17.810/17.811: Final Problem Set 2021

Read the following instructions carefully:

- All answers must be typed or clearly written up and stapled.
- Late submission of the write-up will not be accepted.
- You are NOT allowed to work in groups for the final problem set. All work has to be your own.
- For analytical (proofs) questions, you should include your detailed derivation for all intermediate steps (logical arguments). Partial credits will be given.
- No questions except for pure clarification ones will be taken.
- There are 8 parts in this Final Problem Set, and the maximum you can get from this Final Problem Set would be 85 points. We would convert your scores into 40 as discussed in the syllabus.

Part I - Resource Allocation Across Districts (10 pts)

Set Up

Consider a primary election, which features winner-takes-all system, between two candidates with 10 districts. Each district is indexed and ranked by the number of delegates it has. In district i, there are d_i delegates. We have $d_1 < d_2 < \ldots < d_{10}$.

Each candidate has the same fixed number of campaign surrogates, denoted by n, and $n < 10$. The politician must decide where to send each of their surrogates, with no more than one per district.

If one and only one candidate sends a surrogate to a certain district, the candidate wins the district. This means that the candidate will win all votes of the delegates from this district.

Otherwise, if no candidate sends a surrogate to the district or more than one candidate sends a surrogate to the district, the district delegates will abstain, and no candidate gets any votes of the delegates from this district.

Suppose the candidate who wins the most delegates wins the election and gets payoff 1, while the loser gets -1 (plurality voting system). If there is a tie in delegates, the race will be decided via coinflip. That is each candidate will have a probability of 0.5 to win the election.

Problem 1

Find a (pure strategy) Nash Equilibrium of this game. You have to show that the strategy you find is a Nash Equilibrium.

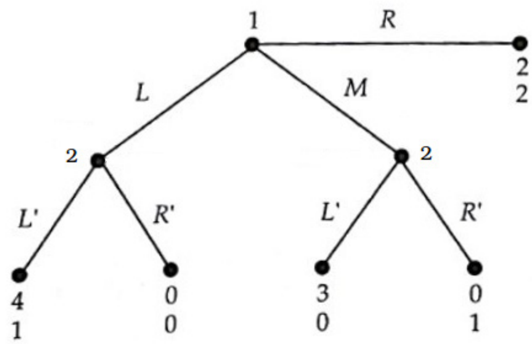
Problem 2

Is this Nash equilibrium unique? If so, prove it. If not, make a statement about the outcomes in the full set of Nash equilibrium.

Part II - Solve the Game (10 pts)

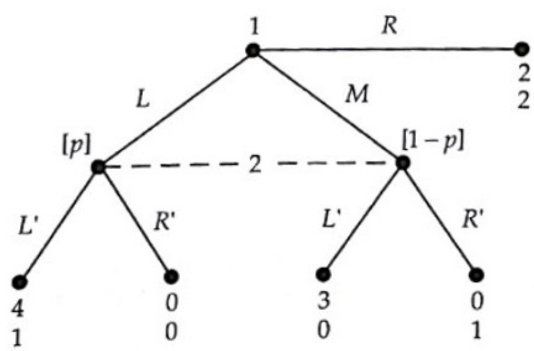
Problem 1

In the following extensive-form game, derive the normal-form game and find all pure strategy and mixed strategy Nash Equilibria. Find all Subgame Perfect Nash Equilibria.



Problem 2

In the following extensive-form game, derive the normal-form game and find all pure strategy Nash Equilibria, pure strategy Subgame Perfect Nash Equilibria and pure strategy Perfect Bayesian Equilibria.



Part III - Finitely Repeated Game (10 pts)

Suppose the following game is played three times. Does this game have a Subgame Perfect Equilibrium in which (A,X) is played in the first stage? Specify the strategy if there is one. Explain your reasons if there is no such SPE. We assume the discount rate $\delta = 1$ in this problem. (Hint: Do not forget about mixed strategies.)

		Player 2			
		X	Y	Z	W
Player 1	A	5,5	0,6	-2,4	-1,5
	B	7.25,-1	3,3	-1,-1	0,0
	C	3,2	2,-1	1,3	0.5,0
	D	-1,0	-1,0.5	-1,0	0,0

Part IV - Infinitely Repeated Game (20 pts)

Problem 1

Let G be the 3×3 game shown below.

Player 2

Player 1		C	D	E
	C	1,1	-1,2	0,0
	D	2,-1	0,0	$0,\frac{1}{2}$
	E	0,0	$\frac{1}{2},0$	0,0

What are the minimum average payoffs Player 1 and Player 2 could get respectively from this infinitely repeated game if they are playing strategies on a Subgame Perfect Equilibrium? Explain your reasons.

Problem 2

Let G be the 3×3 game shown below.

Player 2

Player 1		C	D	E
	C	1,1	-1,2	0,0
	D	2,-1	0,0	$0,\frac{1}{2}$
	E	0,0	$\frac{1}{2},0$	0,0

Consider a strategy profile for the repeated game G^{δ} in which the players play (C; C) in the first period, and in subsequent periods play (C; C) if both players played as they were supposed to in the previous two periods and play (D; D) otherwise.

(To help make this clear, note that if the players follow these strategies (C; C) is played in every period. If a player deviates when he was supposed to play C the "punishment" is that (D; D) is played twice before the players return to playing (C; C). If a player fails to play D when he is supposed to, the punishment is in effect prolonged.)

For what values of the discount factor $\delta \in (0, 1)$ is the profile a subgame-perfect equilibrium? δ is the discount factor across stages.

Part V - To Exchange or Not To Exchange (10 pts)

Asya's night is the most popular TV show in the State of Game Theory. At the end of each show, two lucky players will be drawn randomly from the audience. Each of them will receive a ticket on which there is an integer number from 0 to 100. The number on a player's ticket is the number of dollars the player will receive. The numbers on each of the player's ticket are identically and independently distributed, and the probability that a player is receiving each possible number is $\frac{1}{101}$.

The moderator give each player her ticket. After seeing her own ticket, the player will be asked privately whether she is willing to exchange her ticket with her opponent. If both of them privately tells the moderator that they agree to do the exchange, an exchange will happen, and each player will receive the number of dollars on their new tickets.

We assume that each player maximizes her expect monetary payoff. Formally model this situation as a Bayesian Game, and solve the Bayesian Nash Equilibria.

Part VI - Signalling Game (10 pts)

Set Up

There are two types of workers, a High productivity worker and a Low productivity worker, their productivity $\theta \in \{H, L\}$, with $H > L > 0$. Whether a worker is of type H or type L is privately observed by workers. However, the probability that a worker is of type H is λ, common knowledge to everyone. Workers then choose level of education e, which we assume to have no value but incurs a disutility of $\frac{e}{\theta}$.

Then the employer sees the education level of workers and then chooses wage w for workers. The utility of a worker is $u_1(e, w; \theta) = w - \frac{e}{\theta}$ and the utility of the employer is $u_2(e, w; \theta) = E[\theta|e] - w$, where θ is the type of worker the employer turns out to have hired.

Problem 1 Find a separating Perfect Bayesian Equilibrium of the game.

Probelm 2 Find a pooling Perfect Bayesian Equilibrium of the game.

Part VII - Pandering of Politicians (15 pts)

Set Up

There are two periods, 1, and 2, and a pair of possible actions $\{a, b\}$ in each period. Among the two possible actions, one is better for every voter than the other. However, no voter knows whether it is a or b. The prior belief that a is the optimal choice is p, $p > \frac{1}{2}$.

The politician is more knowledgeable than voters, and we assume the politician knows what is best for the society and what is best for her. There are two types of politicians though, a good type and a bad type. There is a probability π that a politician is a good one, which means the politician inherently shares the same preference with the voters. That is, if nature decides the socially optimal policy is a, the personally optimal policy for the politician is a. On the other hand, there is a probability $1 - \pi$ that a political of bad type, which means the politician has an opposite preference compared with the voters. That is, if nature decides a is the socially optimal policy, b will be the personally optimal policy of the politician.

A politician obtains utility G from selecting her personally preferred policy in each period and utility R from being in office for each period. There is a discounted factor β for the utility received in the second period.

Voters are interested in maximizing their expected utility. That is, the electorate obtains a payoff of 2 if the actions chosen in both periods are optimal, a payoff of 1 if just one action is optimal, and a zero payoff if neither is. That is, the electorate is "risk neutral".

The game is played in the following way:

1. Nature decides the socially optimal policy.
2. Nature decides whether the current politician is of good type or bad type.
3. The politician learns about the socially optimal policy, and decides whether to implement policy a or policy b.
4. The voters observe the policy implemented, and decide whether to reelect the incumbent politician or remove the incumbent politician.
5. If the voters reelect the incumbent politician, the incumbent politician decides whether to implement policy a or policy b. However, if the voters remove the incumbent politician, a challenger will be elected, and nature decides whether the challenger is of good type or bad type. The elected challenger decides whether to implement policy a or policy b.
6. The game ends.

Problem 1 Show that both the incumbent politician and the challenging politician will choose her preferred policy in the second period if elected.

Problem 2 Suppose $\frac{1}{2} < \pi < 1$, show that if $\beta\frac{G+R}{G} < 1$, then in the Perfect Bayesian Equilibrium, a politician chooses her preferred action in period 1, while voters reelect her if the policy implemented is a, and removes her if the policy implemented is b. Calculate the payoffs of voters.

Problem 3 Suppose $\frac{1}{2} < \pi < 1$, show that if $\beta\frac{G+R}{G} > 1$, then in the Perfect Bayesian Equilibrium, a politician always chooses a in period 1, while voters always reelect her. Calculate the payoffs of voters.

Problem 4 If we were in a direct democracy, which means voters decide the policies on their own. The game will be played as follows

1. Nature decides the socially optimal policy.
2. Voters decide the policy to be implemented in the first period.
3. Voters decide the policy to be implemented in the second period, and the game ends.

How will the payoffs voters get change? And compare this payoff with payoffs you get from Problem 2 and 3.

Part VIII - Extra Credit (10 pts)

Problem 1

Propose any game of your own substantive interest with 2 actors and 2 actions, possibly with dynamics and uncertainty.

Problem 2

Use the solution concepts covered in the class to solve your game, and interpret your results in the substantive context.

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