - 1) = \sum_{i=1}^{\ell} h_i^2(x)$
`sdisplay(sosd(F(2))'*sosd(F(2)))`	σ_1

SOS Decomposition ☑ ⇨ **p(x) is nonnegative on the set K**

https://github.com/jasour/rarnop19/blob/master/Lecture3_SOS_NonlinearOptimization/SOS_Decomposition/Example_3.m

SOS Programming

2) Design Problem:



Given, $p(x,c) \in \mathbb{R}[x]$ with unknown parameters$c \in \mathbb{R}^m$, e.g., some unknown coefficients

Find c such that $p(x) \geq 0$

SOS

Find c to satisfy SOS Condition:

$$p(x,c) \in SOS$$

Given, $p(x,c) \in \mathbb{R}[x]$ with unknown parameters $c \in \mathbb{R}^m$

and the set $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., m\}$

Find c such that $p(x) \geq 0 \ \ \forall x \in \mathbf{K}$

SOS

Find c to satisfy SOS Condition:

$$p(x,c) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i}, \ \ i = 1, ..., m$$

SOS Programming

2) Design Problem:

Example: **Lyapunov Function Search Using SOS Programming**

Given a dynamical system $\dot{x} = f(x), x(0) = x_0$

We want to show that solutions $x(t)$ converge to zero for all initial conditions (stability).

- To prove this, we need to find an energy function $V(x)$ with following properties

$$\underbrace{V(x) = 0 \quad on \quad x = 0 \qquad V(x) > 0 \quad on \quad x \neq 0 \qquad -\dot{V}(x) > 0}_{\text{Lyapunov function}}$$

- A. Papachristodoulou and S. Prajna. On the construction of Lyapunov functions using the sum of squares decomposition. In Proceedings of the 41st IEEE Conference on Decision and Control, pages 3482–3487, December 2002.
- Stability of Polynomial Differential Equations: Complexity and Converse Lyapunov Questions A. A. Ahmadi and P. A. Parrilo IEEE Transactions on Automatic Control, Submitted, 2013, http://web.mit.edu/~a_a_a/Public/Publications/poly_stability.pdf
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- We look for polynomial Lyapunov function $V(x) = c^T B(x)$

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- We look for polynomial Lyapunov function $V(x) = c^T B(x)$
- Instead of checking nonnegativity, we check SOS conditions.

$$V(0) = 0 \longrightarrow c(1) = 0 \qquad V(x) \in SOS_{2d} \qquad -\dot{V}(x) \in SOS$$

- A. Papachristodoulou and S. Prajna. On the construction of Lyapunov functions using the sum of squares decomposition. In Proceedings of the 41st IEEE Conference on Decision and Control, pages 3482–3487, December 2002.
- Stability of Polynomial Differential Equations: Complexity and Converse Lyapunov Questions A. A. Ahmadi and P. A. Parrilo IEEE Transactions on Automatic Control, Submitted, 2013, http://web.mit.edu/~a_a_a/Public/Publications/poly_stability.pdf
- A. A. Ahmadi, P. A. Parrilo, “SOS Lyapunov Function”, 2011, http://web.mit.edu/~a_a_a/Public/Presentations/AAA_CDC11_paper1.pdf

SOS Programming

2) Design Problem:

Lyapunov Function Search

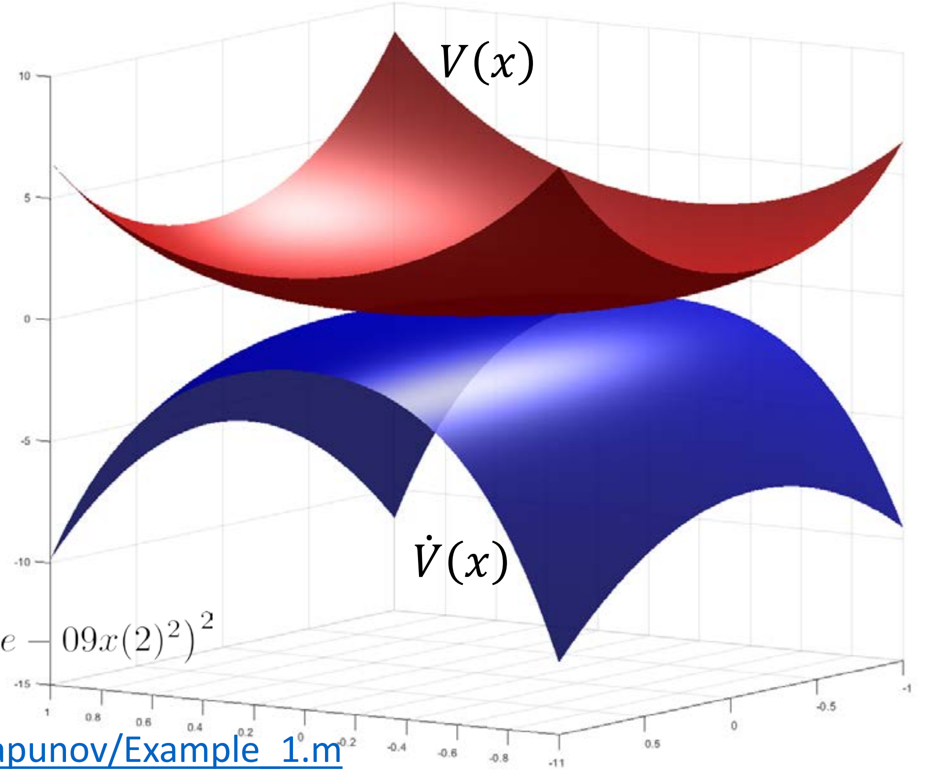
$$\dot{x}_1 = -x_1 + (1 + x_1)x_2$$
$$\dot{x}_2 = -(1 + x_1)x_1$$

SOS Conditions:

$V(x) = c^T B_4(x)$ $\quad V(0) = 0 \longrightarrow c(1) = 0$

$V(x) \in SOS_{2d}$ $\quad -\dot{V}(x) \in SOS$

$$\begin{aligned} V(x) = {} & (-2.46e-06+0.93x(1)-1.19x(2)+0.14x(1)x(2)+0.06x(1)^2+0.09x(2)^2)^2 \\ & +(-4.32e-06+0.03x(1)-0.13x(2)-1.32x(1)x(2)+0.0071x(1)^2+0.01x(2)^2)^2 \\ & +(6.41e-06-0.83x(1)-0.66x(2)+0.041x(1)x(2)-0.26x(1)^2-0.0045x(2)^2)^2 \\ & +(4.99e-05+0.19x(1)+0.046x(2)-0.012x(1)x(2)-0.698x(1)^2-0.756x(2)^2)^2 \\ & +(-1.432e-05+0.12x(1)+0.11x(2)-0.0032x(1)x(2)-0.65x(1)^2+0.645x(2)^2)^2 \\ & +(-0.0001+1.34e-10x(1)-1.74e-10x(2)+3.03e-10x(1)x(2)-2.4456e-09x(1)^2-4.89e-09x(2)^2)^2 \end{aligned}$$



https://github.com/jasour/rarnop19/blob/master/Lecture3_SOS_NonlinearOptimization/SOS_Lyapunov/Example_1.m

SOS Programming

2) Design Problem:

Unconstrained Optimization

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$

$$\begin{aligned} &\underset{\gamma\in\mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x)-\gamma \geq 0, \quad \forall x \in \mathbb{R}^n \end{aligned}$$

SOS

$$\begin{aligned} &\underset{\gamma}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x)-\gamma \in SOS \end{aligned}$$

Constrained Optimization

$$\begin{aligned} &\underset{x\in\mathbb{R}^n}{\text{minimize}} && f(x) \\ &\text{subject to} && g_i(x) \geq 0, \ \ i = 1, ..., n \end{aligned}$$

$$\begin{aligned} &\underset{\gamma}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x)-\gamma \geq 0, \quad \forall x \in \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ \ i = 1, ..., n\} \end{aligned}$$

SOS

$$\begin{aligned} &\underset{\gamma,\sigma_i}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x)-\gamma-\sum_{i=1}^{m}\sigma_i(x)g_i(x) \in SOS \\ & && \sigma_i(x) \in SOS_{2d_i}, \ \ i = 1, ..., m \end{aligned}$$

SOS Programming

$$\underset{x \in \mathbb{R}}{\text{minimize}} \quad x^4 + 2x^3 - 12x^2 - 2x + 6$$

$$\begin{aligned} \underset{\gamma \in \mathbb{R}}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & x^4 + 2x^3 - 12x^2 - 2x + 6 - \gamma \geq 0, \quad \forall x \in \mathbb{R} \end{aligned}$$

SOS

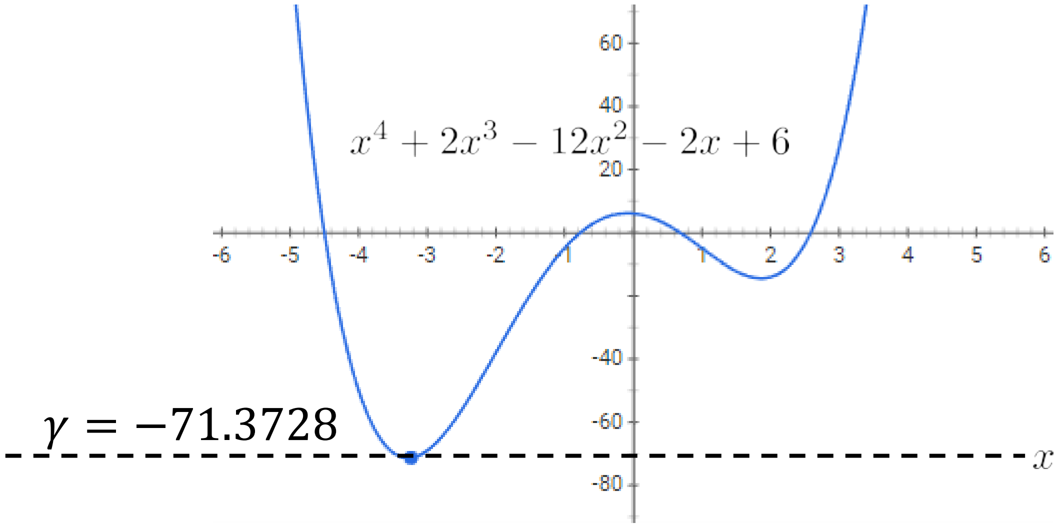
$$\begin{aligned} \underset{\gamma \in \mathbb{R}}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & x^4 + 2x^3 - 12x^2 - 2x + 6 - \gamma \in SOS \end{aligned}$$

SOS

$$\begin{aligned} &\underset{\gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && x^4 + 2x^3 - 12x^2 - 2x + 6 - \gamma \in SOS \end{aligned}$$

SOS Programming in Yalmip

`sdpvar x  gamma`	variables x, γ
`p = x^4+2*x^3-12*x^2-2*x+6;`	$p(x)$
`F = sos(p-gamma);`	$p(x) - \gamma \in SOS$
`ops = sdpsettings('solver','mosek');`	SDP solver
`[sol,v,Q]=solvesos(F,-gamma,ops);`	solve SOS programming
`value(gamma)`	obtained γ
`sdisplay(sosd(F))`	$h(x)$vector in $p(x) - \gamma = h(x)^T h(x) = \sum_{i=1}^{\ell} h_i^2(x)$



https://github.com/jasour/rarnop19/blob/master/Lecture3_SOS_NonlinearOptimization/SOS_Optimization/Example_2_UnconOpt.m

SOS Programming

$$\mathrm{P}^* = \underset{x \in \mathbb{R}^2}{\text{minimize}} \quad -x_1$$

$$\text{subject to} \quad x \in \mathbf{K} = \{x \in \mathbb{R}^2 : 3 - 2x_2 - x_1^2 - x_2^2 \geq 0,\ -x_1 - x_2 - x_1 x_2 \geq 0,\ 1 + x_1 x_2 \geq 0\}$$

$$p(x) - \gamma = \sigma_0(x) + \textstyle\sum_{i=1}^{m} \sigma_i(x) g_i(x) \qquad \sigma_i(x) \in SOS_{2d_i},\ i = 1, 2, 3$$

SOS Programming

$$\mathrm{P}^* = \underset{x\in\mathbb{R}^2}{\text{minimize}} \quad -x_1$$

$$\text{subject to} \quad x \in \mathbf{K} = \{x \in \mathbb{R}^2 : 3 - 2x_2 - x_1^2 - x_2^2 \geq 0,\ -x_1 - x_2 - x_1x_2 \geq 0,\ 1 + x_1x_2 \geq 0\}$$

$$p(x) - \gamma = \sigma_0(x) + \textstyle\sum_{i=1}^{m} \sigma_i(x) g_i(x) \qquad \sigma_i(x) \in SOS_{2d_i},\ i = 1, 2, 3$$

$$\gamma^* = \mathrm{P}^* = -1.6180$$

$$\sigma_0 = 0.126 - 0.114x_1 + 0.1085x_2 + 0.0307x_1^2 + 0.05633x_2^2 - 0.02405x_1x_2$$

$$\sigma_1 = 0.227 - 0.219x_1 + 0.163x_2 + 0.0604x_1^2 + 0.082x_2^2 - 0.0382x_1x_2$$

$$\sigma_2 = 0.413 + 0.10407x_1 + 0.3416x_2 + 0.148x_1^2 + 0.0834x_2^2 + 0.0665x_1x_2$$

$$\sigma_3 = 0.2985 + 0.262x_1 + 0.16294x_2 + 0.18915x_1^2 + 0.0700x_2^2 - 0.0258x_1x2$$

https://github.com/jasour/rarnop19/blob/master/Lecture3_SOS_NonlinearOptimization/SOS_Optimization/Example_3_ConOpt.m

Unconstrained Optimization

$$\mathrm{P} = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$

SOS

SOS Programming

$$\begin{aligned} \mathrm{P}_{sos} = \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in SOS \end{aligned}$$

Constrained Optimization

$$\begin{aligned} \mathrm{P} = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \ldots, n \end{aligned}$$

SOS

SOS Programming

$$\begin{aligned} \mathrm{P}_{sos} = \underset{\gamma, \sigma_i}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma - \sum_{i=1}^{m} \sigma_i g_i(x) \in SOS \\ & \sigma_i \in SOS_{2d_i}, \;\; i = 1, \ldots, m \end{aligned}$$

Nonlinear (nonconvex) Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \ldots, m \end{aligned}$$

Tools: i) Nonnegative Polynomials ii) Semidefinite Programs

Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

Step 2:

2.1 Replace Nonnegative Polynomials with Sum of Squares (SOS) Polynomials ✓

SOS Programming using YALMIP ✓

2.2 Represent **SOS Polynomials** with **Positive Semidefinite** Matrices (PSD)

Reformulate Nonlinear Optimization as **Semidefinite Programs**

Nonlinear Optimization

Nonnegative Polynomials ✓

Nonnegative Matrices

Semidefinite Programs

Semidefinite Program

Semidefinite Program

Semidefinite Program:

$\underset{X\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad C \bullet X \longrightarrow$ linear function

$\text{subject to} \quad A \bullet X = b \longrightarrow$ linear constraints

$X \succcurlyeq 0 \longrightarrow$ linear matrix inequality (LMI)
Positive Semidefinite Matrix (PSD)

Semidefinite Program

Semidefinite Program:

$$\begin{aligned} \underset{X\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad & C\bullet X && \longrightarrow \text{linear function} \\ \text{subject to} \quad & A\bullet X = b && \longrightarrow \text{linear constraints} \\ & X \succcurlyeq 0 && \longrightarrow \begin{array}{l}\text{linear matrix inequality (LMI)} \\ \text{Positive Semidefinite Matrix (PSD)}\end{array} \end{aligned}$$

Example

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$$

$$\begin{aligned} \min\nolimits_{\mathrm{x}}\ & 3x_{11} + 5x_{12} + x_{22} \\ \text{s.t.} \quad & x_{11} + 3x_{12} + 5x_{22} = 2 \\ & x_{11} + 9x_{12} + 4x_{22} = 1 \\ & X \succcurlyeq 0 \end{aligned}$$

Semidefinite Program

Convex Optimization

Semidefinite Program:

$$\begin{array}{lll} \underset{X\in\mathbb{R}^{n\times n}}{\text{minimize}} & C \bullet X & \longrightarrow \text{linear function} \\ \text{subject to} & A \bullet X = b & \longrightarrow \text{linear constraints} \\ & X \succcurlyeq 0 & \longrightarrow \text{linear matrix inequality (LMI)} \\ & & \quad\ \ \text{Positive Semidefinite Matrix (PSD)} \end{array}$$

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Linear Program:

$$\begin{array}{lll} \underset{x\in\mathbb{R}^{n}}{\text{minimize}} & c^T x & \longrightarrow \text{linear function} \\ \text{subject to} & \left.\begin{array}{l} Ax = b \\ x \geq 0 \end{array}\right\} & \text{linear constraints} \end{array}$$

Example

Find $[x_1, x_2, x_3]$ to

$$\begin{aligned} \min_{\mathrm{x}}\ & 3x_1 + 5x_2 + x_3 \\ \text{s.t.}\quad & x_1 + 3x_2 + 5x_3 = 2 \\ & x_1 + 9x_2 + 4x_3 = 1 \\ & x_1 \geq 0,\ \ x_2 \geq 0 \end{aligned}$$

Semidefinite Program

Convex Optimization

Semidefinite Program:

$$\underset{X\in\mathbb{R}^{n\times n}}{\text{minimize}} \quad C \bullet X \longrightarrow$$ linear function

$$\text{subject to} \quad A \bullet X = b \longrightarrow$$ linear constraints

$$X \succcurlyeq 0 \longrightarrow$$ linear matrix inequality (LMI)
Positive Semidefinite Matrix (PSD)

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Element of SDP: Symmetric Square Matrix, Positive Semidefinite Matrix, Linear Function of Matrix

Positive Semidefinite Matrix

- Symmetric Matrix $X \in \mathbb{R}^{n\times n}$ is Positive Semidefinite (**PSD**) dented by $X \succeq 0$ if

for any $x \in \mathbb{R}^n \neq 0 \quad \Rightarrow \quad \underbrace{x^T X x}_{\in \mathbb{R}} \geq 0$

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for any $x \in \mathbb{R}^n \neq 0 \quad \Rightarrow \quad \underbrace{x^T X x}_{\in \mathbb{R}} \geq 0$

Example:

$$X = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x^T X x \geq 0 \longrightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2 \geq 0, \;\; \forall x \neq 0}$$

- Infinite linear constraints in terms of entries of matrix
- Instead we can look at eigenvalues

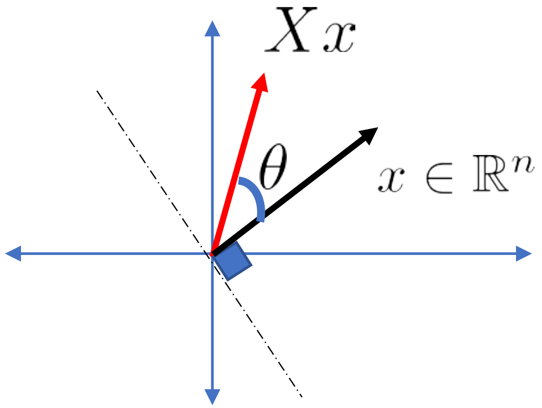
Positive Semidefinite Matrix

- Symmetric Matrix $X \in \mathbb{R}^{n \times n}$ is Positive Semidefinite (**PSD**) dented by $X \succcurlyeq 0$ if

$$\text{for any} \quad x \in \mathbb{R}^n \neq 0 \quad \Rightarrow \quad \underbrace{x^T X x}_{\in \mathbb{R}} \geq 0$$

- Geometrical Interpretation:

$$X \succcurlyeq 0 \quad \Rightarrow \quad |\theta| \leq 90^o$$



Angle between vectors x are Xx is less or equal 90^o

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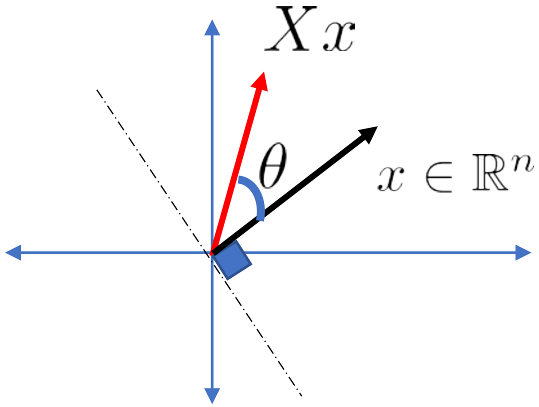

$$x^T X v = \underbrace{< x, Xx >}_{\text{Inner product (dot product) of 2 vector}} \geq 0 \quad \Rightarrow$$

Angle between vectors x are Xx is less or equal 90^o

Positive Semidefinite Matrix

- Symmetric Matrix $X \in \mathbb{R}^{n \times n}$ is Positive Semidefinite (**PSD**) dented by $X \succcurlyeq 0$ if

for any $x \in \mathbb{R}^n \neq 0$ ⇨ $\underbrace{x^T X x}_{\in \mathbb{R}} \geq 0$

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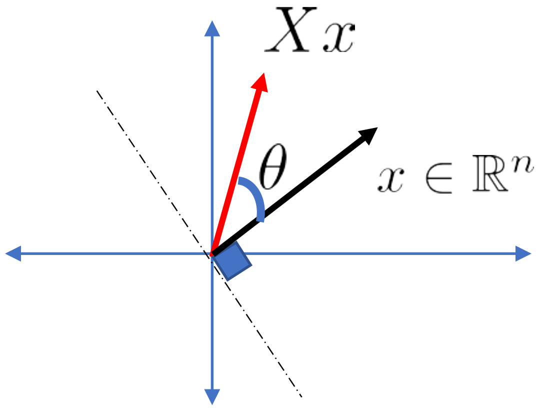


$$x^T X v = \underbrace{< x, Xx >}_{\text{Inner product (dot product) of 2 vector}} \geq 0$$

⇨ Angle between vectors x are Xx is less or equal 90^o

$X \in S^n_+$ Positive Semidefinite (**PSD**)

Eigenvalues of Matrix

- Eigenvalue and Eigenvector of Matrix $X \in \mathbb{R}^{n \times n}$

Eigenvalue $\lambda \in \mathbb{R}$

Eigenvector $v \in \mathbb{R}^n$

Eigenvalue: $\det(X - \lambda I) = 0$

$$Xv = \lambda v$$

eigenvalue **eigenvector**

Eigenvalues of Matrix

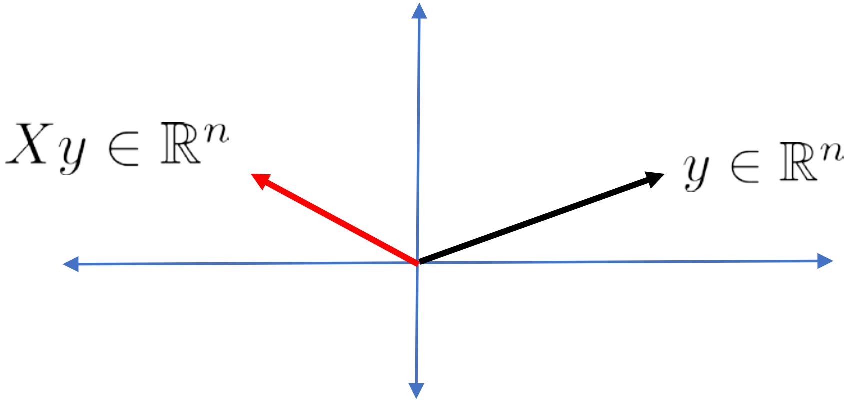
- Eigenvalue and Eigenvector of Matrix $X \in \mathbb{R}^{n \times n}$

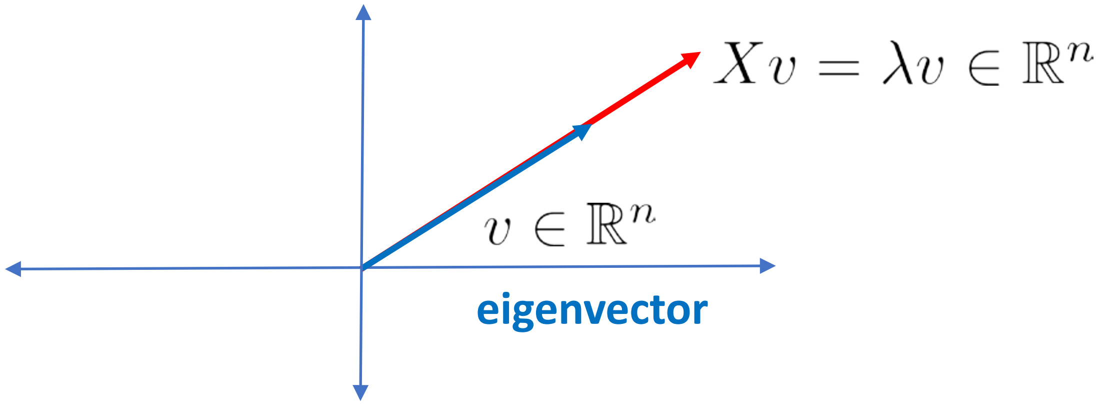
Eigenvalue $\lambda \in \mathbb{R}$

Eigenvector $v \in \mathbb{R}^n$

Eigenvalue: $\det(X - \lambda I) = 0$

$$Xv = \underbrace{\lambda}_{\text{eigenvalue}} \underbrace{v}_{\text{eigenvector}}$$





X Linear Map

y Input vector

Xy Output vector

X Linear Map

v Input vector

Xv Output vector

Eigenvalues of Matrix

- ➢ Eigenvalue and Eigenvector of Matrix $X \in \mathbb{R}^{n\times n}$

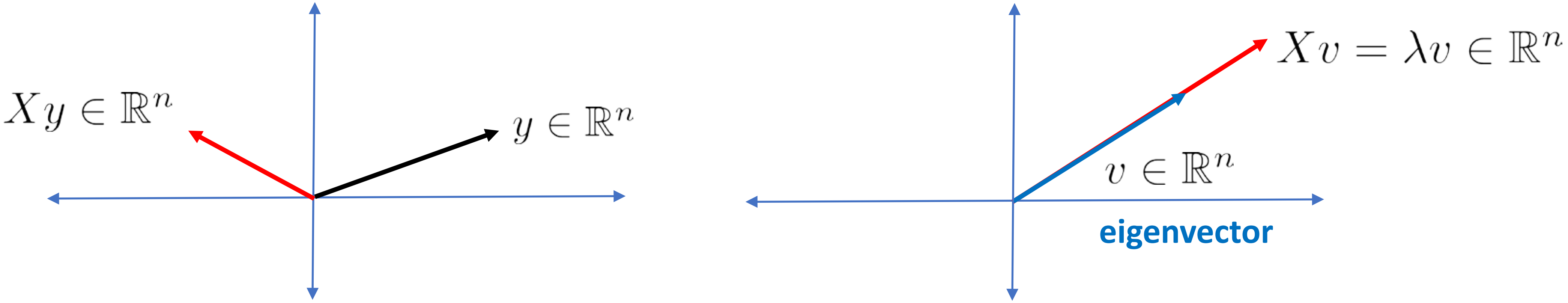
Eigenvalue $\lambda \in \mathbb{R}$

Eigenvector $v \in \mathbb{R}^n$

Eigenvalue: $\det(X - \lambda I) = 0$

$$Xv = \lambda v$$

eigenvalue **eigenvector**



- ➢ If $X \in \mathbb{R}^{n\times n}$ is symmetric: all eigenvalues are real numbers.
- ➢ PSD matrix: Eigenvalues are all nonnegative real numbers.

Eigenvalues of Matrix

- Eigenvalue Decomposition: $X = VDV^{-1}$

D : diagonal matrix of eigenvalues

V : matrix whose columns are the corresponding eigenvectors

(MATLAB: $[V, D]$=eig(X))

Eigenvalues of Matrix

- Eigenvalue Decomposition: $X = VDV^{-1}$

D : diagonal matrix of eigenvalues

V : matrix whose columns are the corresponding eigenvectors

(MATLAB: $[V, D]$=eig(X))

Example: $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Eigenvalues:

$$|X - \lambda I| = 0 \qquad \left|\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\right| = 0 \qquad \left|\begin{bmatrix} 1-\lambda_1 & 2 \\ 3 & 4-\lambda_2 \end{bmatrix}\right| = (1-\lambda_1)(4-\lambda_2) - 3 \times 6 = 0$$

$$\lambda_1 = -0.37$$

$$\lambda_2 = 5.37$$

Eigenvectors:

$$Xv = \lambda v$$

Eigenvalue Decomposition

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -0.8246 & -0.4160 \\ 0.5657 & -0.9094 \end{bmatrix} \begin{bmatrix} -0.3723 & 0 \\ 0 & 5.3723 \end{bmatrix} \begin{bmatrix} -0.8246 & -0.4160 \\ 0.5657 & -0.9094 \end{bmatrix}^{-1}$$

Eigenvalues of Matrix

- Eigenvalue Decomposition: $X = VDV^{-1}$

D : diagonal matrix of eigenvalues

V : matrix whose columns are the corresponding eigenvectors

(MATLAB: $[V, D]$=eig(X))

- If $X \in \mathbb{R}^{n \times n}$ is symmetric: all eigenvalues are real numbers and matrix V is orthogonal matrix.

Eigenvalue Decomposition: $X = VDV^T$ $(V^{-1} = V^T)$

PSD Matrix Decomposition $X \in \mathbb{R}^{n\times n}$

Gramian matrix

Given $L \in \mathbb{R}^{n\times k}$ → Gram matrix of L: $X = LL^T \in \mathbb{R}^{n\times n}$

PSD Matrix Decomposition $X \in \mathbb{R}^{n \times n}$

Gramian matrix

Given $L \in \mathbb{R}^{n \times k}$ $\longrightarrow$ Gram matrix of L: $X = LL^T \in \mathbb{R}^{n \times n}$

- The Gramian matrix is PSD $x^T X x \geq 0$ $x^T L L^T x = \underbrace{(x^T L)(x^T L)^T}_{\in \mathbb{R}} \geq 0$

PSD Matrix Decomposition $X \in \mathbb{R}^{n\times n}$

Gramian matrix

Given $L \in \mathbb{R}^{n\times k}$ → Gram matrix of L: $X = LL^T \in \mathbb{R}^{n\times n}$

- The Gramian matrix is PSD $x^TXx \geq 0$ $x^TLL^Tx = \underbrace{(x^TL)(x^TL)^T}_{\in \mathbb{R}} \geq 0$
- Every PSD matrix is the Gramian matrix for some set of vectors.

$$X \in \mathcal{S}_+^n \longrightarrow X = VDV^T = V\underbrace{\sqrt{D}\sqrt{D}}V^T = (V\sqrt{D})(V\sqrt{D})^T \longrightarrow X \text{ is a Gram matrix of } V\sqrt{D}$$

Eigenvalue Decomposition

Nonnegative eigenvalues

PSD Matrix Decomposition $X \in \mathbb{R}^{n\times n}$

Given $L \in \mathbb{R}^{n\times k}$ → Gram matrix of L: $X = LL^T \in \mathbb{R}^{n\times n}$

Example

$$X = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 5 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

Eigenvalues: 0,5,7

PSD Matrix Decomposition $X \in \mathbb{R}^{n \times n}$

Given $L \in \mathbb{R}^{n \times k}$ ⟶ Gram matrix of L: $X = LL^T \in \mathbb{R}^{n \times n}$

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Eigenvalue Decomposition:

$$X = VDV^T$$

$$X = \overbrace{\begin{bmatrix} -0.5071 & 0.3162 & -0.8018 \\ 0.1690 & 0.9487 & 0.2673 \\ -0.8452 & 0 & 0.5345 \end{bmatrix}}^{\text{Eigenvectors}} \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}}^{\text{Eigenvalues}} \begin{bmatrix} -0.5071 & 0.3162 & -0.8018 \\ 0.1690 & 0.9487 & 0.2673 \\ -0.8452 & 0 & 0.5345 \end{bmatrix}^T$$

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$$X = V\sqrt{D}\sqrt{D}V^T = (V\sqrt{D})(V\sqrt{D})^T$$

$$X = \begin{bmatrix} 0 & 0.7071 & -2.1213 \\ 0 & 2.1213 & 0.7071 \\ 0 & 0 & 1.4142 \end{bmatrix} \begin{bmatrix} 0 & 0.7071 & -2.1213 \\ 0 & 2.1213 & 0.7071 \\ 0 & 0 & 1.4142 \end{bmatrix}^T$$

PSD Matrix Decomposition $X \in \mathbb{R}^{n\times n}$

Given $L \in \mathbb{R}^{n\times k}$ $\longrightarrow$ Gram matrix of L: $X = LL^T \in \mathbb{R}^{n\times n}$

Example $X = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 5 & 1 \\ -3 & 1 & 2 \end{bmatrix}$

Eigenvalues: 0,5,7

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$$X = \overbrace{\begin{bmatrix} -0.5071 & 0.3162 & -0.8018 \\ 0.1690 & 0.9487 & 0.2673 \\ -0.8452 & 0 & 0.5345 \end{bmatrix}}^{\text{Eigenvectors}} \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}}^{\text{Eigenvalues}} \begin{bmatrix} -0.5071 & 0.3162 & -0.8018 \\ 0.1690 & 0.9487 & 0.2673 \\ -0.8452 & 0 & 0.5345 \end{bmatrix}^T$$

$$X = V\sqrt{D}\sqrt{D}V^T = (V\sqrt{D})(V\sqrt{D})^T \quad X = \begin{bmatrix} 0 & 0.7071 & -2.1213 \\ 0 & 2.1213 & 0.7071 \\ 0 & 0 & 1.4142 \end{bmatrix} \begin{bmatrix} 0 & 0.7071 & -2.1213 \\ 0 & 2.1213 & 0.7071 \\ 0 & 0 & 1.4142 \end{bmatrix}^T$$

$$X = \underbrace{\begin{bmatrix} 0.7071 & -2.1213 \\ 2.1213 & 0.7071 \\ 0 & 1.4142 \end{bmatrix}}_{L \in \mathbb{R}^{3\times 2}} \begin{bmatrix} 0.7071 & -2.1213 \\ 2.1213 & 0.7071 \\ 0 & 1.4142 \end{bmatrix}^T$$

Linear Function of Matrix *X*

- Inner product of matrixes $A \bullet X = trace(A^T X)$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 1 & 6 \end{bmatrix} = trace\left(\begin{bmatrix} 6 & 18 \\ 10 & 24 \end{bmatrix}\right) = 30$

Linear Function of Matrix X

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- $A(X)$: Linear function of matrix X

$$A(X) \longrightarrow A \bullet X = trace(A^T X) \in \mathbb{R}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \qquad A(X) = A \bullet X = trace(\begin{bmatrix} x_{11} + 2x_{12} & x_{12} + 2x_{22} \\ 2x_{11} + 3x_{12} & 2x_{12} + 3x_{22} \end{bmatrix}) = x_{11} + 4x_{12} + 3x_{22}$$

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- If X is a symmetric matrix, without loss of generality, we assume that the matrix A is also symmetric.

Semidefinite Program

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times n}}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \\ & X \succcurlyeq 0. \end{aligned}$$

- We are looking for symmetric PSD matrix $X \in \mathbb{S}_+^n$ to minimize the linear function $C(X)$ with respect to linear constraints $A_i(X) = b_i$.

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- We are looking for symmetric PSD matrix $X \in \mathbb{S}^n_+$ to minimize the linear function $C(X)$ with respect to linear constraints $A_i(X) = b_i$.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 11 \\ 19 \end{bmatrix} \qquad X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}$$

$$\begin{aligned} \underset{X \in \mathbb{R}^{3 \times 3}}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, 2. \\ & X \succcurlyeq 0. \end{aligned}$$

$$\Rightarrow$$

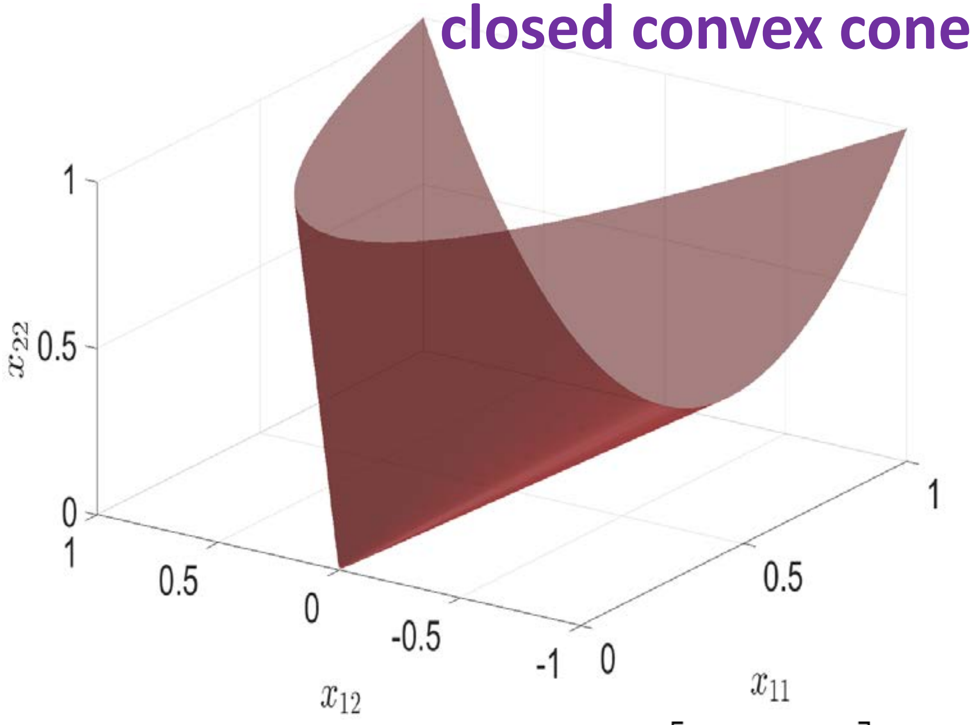
$$\begin{aligned} \underset{X \in \mathbb{R}^{3 \times 3}}{\text{minimize}} \quad & x_{11} + 4x_{12} + 6x_{13} + 9x_{22} + 7x_{13} \\ \text{subject to} \quad & x_{11} + 2x_{13} + 3x_{22} + 14x_{23} + 5x_{33} = 11 \\ & 4x_{12} + 16x_{13} + 6x_{22} + 4x_{33} = 19 \\ & X \succcurlyeq 0 \end{aligned}$$

Semidefinite Program

- **Cone of PSD Matrixes:** Set of PSD symmetric matrix $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n : X \succcurlyeq 0\}$

We need to show that $X_1, X_2 \in \mathbb{S}^n_+ \xrightarrow{\alpha, \beta \geq 0} \alpha X_1 + \beta X_2 \in \mathbb{S}^n_+$

closed convex cone



Set of all x_{11}, x_{12}, x_{22} that $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \succcurlyeq 0$

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$$X_1, X_2 \in \mathbb{S}^n_+ \xrightarrow[v \in \mathbb{R}^n \neq 0]{\alpha, \beta \geq 0} v^T(\alpha X_1 + \beta X_2)v = \alpha v^T X_1 v + \beta v^T X_2 v \succcurlyeq 0$$

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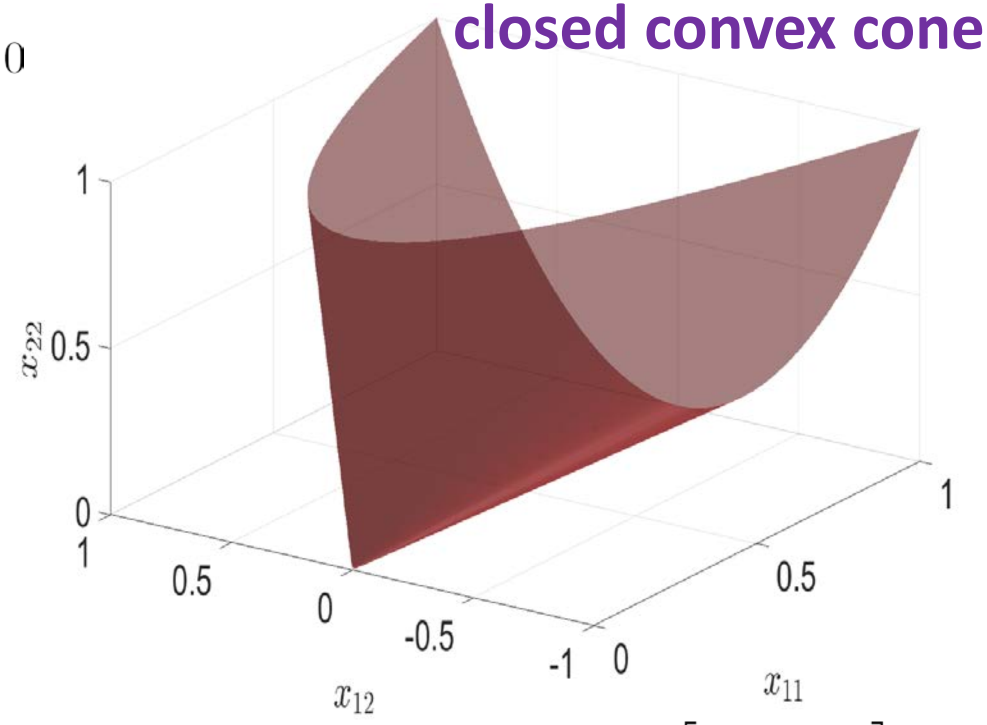

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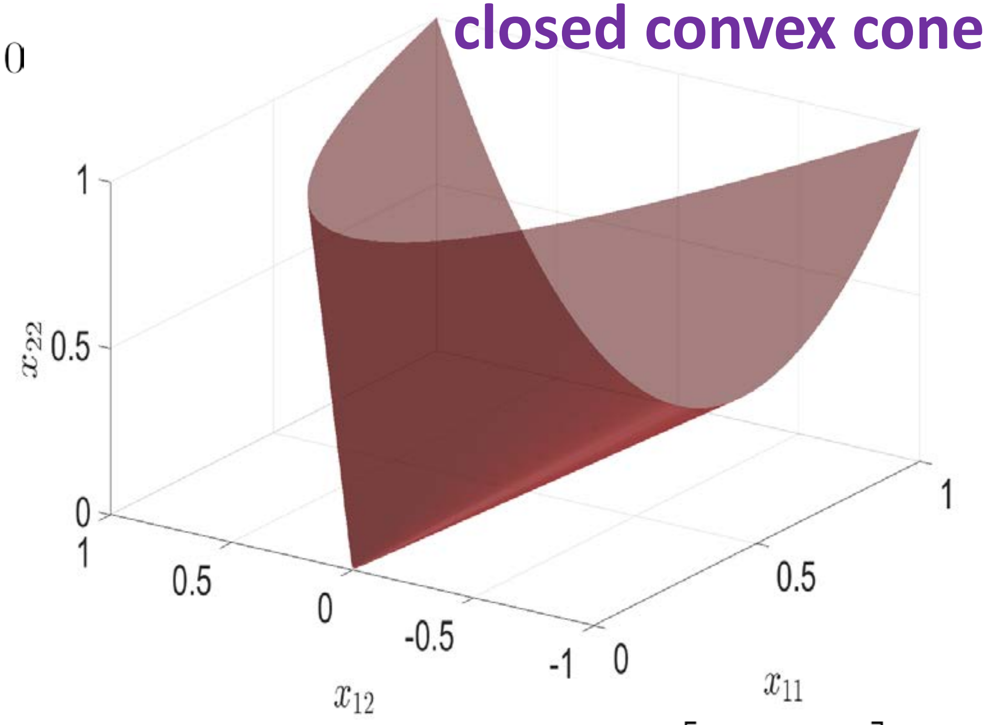
- Hence, SDP is a convex optimization

$$\begin{aligned} \underset{X \in \mathbb{R}^{n \times n}}{\text{minimize}} \quad & C \bullet X \longrightarrow \text{Linear function} \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, \ldots, m. \longrightarrow \text{Linear constraints} \\ & X \in \mathbb{S}_+^n. \longrightarrow \text{Convex Cone} \end{aligned}$$

closed convex cone



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YALMIP: J. Lofberg,”YALMIP : A Toolbox for Modeling and Optimization in MATLAB”, In Proceedings of the CACSD Conference, 2004 https://yalmip.github.io/

CVX: Matlab Software for Disciplined Convex Programming, http://cvxr.com/cvx/

Input: SDP

- Solves SDP’s using SDP solvers

SDP solvers: e.g.,

MOSEK https://www.mosek.com
SEDUMI http://sedumi.ie.lehigh.edu
SDPT3 http://www.math.nus.edu.sg/~mattohkc/sdpt3.html

Rely on **interior point** methods

$$\begin{aligned} \underset{X\in\mathbb{R}^{3\times 3}}{\text{minimize}} \quad & C \bullet X \\ \text{subject to} \quad & A_i \bullet X = b_i \quad i = 1, 2. \\ & X \succcurlyeq 0. \end{aligned}$$

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$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 11 \\ 19 \end{bmatrix}$$

```
A1 = [1 0 1;0 3 7;1 7 5];
A2=[0 2 8;2 6 0;8 0 4];
C=[1 2 3;2 9 0;3 0 7];
b=[11;19];
X = sdpvar(3,3);
F = [trace(A1*X)==b(1); trace(A2*X)==b(2);X >= 0 ];
ops = sdpsettings('solver','sedumi');
optimize(F,trace(C'*X),ops);
value(X)
```

$X \in \mathbb{R}^{3\times 3}$

Constraints

SDP solvers: MOSEK, SEDUMI or SDPT3.

SDP

Obtained Solution

- Theory and applications of semidefinite programs, and an introduction to primal-dual **interior-point methods:**

L. Vandenberghe and S. Boyd," SEMIDEFINITE PROGRAMMING" SIAM Review, 38(1): 49-95, March 1996.
https://web.stanford.edu/~boyd/papers/sdp.html

- Lieven Vandenberghe "*Nonnegative polynomials, SDP formulations, and primal-dual **interior point methods***",
  http://www.mit.edu/~parrilo/cdc03_workshop/Vandenberghe.pdf

- **Comparison** of SDP solvers:

H. D. Mittelmann " The State-of-the-Art in Conic Optimization Software"
http://www.optimization-online.org/DB_FILE/2010/08/2694.pdf

- A. Majumdar, G. Hall, and A. A. Ahmadi, "*A **Survey** of Recent Scalability Improvements for Semidefinite Programming with Applications in Machine Learning, Control, and Robotics*" Annual Reviews in Control, Robotics, and Autonomous Systems, 2019, https://arxiv.org/pdf/1908.05209.pdf

From SOS Program
To Semidefinite Program

From SOS to SDP

Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if :

it can be written as a finite sum of squares of other polynomials.

$$p(x) \in \mathbb{R}[x] \xrightarrow{\text{SOS}} \boxed{p(x) = \sum_{i=1}^{\ell} h_i^2(x) \qquad h_i(x) \in \mathbb{R}[x], \quad i = 1, \ldots, \ell}$$

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$$\Updownarrow$$

PSD Matrix representation of **SOS polynomials**

$$\boxed{\underbrace{Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0}_{\text{PSD Matrix}} \qquad p(x) = B(x)^T Q B(x) \qquad \textit{where} \quad B(x) \text{ :vector of monomials in } x}$$

Example: $p(x) = 2x_1^4 + 5x_2^4 - x_1^2x_2^2 + 2x_1^3x_2$

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SOS Form $p(x) = \sum_{i=1}^{\ell} h_i^2(x)$

$$p(x) = \left(\overset{h_1(x)}{\boxed{\frac{1}{\sqrt{2}}(2x_1^2 - 3x_2^2 + x_1x_2)}}\right)^2 + \left(\overset{h_2(x)}{\boxed{\frac{1}{\sqrt{2}}(x_2^2 + 3x_1x_2)}}\right)^2$$

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Goal: $p(x) = B(x)^T Q B(x)$

$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$

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vector of coefficients

vector of monomials in x_1 and x_2

$h_1(x) = C_1^T B(x)$

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vector of coefficients   vector of monomials in $x_1$and x_2

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$$= \begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}^T \underbrace{\begin{bmatrix}5 & 0 & -3\\ 0 & 5 & 1\\ -3 & 1 & 2\end{bmatrix}}_{Q = LL^T} \begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}$$

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$h_2(x) = C_2^T B(x)$

$\begin{bmatrix}h_1(x)\\ h_2(x)\end{bmatrix}$, $\quad h_1^2(x) + h_2^2(x) = \begin{bmatrix}h_1(x)\\ h_2(x)\end{bmatrix}^T \begin{bmatrix}h_1(x)\\ h_2(x)\end{bmatrix}$

$$\begin{bmatrix}h_1(x)\\ h_2(x)\end{bmatrix} = \begin{bmatrix}C_1^T B(x)\\ C_1^T B(x)\end{bmatrix} = \begin{bmatrix}C_1^T\\ C_1^T\end{bmatrix} B(x) = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}-3 & 1 & 2\\ 1 & 3 & 0\end{bmatrix}\begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}\right) \in \mathbb{R}^2$$

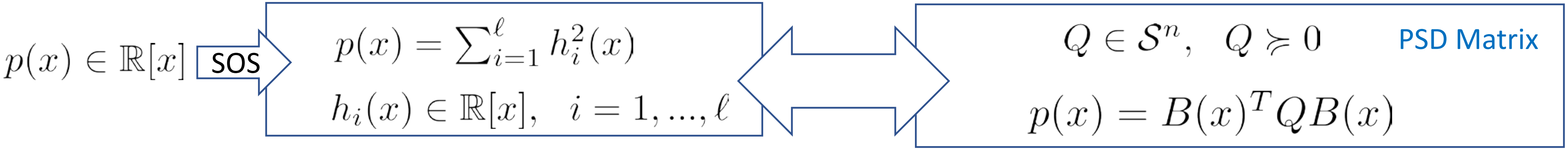
$$= \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix}-3 & 1 & 2\\ 1 & 3 & 0\end{bmatrix}\begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}\right)^T}_{\begin{bmatrix}h_1(x)\\ h_2(x)\end{bmatrix}^T} \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix}-3 & 1 & 2\\ 1 & 3 & 0\end{bmatrix}\begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}\right)}_{\begin{bmatrix}h_1(x)\\ h_2(x)\end{bmatrix}} = \begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}^T \left(\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}-3 & 1 & 2\\ 1 & 3 & 0\end{bmatrix}^T}_{L} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}-3 & 1 & 2\\ 1 & 3 & 0\end{bmatrix}}_{L^T}\right)\begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}$$

$$= \begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}^T \underbrace{\begin{bmatrix}5 & 0 & -3\\ 0 & 5 & 1\\ -3 & 1 & 2\end{bmatrix}}_{Q = LL^T} \begin{bmatrix}x_2^2\\ x_1x_2\\ x_1^2\end{bmatrix}$$

Eigenvalues of Q = 0, 5, 7 ⇨ $Q \succcurlyeq 0$

From SOS to SDP

- Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if it can be written as a finite sum of squares of other polynomials.



$p(x) \in \mathbb{R}[x]$ SOS

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, ..., \ell$$

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$

PSD Matrix

$$p(x) = B(x)^T Q B(x)$$

PSD Matrix representation of **SOS polynomials**

From SOS to SDP

- Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if it can be written as a finite sum of squares of other polynomials.

$p(x) \in \mathbb{R}[x]$ $\xrightarrow{\text{SOS}}$ ①

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \dots, \ell$$

$\Longleftrightarrow$ ②

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$

PSD Matrix

$$p(x) = B(x)^T Q B(x)$$

PSD Matrix representation of **SOS polynomials**

① $p(x) = \sum_{i=1}^{\ell} h_i^2(x) \geq 0$ $\Longrightarrow$ ②

② $Q \succcurlyeq 0$ $\Longrightarrow$ ①

From SOS to SDP

- Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if it can be written as a finite sum of squares of other polynomials.

$p(x) \in \mathbb{R}[x]$ $\xrightarrow{\text{SOS}}$ ①

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \dots, \ell$$

$\Longleftrightarrow$ ②

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$

PSD Matrix

$$p(x) = B(x)^T Q B(x)$$

PSD Matrix representation of **SOS polynomials**

① $p(x) = \sum_{i=1}^{\ell} h_i^2(x) \geq 0$ $\Longrightarrow$ ②

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x) = \sum_{i=1}^{\ell} (C_i^T B(x))^2$$

Coefficient vector of $h_i(x)$

$h_i(x) = C_i^T B(x)$

② $Q \succcurlyeq 0$ $\Longrightarrow$ ①

From SOS to SDP

- Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if it can be written as a finite sum of squares of other polynomials.

$p(x) \in \mathbb{R}[x]$ $\xrightarrow{\text{SOS}}$ ① $p(x) = \sum_{i=1}^{\ell} h_i^2(x)$, $h_i(x) \in \mathbb{R}[x], \quad i = 1, \dots, \ell$ $\Longleftrightarrow$ ② $Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$, $p(x) = B(x)^T Q B(x)$ PSD Matrix

PSD Matrix representation of **SOS polynomials**

① $p(x) = \sum_{i=1}^{\ell} h_i^2(x) \geq 0 \Longrightarrow$ ②

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x) = \sum_{i=1}^{\ell} (C_i^T B(x))^2$$

Coefficient vector of $h_i(x)$

$$h_i(x) = C_i^T B(x)$$

$$\begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix} = \begin{bmatrix} C_1^T B(x) \\ \vdots \\ C_\ell^T B(x) \end{bmatrix} = \begin{bmatrix} C_1^T \\ \vdots \\ C_\ell^T \end{bmatrix} B(x) = C^T B(x)$$

② $Q \succcurlyeq 0 \Longrightarrow$ ①

From SOS to SDP

- Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if it can be written as a finite sum of squares of other polynomials.

$$p(x) \in \mathbb{R}[x] \xrightarrow{\text{SOS}} \boxed{① \quad \begin{gathered} p(x) = \sum_{i=1}^{\ell} h_i^2(x) \\ h_i(x) \in \mathbb{R}[x], \quad i = 1, \dots, \ell \end{gathered}} \Longleftrightarrow \boxed{② \quad \begin{gathered} Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0 \qquad \text{PSD Matrix} \\ p(x) = B(x)^T Q B(x) \end{gathered}}$$

PSD Matrix representation of **SOS polynomials**

① $p(x) = \sum_{i=1}^{\ell} h_i^2(x) \geq 0 \Longrightarrow$ ②

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x) = \sum_{i=1}^{\ell} (C_i^T B(x))^2$$

Coefficient vector of $h_i(x)$

$h_i(x) = C_i^T B(x)$

$$\begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix} = \begin{bmatrix} C_1^T B(x) \\ \vdots \\ C_\ell^T B(x) \end{bmatrix} = \begin{bmatrix} C_1^T \\ \vdots \\ C_\ell^T \end{bmatrix} B(x) = C^T B(x)$$

$$= \underbrace{(C^T B(x))^T (C^T B(x))}_{\begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix}^T \begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix}}$$

② $Q \succcurlyeq 0 \Longrightarrow$ ①

From SOS to SDP

- Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if it can be written as a finite sum of squares of other polynomials.

$p(x) \in \mathbb{R}[x]$ SOS ⇒ ①

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \dots, \ell$$

⇔ ②

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$

PSD Matrix

$$p(x) = B(x)^T Q B(x)$$

PSD Matrix representation of **SOS polynomials**

① $p(x) = \sum_{i=1}^{\ell} h_i^2(x) \geq 0$ ⇒ ②

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x) = \sum_{i=1}^{\ell} (C_i^T B(x))^2$$

Coefficient vector of $h_i(x)$

$$h_i(x) = C_i^T B(x)$$

$$\begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix} = \begin{bmatrix} C_1^T B(x) \\ \vdots \\ C_\ell^T B(x) \end{bmatrix} = \begin{bmatrix} C_1^T \\ \vdots \\ C_\ell^T \end{bmatrix} B(x) = C^T B(x)$$

$$= \underbrace{(C^T B(x))^T (C^T B(x))}_{\begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix}^T \begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix}} = B^T(x) CC^T B(x) = B(x)^T Q B(x)$$

⇒ $Q \succcurlyeq 0$

② $Q \succcurlyeq 0$ ⇒ ①

From SOS to SDP

- Polynomial $p(x)$ is **sum of squares** (SOS) polynomial if it can be written as a finite sum of squares of other polynomials.

$p(x) \in \mathbb{R}[x]$ SOS ⇒ ① $p(x) = \sum_{i=1}^{\ell} h_i^2(x)$, $h_i(x) \in \mathbb{R}[x], \quad i = 1, \dots, \ell$ ⇔ ② $Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$ PSD Matrix; $p(x) = B(x)^T Q B(x)$

PSD Matrix representation of **SOS polynomials**

① $p(x) = \sum_{i=1}^{\ell} h_i^2(x) \geq 0$ ⇒ ②

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x) = \sum_{i=1}^{\ell} (C_i^T B(x))^2$$

Coefficient vector of $h_i(x)$

$h_i(x) = C_i^T B(x)$

$$\begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix} = \begin{bmatrix} C_1^T B(x) \\ \vdots \\ C_\ell^T B(x) \end{bmatrix} = \begin{bmatrix} C_1^T \\ \vdots \\ C_\ell^T \end{bmatrix} B(x) = C^T B(x)$$

$$= \underbrace{(C^T B(x))^T (C^T B(x))}_{\begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix}^T \begin{bmatrix} h_1(x) \\ \vdots \\ h_\ell(x) \end{bmatrix}} = B^T(x) C C^T B(x) = B(x)^T Q B(x)$$

⇒ $Q \succcurlyeq 0$

② $Q \succcurlyeq 0$ ⇒ ①

$$p(x) = B^T(x) Q B(x) = B^T(x) \overbrace{(LL^T)}^{Q = LL^T,\, L \in R^{n \times \ell}} B(x) = (L^T B(x))^T (L^T B(x)) = \sum_{i=1}^{\ell} \overbrace{(L_i^T B(x))}^{i-th \text{ element of vector } L^T B(x)}{}^2 = \sum_{i=1}^{\ell} h_i^2(x)$$

⇒ $p(x)$ is **SOS**

From SOS to SDP

- SOS Decomposition

$p(x) \in \mathbb{R}[x]$ SOS

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \ldots, \ell$$

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$

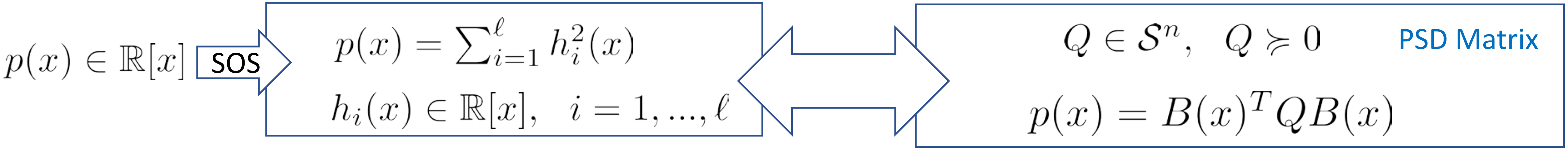
PSD Matrix

$$p(x) = B(x)^T Q B(x)$$

PSD Matrix representation of **SOS** **polynomials**

From SOS to SDP

- SOS Decomposition



$p(x) \in \mathbb{R}[x]$ SOS

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$
$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \ldots, \ell$$

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$
PSD Matrix
$$p(x) = B(x)^T Q B(x)$$

PSD Matrix representation of **SOS polynomials**

- In general, SOS decomposition is NOT unique.

From SOS to SDP

Example : $p(x) = 2x_1^4 + 5x_2^4 - x_1^2x_2^2 + 2x_1^3x_2$

SOS Decomposition 1

⇩

$$p(x) = \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1x_2)^2 + \frac{1}{2}(x_2^2 + 3x_1x_2)^2$$

$$L = \frac{1}{\sqrt{2}}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}^T$$

⇩

$$p(x) = \begin{bmatrix} x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}^T \underbrace{\begin{bmatrix} 5 & 0 & -3 \\ 0 & 5 & 1 \\ -3 & 1 & 2 \end{bmatrix}}_{Q = LL^T} \underbrace{\begin{bmatrix} x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}}_{B(x)}$$

eigenvalues = 0, 5, 7 $\Rightarrow$ $Q \succeq 0$

From SOS to SDP

Example : $p(x) = 2x_1^4 + 5x_2^4 - x_1^2x_2^2 + 2x_1^3x_2$

SOS Decomposition 1

$$p(x) = \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1x_2)^2 + \frac{1}{2}(x_2^2 + 3x_1x_2)^2$$

$$L = \frac{1}{\sqrt{2}}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}^T$$

$$p(x) = \begin{bmatrix} x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}^T \underbrace{\begin{bmatrix} 5 & 0 & -3 \\ 0 & 5 & 1 \\ -3 & 1 & 2 \end{bmatrix}}_{Q = LL^T} \underbrace{\begin{bmatrix} x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}}_{B(x)}$$

eigenvalues = 0, 5, 7 $\Rightarrow$ $Q \succcurlyeq 0$

SOS Decomposition 2

$$\begin{aligned} p(x) = &(1.0262x_1^2 - 2.1569x_2^2 + 0.2967x_1x_2)^2 \\ &+(-0.6889x_1^2 - 0.5253x_2^2 - 1.4364x_1x_2)^2 \\ &+(0.6873x_1^2 + 0.2682x_2^2 - 0.4277x_1x_2)^2 \end{aligned}$$

$$L = \begin{bmatrix} 0.2682 & 0.5253 & -2.1569 \\ -0.4277 & 1.4364 & 0.2967 \\ 0.6873 & 0.6889 & 1.0262 \end{bmatrix}$$

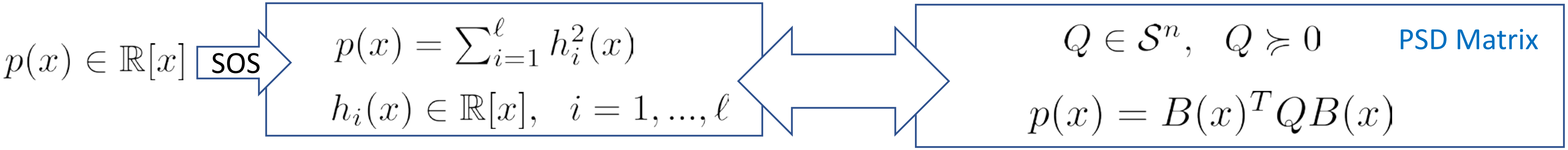
$$p(x) = \begin{bmatrix} x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}^T \underbrace{\begin{bmatrix} 5 & 0 & -1.667 \\ 0 & 2.334 & 1 \\ -1.667 & 1 & 2 \end{bmatrix}}_{Q = LL^T} \underbrace{\begin{bmatrix} x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}}_{B(x)}$$

eigenvalues = 0.72, 2.81, 5.79 $\Rightarrow$ $Q \succcurlyeq 0$

https://github.com/jasour/rarnop19/blob/master/Lecture3_SOS_NonlinearOptimization/SOS_Decomposition/Example_2.m

From SOS to SDP

- SOS Decomposition



$p(x) \in \mathbb{R}[x]$ SOS

$$p(x) = \sum_{i=1}^{\ell} h_i^2(x)$$

$$h_i(x) \in \mathbb{R}[x], \quad i = 1, \ldots, \ell$$

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$

PSD Matrix

$$p(x) = B(x)^T Q B(x)$$

PSD Matrix representation of **SOS polynomials**

From SOS To SDP

- Verification Problems
- Design Problems
- Optimization

From SOS to SDP

1) Nonnegativity Verification:

Given, $p(x) \in \mathbb{R}[x]$

Check if $p(x) \geq 0$

SOS

SOS Condition

$p(x) \in SOS$

PSD

PSD Matrix

$$Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$$

$$p(x) = B(x)^T Q B(x)$$

SDP

SDP

Find $Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$ (PSD)

Such that,

$$\underbrace{\text{Coefficient of polynomial } p(x) \text{ and } B(x)^T Q B(x) \text{ matches.}}_{\text{Linear constraints to satisfy } p(x) = B^T(x) Q B(x)}$$

From SOS to SDP

Example: Check the nonnegativity of polynomial $p(x)$

$$p(x) = x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 \quad \Longrightarrow \quad p(x) = \begin{bmatrix} 1 \\ x_1 \\ x_1^2 \end{bmatrix}^T \underbrace{\begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{01} & q_{11} & q_{12} \\ q_{02} & q_{12} & q_{22} \end{bmatrix}}_{Q \in \mathcal{S}^n} \underbrace{\begin{bmatrix} 1 \\ x_1 \\ x_1^2 \end{bmatrix}}_{B_2(x)}$$

From SOS to SDP

Example: Check the nonnegativity of polynomial $p(x)$

$$p(x) = x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 \Longrightarrow p(x) = \begin{bmatrix} 1 \\ x_1 \\ x_1^2 \end{bmatrix}^T \underbrace{\begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{01} & q_{11} & q_{12} \\ q_{02} & q_{12} & q_{22} \end{bmatrix}}_{Q \in \mathcal{S}^n} \underbrace{\begin{bmatrix} 1 \\ x_1 \\ x_1^2 \end{bmatrix}}_{B_2(x)}$$

$$p(x) = B_2(x)^T Q B_2(x)$$

$$x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 = q_{22}x_1^4 + 2q_{12}x_1^3 + (q_{11} + 2q_{02})x_1^2 + 2q_{01}x_1 + q_{00}$$

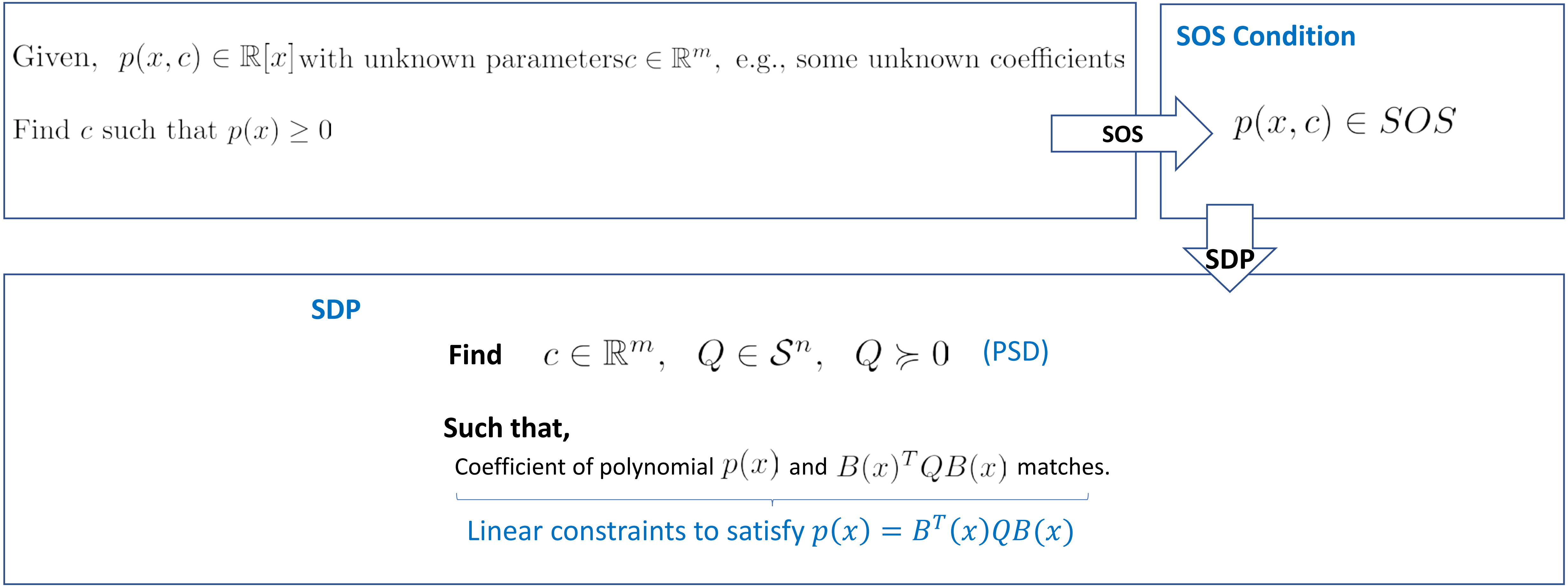
SDP

Find $Q \succcurlyeq 0$ **Such that,** $\underbrace{q_{22} = 1,}_{x_1^4} \ \underbrace{2q_{12} = 4,}_{x_1^3} \ \underbrace{q_{11} + 2q_{02} = 6,}_{x_1^2} \ \underbrace{2q_{01} = 4,}_{x_1} \ \underbrace{q_{00} = 5}_{x_1^0}$ (coefficients of monomials)

Linear constraints to satisfy $p(x) = B_2^T(x) Q B_2(x)$

From SOS to SDP

2) Design Problem:



Given, $p(x,c) \in \mathbb{R}[x]$ with unknown parameters$c \in \mathbb{R}^m$, e.g., some unknown coefficients

Find c such that $p(x) \geq 0$

SOS

SOS Condition

$$p(x,c) \in SOS$$

SDP

SDP

Find $c \in \mathbb{R}^m, \quad Q \in \mathcal{S}^n, \quad Q \succcurlyeq 0$ (PSD)

Such that,

$$\underbrace{\text{Coefficient of polynomial } p(x) \text{ and } B(x)^T Q B(x) \text{ matches.}}_{\text{Linear constraints to satisfy } p(x) = B^T(x) Q B(x)}$$

From SOS to SDP

Example : Design γ such that $p(x) \geq 0$

$$p(x) = x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \Longrightarrow p(x) = \begin{bmatrix} 1 \\ x_1 \\ x_1^2 \end{bmatrix}^T \underbrace{\begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{01} & q_{11} & q_{12} \\ q_{02} & q_{12} & q_{22} \end{bmatrix}}_{Q \in \mathcal{S}^n} \underbrace{\begin{bmatrix} 1 \\ x_1 \\ x_1^2 \end{bmatrix}}_{B_2(x)}$$

SDP

Find $\gamma \in \mathbb{R},\ Q \succcurlyeq 0$ **Such that,** $\underbrace{q_{22} = 1,}_{x_1^4}\ \underbrace{2q_{12} = 4,}_{x_1^3}\ \underbrace{q_{11} + 2q_{02} = 6,}_{x_1^2}\ \underbrace{2q_{01} = 4,}_{x_1}\ \underbrace{q_{00} = 5 - \gamma}_{x_1^0}$ (coefficients of monomials)

Linear constraints to satisfy $p(x) = B_2^T(x)QB_2(x)$

From SOS to SDP

Lyapunov Function Search

Example:

$$\dot{x}_1 = -x_1 + (1 + x_1)x_2$$
$$\dot{x}_2 = -(1 + x_1)x_1$$

SOS Conditions: $V(x) = c^T B_4(x)$ $\quad V(0) = 0 \longrightarrow c(1) = 0$ $\quad V(x) \in SOS_{2d}$ $\quad -\dot{V}(x) \in SOS$

$$V(x) = B_2(x)QB_2(x) = \begin{bmatrix} x_2 \\ x_1 \\ x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}^T \begin{bmatrix} 1.8991 & -0.5393 & -0.0812 & -0.0294 & -0.0064 \\ -0.5393 & 1.6216 & 0.0294 & 0.0506 & 0.0747 \\ -0.0812 & 0.0294 & 0.9981 & 0.0000 & 0.1118 \\ -0.0294 & 0.0506 & 0.0000 & 1.7727 & 0.0000 \\ -0.0064 & 0.0747 & 0.1118 & 0.0000 & 0.9981 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}$$

$$-\dot{V}(x) = \begin{bmatrix} x_2 \\ x_1 \\ x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}^T \begin{bmatrix} 1.0786 & -0.2618 & -0.0000 & 0.2073 & 0.1063 \\ -0.2618 & 2.1645 & -0.0000 & 0.0357 & -0.2708 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0001 \\ 0.2073 & 0.0357 & -0.0000 & 3.3280 & -0.2241 \\ 0.1063 & -0.2708 & 0.0001 & -0.2241 & 4.0809 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_2^2 \\ x_1x_2 \\ x_1^2 \end{bmatrix}$$

From SOS to SDP

Unconstrained Optimization

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$

$$\begin{aligned} &\underset{\gamma\in\mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x)-\gamma \geq 0, \quad \forall x\in\mathbb{R}^n \end{aligned}$$

SOS

$$\begin{aligned} &\underset{\gamma}{\text{maximize}} && \gamma \\ &\text{subject to} && p(x)-\gamma \in SOS \end{aligned}$$

SDP

SDP

$$\begin{aligned} &\underset{Q\in\mathcal{S}^n,\gamma}{\text{maximize}} && \gamma \longrightarrow \text{linear objective} \\ &\text{subject to} && \text{coefficients of } p(x)-\gamma = \text{coefficients of } B^T(x)QB(x) \longrightarrow \text{linear constraints} \\ & && Q \succcurlyeq 0 \longrightarrow \text{PSD} \end{aligned}$$

Example:

$$\underset{x\in\mathbb{R}}{\text{minimize}} \quad x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5$$

$$\begin{array}{ll} \underset{\gamma\in\mathbb{R}}{\text{maximize}} & \gamma \\ \text{subject to} & x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n \end{array}$$

SOS

$$\begin{array}{ll} \underset{\gamma\in\mathbb{R}}{\text{maximize}} & \gamma \\ \text{subject to} & x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \in SOS \end{array}$$

Example:

$$\underset{x \in \mathbb{R}}{\text{minimize}} \quad x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5$$

$$\begin{array}{ll} \underset{\gamma \in \mathbb{R}}{\text{maximize}} & \gamma \\ \text{subject to} & x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n \end{array}$$

SOS

$$\begin{array}{ll} \underset{\gamma \in \mathbb{R}}{\text{maximize}} & \gamma \\ \text{subject to} & x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \in SOS \end{array}$$

SDP

SDP

$$\begin{array}{ll} \underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} & \gamma \\ \text{subject to} & \text{coefficients of } (x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma) = \text{coefficients of } B^T(x)QB(x) \\ & Q \succcurlyeq 0 \end{array}$$

Example:

$$\underset{x\in\mathbb{R}}{\text{minimize}} \quad x_1^4+4x_1^3+6x_1^2+4x_1+5$$

$$\begin{aligned}&\underset{\gamma\in\mathbb{R}}{\text{maximize}} && \gamma\\ &\text{subject to} && x_1^4+4x_1^3+6x_1^2+4x_1+5-\gamma\geq 0,\quad \forall x\in\mathbb{R}^n\end{aligned}$$

SOS

$$\begin{aligned}&\underset{\gamma\in\mathbb{R}}{\text{maximize}} && \gamma\\ &\text{subject to} && x_1^4+4x_1^3+6x_1^2+4x_1+5-\gamma\in SOS\end{aligned}$$

SDP

SDP

$$\begin{aligned}&\underset{Q\in\mathcal{S}^n,\gamma\in\mathbb{R}}{\text{maximize}} && \gamma\\ &\text{subject to} && \text{coefficients of } (x_1^4+4x_1^3+6x_1^2+4x_1+5-\gamma) = \text{coefficients of } B^T(x)QB(x\\ &&& Q\succcurlyeq 0\end{aligned}$$

SDP

$$\begin{aligned}&\underset{Q\in\mathcal{S}^n,\gamma\in\mathbb{R}}{\text{maximize}} && \gamma\\ &\text{subject to} && q_{22}=1,\quad 2q_{12}=4,\quad q_{11}+2q_{02}=6,\quad 2q_{01}=4,\quad q_{00}=5-\gamma\\ &&& Q=\begin{bmatrix} q_{00} & q_{01} & q_{02}\\ q_{01} & q_{11} & q_{12}\\ q_{02} & q_{12} & q_{22}\end{bmatrix}\succcurlyeq 0\end{aligned}$$

Example:

$$\underset{x \in \mathbb{R}}{\text{minimize}} \quad x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5$$

$$\begin{aligned} &\underset{\gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \geq 0, \quad \forall x \in \mathbb{R}^n \end{aligned}$$

SOS

$$\begin{aligned} &\underset{\gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma \in SOS \end{aligned}$$

SDP

SDP

$$\begin{aligned} &\underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && \text{coefficients of } (x_1^4 + 4x_1^3 + 6x_1^2 + 4x_1 + 5 - \gamma) = \text{coefficients of } B^T(x)QB(x) \\ & && Q \succcurlyeq 0 \end{aligned}$$

SDP

$$\begin{aligned} &\underset{Q \in \mathcal{S}^n, \gamma \in \mathbb{R}}{\text{maximize}} && \gamma \\ &\text{subject to} && q_{22} = 1, \quad 2q_{12} = 4, \quad q_{11} + 2q_{02} = 6, \quad 2q_{01} = 4, \quad q_{00} = 5 - \gamma \\ & && Q = \begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{01} & q_{11} & q_{12} \\ q_{02} & q_{12} & q_{22} \end{bmatrix} \succcurlyeq 0 \end{aligned}$$

$$\gamma = 4$$

$$Q = \begin{bmatrix} 1 & 2 & 0.9998 \\ 2 & 4.0004 & 2 \\ 0.9998 & 2 & 1 \end{bmatrix}$$

https://github.com/jasour/rarnop19/blob/master/Lecture3_SOS_NonlinearOptimization/SOS_Optimization/Example_1_UnconOpt.m

From SOS to SDP

3) Constrained Nonnegativity Verification:

Given, $p(x) \in \mathbb{R}[x]$ and the sset $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., m\}$

Check if $p(x) \geq 0 \ \ \forall x \in \mathbf{K}$

SOS

SOS condition

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i},\ \ i = 1, ..., m$$

From SOS to SDP

3) Constrained Nonnegativity Verification:

Given, $p(x) \in \mathbb{R}[x]$ and the sset $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., m\}$

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SOS

SOS condition

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i},\ i = 1, ..., m$$

$$\sigma_i \in SOS_{2d_i},\ i = 1, ..., m \xrightarrow{\sigma_i = B_i(x)^T Q_i B_i(x),\quad i = 1, ..., m} Q_i \in \mathcal{S}^n,\quad Q_i \succcurlyeq 0,\quad i = 1, ..., m$$

Vector monomials up to order d_i

From SOS to SDP

3) Constrained Nonnegativity Verification:

Given, $p(x) \in \mathbb{R}[x]$ and the sset $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., m\}$

Check if $p(x) \geq 0 \ \ \forall x \in \mathbf{K}$

SOS

SOS condition

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i},\ i = 1, ..., m$$

$$\sigma_i \in SOS_{2d_i},\ i = 1, ..., m \xrightarrow{\sigma_i = B_i(x)^T Q_i B_i(x),\quad i = 1, ..., m} Q_i \in \mathcal{S}^n,\quad Q_i \succcurlyeq 0,\quad i = 1, ..., m$$

Vector monomials up to order d_i

$$p(x) - \sum_{i=1}^{m} \sigma_i g_i(x) \in SOS \xrightarrow{p(x) - \sum_{i=1}^{m} \sigma_i g_i(x) = B(x)^T Q_0 B(x)} Q_0 \in \mathcal{S}^n,\quad Q_0 \succcurlyeq 0$$

From SOS to SDP

3) Constrained Nonnegativity Verification:

Given, $p(x) \in \mathbb{R}[x]$ and the sset $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., m\}$

Check if $p(x) \geq 0 \ \ \forall x \in \mathbf{K}$

SOS

SOS condition

$$p(x) - \sum_{i=1}^{m} \sigma_i(x) g_i(x) \in SOS$$

$$\sigma_i(x) \in SOS_{2d_i},\ \ i = 1, ..., m$$

$$\sigma_i \in SOS_{2d_i},\ \ i = 1, ..., m \xrightarrow{\sigma_i = B_i(x)^T Q_i B_i(x),\ \ i = 1, ..., m} Q_i \in \mathcal{S}^n,\ \ Q_i \succcurlyeq 0,\ \ i = 1, ..., m$$

Vector monomials up to order d_i

$$p(x) - \sum_{i=1}^{m} \sigma_i g_i(x) \in SOS \xrightarrow{p(x) - \sum_{i=1}^{m} \sigma_i g_i(x) = B(x)^T Q_0 B(x)} Q_0 \in \mathcal{S}^n,\ \ Q_0 \succcurlyeq 0$$

SDP

Find $Q_i \in \mathcal{S}^n,\ \ Q_i \succcurlyeq 0,\ \ i = 0, ..., m$ (Linear Matrix inequality)

coefficients of polynomial $p(x) - \sum_{i=1}^{m} \sigma_i g_i(x) =$ coefficients of $B^T(x) Q_0 B(x)$ (Linear Constraint)

$$\sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x),\ \ i = 1, ..., m$$

Example: Check the nonnegativity of polynomial $p(x)$ on the set $\mathbf{K}$

$p(x) = x_1^3 - 4x_1^2 + 2x_1x_2 - x_2^2 + x_2^3$

$\mathbf{K} = \{x : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 - 1 \geq 0\}$

Example: Check the nonnegativity of polynomial $p(x)$ on the set $\mathbf{K}$

$$p(x) = x_1^3 - 4x_1^2 + 2x_1x_2 - x_2^2 + x_2^3 \qquad \mathbf{K} = \{x : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 - 1 \geq 0\}$$

Relaxed to

We need to show that $p(x)$ can be written as:

$$p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3(x_1 + x_2 - 1) \in SOS$$

(SOS condition)

where, $\sigma_i \in SOS_2,\ i = 1, 2, 3$

Example: Check the nonnegativity of polynomial $p(x)$ on the set $\mathbf{K}$

$$p(x) = x_1^3 - 4x_1^2 + 2x_1x_2 - x_2^2 + x_2^3 \qquad \mathbf{K} = \{x : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 - 1 \geq 0\}$$

Relaxed to

We need to show that $p(x)$ can be written as:

$$p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3(x_1 + x_2 - 1) \in SOS$$

where, $\sigma_i \in SOS_2,\ i = 1, 2, 3$

(SOS condition)

SDP

Find $Q_i \in \mathcal{S}^2,\ \ Q_i \succcurlyeq 0,\ \ i = 0, 1, 2, 3$ (Linear Matrix inequality)

$$\sigma_i = B_i(x)^T Q_i B_i(x), \quad i = 1, 2, 3, m$$

Vector monomials in terms of $x_1$and x_2 up to order 2

coefficients of polynomial $p(x) - \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3(x_1 + x_2 - 1)$ = coefficients of $B^T(x) Q_0 B(x)$

From SOS to SDP

4) Constrained Design Problem:

Given, $p(x, c) \in \mathbb{R}[x]$ with unknown parameters $c \in \mathbb{R}^m$ and the set $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ \ i = 1, ..., m\}$

Find c such that $p(x) \geq 0 \ \ \forall x \in \mathbf{K}$

SOS

$$p(x, c) - \sum_{i=1}^{m} \sigma_i g_i(x) \in SOS$$

$$\sigma_i \in SOS_{2d_i},\ \ i = 1, ..., m$$

SDP

Find $c \in \mathbb{R}^m,\ \ Q_i \in \mathcal{S}^n,\ \ Q_i \succcurlyeq 0,\ \ i = 0, ..., m$ (Linear Matrix inequality)

coefficients of polynomial $p(x, c) - \sum_{i=1}^{m} \sigma_i g_i(x) =$ coefficients of $B^T(x) Q_0 B(x)$ (Linear Constraint)

$$\sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x),\ \ i = 1, ..., m$$

From SOS to SDP

Constrained Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \geq 0, \;\; \forall x \in \{x \in \mathbb{R}^n : g_i(x) \geq 0, \;\; i = 1, \dots, n\} \end{aligned}$$

SOS

$$\begin{aligned} \underset{\gamma,\sigma_i}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma - \sum_{i=1}^{m} \sigma_i g_i(x) \in SOS \\ & \sigma_i \in SOS_{2d_i}, \;\; i = 1, \dots, m \end{aligned}$$

SDP

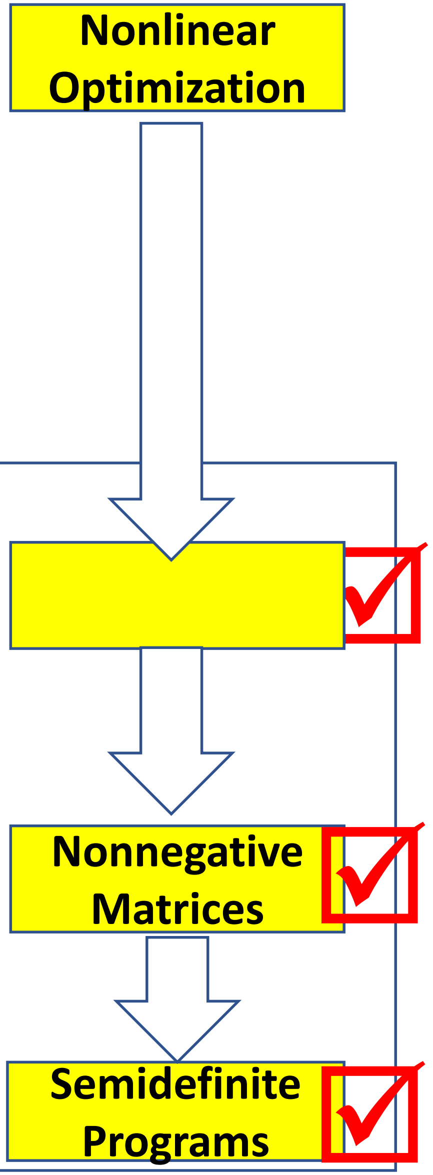
SDP

$$\begin{aligned} \underset{\gamma,Q_i|_{i=0}^m}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & \text{coefficients of polynomial } p(x) - \textstyle\sum_{i=1}^m \sigma_i g_i(x) = \text{coefficients of } B^T(x) Q_0 B(x) \\ & \sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, \dots, m \end{aligned}$$

Nonlinear (nonconvex) Optimization

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \ldots, m \end{aligned}$$

Tools: i) Nonnegative Polynomials ii) Semidefinite Programs


Nonlinear Optimization

✓

Nonnegative Matrices ✓

Semidefinite Programs ✓

Step 1:

Reformulate **Nonlinear Optimization** problem in terms of **Nonnegative Polynomials**

Step 2:

2.1 Replace Nonnegative Polynomials with Sum of Squares (SOS) Polynomials ✓

SOS Programming using YALMIP ✓

2.2 Represent **SOS Polynomials** with **Positive Semidefinite** Matrices (PSD) ✓

Reformulate Nonlinear Optimization as **Semidefinite Programs** ✓

Unconstrained Optimization

$$\mathrm{P} = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$

Constrained Optimization

$$\begin{aligned} \mathrm{P} = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \ldots, n \end{aligned}$$

Unconstrained Optimization

$$\mathrm{P} = \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$

SOS

SOS Programming

$$\begin{aligned} \mathrm{P}_{sos} = \underset{\gamma}{\text{maximize}} & \quad \gamma \\ \text{subject to} & \quad p(x) - \gamma \in SOS \end{aligned}$$

Constrained Optimization

$$\begin{aligned} \mathrm{P} = \underset{x \in \mathbb{R}^n}{\text{minimize}} & \quad f(x) \\ \text{subject to} & \quad g_i(x) \geq 0, \;\; i = 1, \ldots, n \end{aligned}$$

SOS

SOS Programming

$$\begin{aligned} \mathrm{P}_{sos} = \underset{\gamma, \sigma_i}{\text{maximize}} & \quad \gamma \\ \text{subject to} & \quad p(x) - \gamma - \sum_{i=1}^{m} \sigma_i g_i(x) \in SOS \\ & \quad \sigma_i \in SOS_{2d_i}, \;\; i = 1, \ldots, m \end{aligned}$$

Unconstrained Optimization

$$\mathrm{P} = \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$

SOS

SOS Programming

$$\begin{aligned} \mathrm{P}_{sos} = \underset{\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in SOS \end{aligned}$$

SDP

SDP

$$\begin{aligned} \underset{Q\in\mathcal{S}^n,\gamma}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & \text{coefficients of } p(x) - \gamma = \text{coefficients of } B^T(x)QB(x) \\ & Q \succcurlyeq 0 \end{aligned}$$

Constrained Optimization

$$\begin{aligned} \mathrm{P} = \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0, \;\; i = 1, \ldots, n \end{aligned}$$

SOS

SOS Programming

$$\begin{aligned} \mathrm{P}_{sos} = \underset{\gamma,\sigma_i}{\text{maximize}} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma - \sum_{i=1}^{m} \sigma_i g_i(x) \in SOS \\ & \sigma_i \in SOS_{2d_i}, \;\; i = 1, \ldots, m \end{aligned}$$

SDP

SDP

$$\underset{\gamma,Q_i|_{i=0}^m}{\text{maximize}} \quad \gamma$$

subject to

$$\text{coefficients of polynomial } p(x) - \textstyle\sum_{i=1}^m \sigma_i g_i(x) = \text{coefficients of } B^T(x)Q_0B(x)$$

$$\sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, \ldots, m$$

Unconstrained Optimization

$$\mathrm{P} = \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$

SDP

SDP

$$\begin{array}{ll} \underset{Q\in\mathcal{S}^n,\gamma}{\text{maximize}} & \gamma \\ \text{subject to} & \text{coefficients of } p(x)-\gamma = \text{coefficients of } B^T(x)QB(x) \\ & Q \succcurlyeq 0 \end{array}$$

Constrained Optimization

$$\begin{array}{ll} \mathrm{P} = \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \;\; i = 1, \dots, n \end{array}$$

SDP

SDP

$$\underset{\gamma, Q_i|_{i=0}^{m}}{\text{maximize}} \quad \gamma$$

subject to

$$\text{coefficients of polynomial } p(x) - \sum_{i=1}^{m} \sigma_i g_i(x) = \text{coefficients of } B^T(x)Q_0B(x)$$

$$\sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, \dots, m$$

Optimal solution x^*

At optimal solution x^*:

$$\left.\begin{array}{lll} \text{Unconstrained Optimization} & p(x^*) = \gamma^* & \\ \text{Constrained Optimization} & p(x^*) = \gamma^* & g_i(x^*) \geq 0, \;\; i = 1, \dots, m \end{array}\right\}$$

System of nonlinear equations and inequalities

Unconstrained Optimization

$$P = \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad p(x)$$

SDP

SDP

$$\begin{array}{ll} \underset{Q\in\mathcal{S}^n,\gamma}{\text{maximize}} & \gamma \\ \text{subject to} & \text{coefficients of } p(x)-\gamma = \text{coefficients of } B^T(x)QB(x) \\ & Q \succcurlyeq 0 \end{array}$$

Constrained Optimization

$$\begin{array}{ll} P = \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \;\; i = 1, \ldots, n \end{array}$$

SDP

SDP

$$\underset{\gamma, Q_i|_{i=0}^{m}}{\text{maximize}} \quad \gamma$$

subject to

$$\text{coefficients of polynomial } p(x) - \textstyle\sum_{i=1}^{m} \sigma_i g_i(x) = \text{coefficients of } B^T(x)Q_0B(x$$

$$\sigma_i = B_{d_i}(x)^T Q_i B_{d_i}(x), \quad i = 1, \ldots, m$$

Optimal solution x^*

At optimal solution x^*:

Unconstrained Optimization $\quad p(x^*) = \gamma^*$

Constrained Optimization $\quad p(x^*) = \gamma^* \qquad g_i(x^*) \geq 0, \;\; i = 1, \ldots, m$

System of nonlinear equations and inequalities

- To obtain optimal solutions x^*,we will look at **dual optimization problem** (dual SDP)
  (Complementary slackness in KKT optimality condition)

By looking at the **Dual SDP** of SOS SDP:

- Obtain Optimal Solution x^*

- Monotonic Nondecreasing Convergence

$$\mathrm{P}^{*d}_{SDP} \le \mathrm{P}^{*d+1}_{SDP} \le \dots \le \mathrm{P}^{*\infty}_{SDP} = \mathrm{P}^*$$

  - Optimal Objective function of SOS SDP/ Dual SDP with relaxation order d

Optimal Objective function of Original Optimization

- Finite Convergence $\exists\ d^* \quad \mathrm{P}^{*d^*}_{SDP} = \mathrm{P}^* \quad d \ge d^*$

Theory of Sum of Squares

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- Pablo A. Parrilo, Sanjay Lall , “Semidefinite Programming Relaxations and Algebraic Optimization in Control” European Journal of Control, V. 9, No. 2-3, pp. 307–321, 2003, http://www.mit.edu/~parrilo/cdc03_workshop/ejc03_comp.pdf

- Workshop: SDP Relaxations and Algebraic Optimization in Control, 2003 http://www.mit.edu/~parrilo/cdc03_workshop/index.html

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- Georgina Hall ,”Engineering and Business Applications of Sum of Squares Polynomials”, 2019, https://arxiv.org/pdf/1906.07961.pdf

- Section 4: Applications of Sum of Squares Programming, A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, P. A. Parrilo, “SOSTOOLS Sum of Squares Optimization Toolbox for MATLAB”, 2013, http://www.cds.caltech.edu/sostools/sostools.pdf

Application in Nonlinear Optimization

- *Sections 2 and 5*: Jean Bernard Lasserre, “Moments, Positive Polynomials and Their Applications” Imperial College Press Optimization Series, V. 1, 2009.

- Section 3: Monique Laurent, “Sums Of Squares, Moment Matrices and Optimization Over Polynomials”, 2010, https://homepages.cwi.nl/~monique/files/moment-ima-update-new.pdf

SOS Programming Using YALMIP

https://yalmip.github.io/tutorial/sumofsquaresprogramming/ https://yalmip.github.io/example/moresos/

16.S498 Risk Aware and Robust Nonlinear Planning
Fall 2019