17.810/17.811 – Game Theory

Lecture 3: Mixed Strategy Nash Equilibrium

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Where We Are

- In the last lecture, we learned about Nash equilibrium: what it means and how to solve for it
- We focused on equilibrium in pure strategies, meaning actions were mapped to certain outcomes
- We will now consider mixed strategies: probabilistic play
- But first, we have to develop a notion of preferences over uncertain outcomes: expected utility theory

Reading

These slides will focus on the following readings:

- Choice Under Uncertainty
  - These slides should contain all the information you need to know. However, if you wish for more technical detail, see McCarty and Meirowitz, Chapter 3.1.
- Mixed Strategy Nash Equilibrium
  - Gibbons, 1.3A

Choice Under Uncertainty

So far we have been talking about preferences over certain alternatives.

Let's think about preferences over what might be called "risky" alternatives.

It is not difficult to imagine a world where decisionmakers make choices that lead to chances of different outcomes.

Choice Under Uncertainty

- Agents understand that outcomes are generated probabilistically from their choices: different actions increase or decrease the likelihood of particular outcomes.

- Recall the earlier example:

$$A = \{\text{send in troops, try negotiations, do nothing}\}$$

$$X = \{\text{win large concessions, win small concessions, status quo}\}$$

- The agents might believe that large concessions are more likely when the troops are deployed than when negotiations are initiated.

Choice Under Uncertainty

There are two key elements of this decision problem:

1. Beliefs that we model as probability distributions or “lotteries” over outcomes associated with each action.

2. Payoffs associated with each outcome.

Simple Lotteries

- Let X be a set of outcomes with elements $\{x_1, x_2, \ldots x_n\}$.
- Throughout this section, we will assume common knowledge of the probabilities of outcomes associated with each action.
- This knowledge could come from repeated observation, e.g., "Over the many times I've flipped a quarter in my life, it has come up heads roughly half of the time."

Simple Lotteries

The basic building block of expected utility is the lottery.

Definition

A **simple lottery** L is a list $L = (p_1, \ldots, p_n)$ with $0 \leq p_k \leq 1$ and $p_1 + p_2 + p_3 + \ldots + p_n = 1$.

(Compact notation: $\sum_{k=1}^{n} p_k = 1$.)

Example of a Simple Lottery

Simple lotteries come in many forms: lotteries that assign positive probabilities to all outcomes, lotteries that assign positive probability to only some outcomes, and degenerate lotteries.

Example

Let $X = \{1, 2, 3, 4, 5, 6\}$.

- All positive probabilities: a fair six-sided die
  $L = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$
- Some positive probabilities: a die with only even numbers
  $L = \left(0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}\right)$
- Degenerate lottery: a loaded die that always comes up 6
  $L = (0, 0, 0, 0, 0, 1)$

Compound Lotteries

- In a simple lottery, the outcomes that result are certain.
- This need not be the case generally.
- In fact, we may think about **compound lotteries** where the lottery outcome is another lottery.

Example (War)

- Lottery over winning and losing.
- If lose, lose. If win, a lottery over winning and losing the peace.

Formalizing the Idea of Compound Lotteries

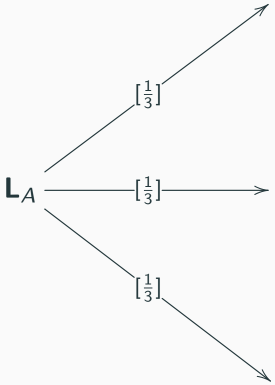
Definition

Given M simple lotteries $L_m = (p_1^m, \dots p_n^m)$, $m = 1, \dots M$ and $0 \leq \alpha_m \leq 1$ with $\sum_{i=1}^{M} \alpha_m = 1$, the **compound lottery** $\mathbf{L} = (L_1, \dots, L_M; \alpha_1, \dots, \alpha_m)$ is the risky alternative that yields the simple lottery L_m with probability α_m.

Compound Lotteries

- In fact, you might think that there is an infinite hierarchy of such compound lotteries, making life very difficult for us when thinking about preferences with uncertainty.
- However, for every compound lottery, we can calculate a corresponding **reduced lottery** that is a simple lottery.

Suppose there are three possible outcomes: $X = \{win, lose, draw\}$



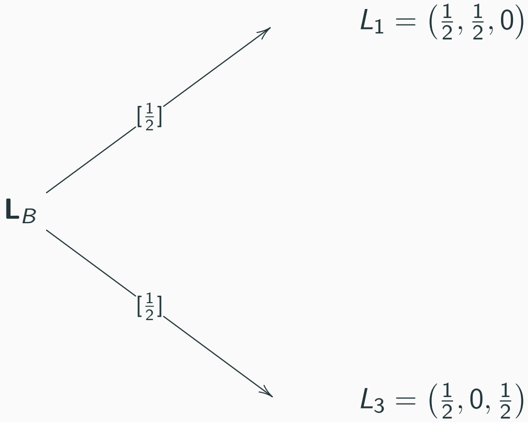
$$L_1 = (1, 0, 0)$$

$$L_2 = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$$

$$L_3 = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$$

Reduced lottery $= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$

Reducing Compound Lotteries



$$\textbf{Reduced lottery} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

Reducing Compound Lotteries

Since we care about preferences over outcomes, we treat these two lotteries ($\mathbf{L}_A$ and $\mathbf{L}_B$) as having equal value.

Utility Representations of Lotteries

As before, we seek a simple numeric representation of preferences over lotteries. We call this expected utility.

Definition (von Neumann–Morgenstern Utility Function)

The expected utility of a lottery L is given by:

$$EU(L) = p_1 u(x_1) + p_2 u(x_2) + ... + p_n u(x_n) = \sum_{k=1}^{n} p_k u(x_k)$$

Axioms of Expected Utility

Under what conditions can preferences over lotteries be represented by expected utility functions?

We need four axioms:

1. Reduction of compound lotteries
2. Our old friends the rationality axioms (completeness and transitivity)
3. Continuity
4. Independence

Let's go through these one by one.

Axiom 1: Reduction of Compound Lotteries

Axiom

All that matters for preferences is the reduced lottery.

We do not care about the paths traveled to get there.

Axiom 2: Transitivity and Completeness

Let's consider the set of simple lotteries $\mathcal{L}$ over certain outcomes in some set X. We can impose our old weak preference relation R on $\mathcal{L}$, and further demand:

Axiom

Individuals have complete and transitive preferences over $\mathcal{L}$.

Axiom 3: Continuity

Axiom

The preference relation R on $\mathcal{L}$ is continuous.

Intuition for continuous preferences:

- If pRq, then there are neighborhoods $B(p)$ and $B(q)$ such that for all $p' \in B(p)$ and $q' \in B(q)$, $p'Rq'$
- Stable preference orderings with very small perturbations

For example, if a trip to Hawaii is preferred to staying home for vacation, then a lottery between having a great trip to Hawaii and an arbitrarily small probability of a plane crash is still better than staying home.

Axiom 4: Independence

Axiom

The preference relation R on $\mathcal{L}$ satisfies the independence axiom.

Definition

The preference relation R on $\mathcal{L}$ satisfies the **independence axiom** if and only if for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$, we have

$$L \succeq L' \text{ if and only if } \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''.$$

In words, if we mix each of two lotteries with a third one, then the preference ordering of the two resulting mixtures is independent of the particular third lottery used.

Axiom 4: Independence

Uncontroversial Example:

- I weakly prefer an entire apple pie to a dozen chocolate chip cookies
- Therefore I must weakly prefer a slice of pie (equal to 1/12 of the pie) with a glass of milk to one cookie with a glass of milk

Controversial Example:

- I weakly prefer salami to peanut butter
- Therefore I must weakly prefer a salami and jelly sandwich to a peanut butter and jelly sandwich

(This axiom gets violated by interactions between objects.)

Theorem: von Neumann–Morgenstern

Theorem (von Neumann–Morgenstern)

If Axioms 1-4 hold, then there exists a function $u(x)$ such that:

1. $$EU(L_i) = \sum_{k=1}^{n} p_{ik} u(x_k)$$

   where L_i is the lottery over outcomes induced by action i

2. $L_i R L_j$ *if and only if* $EU(L_i) \geq EU(L_j)$.

Example

- Suppose there are 3 possible outcomes: receiving 0, 1, or 5 dollars.
- Let's assume you prefer more money to less.
- Say prefer the lottery $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ to $\left(0, \frac{3}{4}, \frac{1}{4}\right)$.
- What utility function can represent these preferences?
- Here's one that will do: $u(0) = 0$, $u(1) = 1$, and $u(5) = 4$

$$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 4 \geq \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 4$$

- $u(x) = x$ will also do just fine.

A Pure Strategy Nash Equilibrium Does Not Always Exist

In the "Matching Pennies" game, Alice and Bob choose which side of their coin to display, H or T. If they match, Alice wins; if they don't match, Bob wins.

Example

		Bob H	Bob T
Alice	H	1,-1	-1,1
	T	-1,1	1,-1

- If pennies match, Bob wants to deviate
- If pennies don't match, Alice wants to deviate

“Random” Strategies

- Let’s think about stochastic equilibria where players mix probabilistically over their pure strategies.

- To give some intuition for mixed strategies, think about Rock, Paper, Scissors. What is the best way to play this game?

Mixed Strategies

We will call these probabilistic or "random" strategies **mixed strategies**.

Definition

In the normal form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, suppose $S_i = \{s_{i1}, ..., s_{iK}\}$. Then a mixed strategy for player i is a probability distribution $p_i = (p_{i1}, .., p_{iK})$, where $0 \leq p_{ik} \leq 1$ for $k = 1, ..., K$ and $p_{i1} + ... + p_{iK} = 1$.

Nash Equilibrium in Mixed Strategies

We now define the concept of Nash equilibrium in mixed strategies:

Definition

In the normal form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, the mixed strategies $\{p_1^*, ..., p_n^*\}$ are a Nash equilibrium if and only if each player's mixed strategy is a best response to the other players' mixed strategies, that is, given p_{-i}^*:

$$EU(p_i^*, p_{-i}^*) \geq EU(\hat{p}_i, p_{-i}^*)$$

for all i, where $\hat{p}_i$ is any other possible mixed strategy for player i.

Back to Matching Pennies

Now we will show that matching pennies has a MSNE where each player randomizes 1/2, 1/2 over heads and tails.

1. Suppose Bob mixes and plays heads with probability 1/2 and tails with probability 1/2.
2. We have to show that doing the same is a best response for Alice (the reverse will follow by symmetry).
3. Let's write down Alice's utility function for a mixed strategy where she plays heads with probability p and tails with probability $1 - p$:

$$\begin{aligned} EU_A = & \; Pr(H,H)u_A(H,H) \; + \; Pr(H,T)u_A(H,T) + \\ & \; Pr(T,H)u_A(T,H) \; + \; Pr(T,T)u_A(T,T) \\ = & \; \left(\frac{1}{2}\right) p(1) + \left(\frac{1}{2}\right) p(-1) + \left(\frac{1}{2}\right) (1-p)(-1) + \left(\frac{1}{2}\right) (1-p)(1) \end{aligned}$$

Back to Matching Pennies

4. What p maximizes this utility function? Any p at all, including $\frac{1}{2}$.

5. We have shown that Bob playing $\left(\frac{1}{2}, \frac{1}{2}\right)$ makes Alice indifferent between any choice of p. By the symmetry of the game, the same is true for Bob: Alice playing $\left(\frac{1}{2}, \frac{1}{2}\right)$ makes Bob indifferent between any mixed strategy.

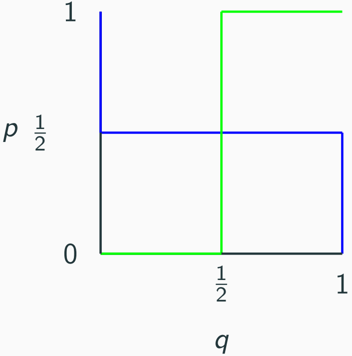
6. The intersection of their best responses is where Bob plays $\left(\frac{1}{2}, \frac{1}{2}\right)$ and Alice plays $\left(\frac{1}{2}, \frac{1}{2}\right)$. This is the MSNE.

Let's see this graphically.

Back to Matching Pennies: Best Response

Alice plays heads with probability p and Bob plays heads with probability q.

$$Eu_A(H) = q(1) + (1-q)(-1) \qquad Eu_B(H) = p(-1) + (1-p)(1)$$
$$Eu_A(T) = q(-1) + (1-q)(1) \qquad Eu_B(T) = p(1) + (1-p)(-1)$$
$$Eu_A(H) \leq Eu_A(T) \text{ if } q \leq \tfrac{1}{2} \qquad Eu_B(H) \leq Eu_B(T) \text{ if } p \geq \tfrac{1}{2}$$
$$Eu_A(H) \geq Eu_A(T) \text{ if } q \geq \tfrac{1}{2} \qquad Eu_B(H) \geq Eu_B(T) \text{ if } p \leq \tfrac{1}{2}$$



Best Responses and the Linearity of Expected Payoffs

Recall that:

Definition

A best response is the set of strategies that player i can play to maximize her payoff, given the strategy of the other players in the game $(-i)$.

Like in PSNE, in MSNE players are always playing a best response.

Best Responses and the Linearity of Expected Payoffs

Example (Outcome Probabilities in Matching Pennies)

		Bob	
		H(q)	T(1-q)
Alice	H(p)	pq	p(1-q)
	T(1-p)	(1-p)q	(1-p)(1-q)

Alice's expected utility from a mixed strategy profile p is:

$$pqu_A(H,H) + p(1-q)u_A(H,T) + (1-p)qu_A(T,H) + (1-p)(1-q)u_A(T,T)$$

which can be written:

$$p[qu_A(H,H) + (1-q)u_A(H,T)] + (1-p)[qu_A(T,H) + (1-q)u_A(T,T)].$$

Best Responses and the Linearity of Expected Payoffs

$$p[qu_A(H,H) + (1-q)u_A(H,T)] + (1-p)[qu_A(T,H) + (1-q)u_A(T,T)]$$

Notice that the term in the first bracket is Alice's expected payoff when she uses the pure strategy H and the term in the second bracket is her expected payoff when she uses the pure strategy T.

- That is, Alice's expected payoff to a mixed strategy profile is a weighted average of her pure strategy payoffs, given s_{-i}.

Expected utility is linear in p.

- Linearity implies either a pure strategy or any mixture.
- Equilibrium in mixed strategies requires both players to be indifferent between the pure strategies that they mix over.

Back to "Bach or Stravinsky"

There may be MSNE in games where PSNE also exist.

Example

		Bob	
		Mountain	Lake
Alice	Mountain	2,1	0,0
	Lake	0,0	1,2

What are the pure strategy Nash equilibria?

- (Mountain, Mountain)
- (Lake, Lake)

Back to "Bach or Stravinsky"

There is an additional equilibrium here. To see it, first construct Alice's best response function.

- Let Bob play Mountain with probability q.
- Remembering that we need to find Alice's best response, we must write down the expected utility of each pure strategy:

$$EU_A(M) = 2 \cdot q + 0 \cdot (1 - q) = 2q$$
$$EU_A(L) = 0 \cdot q + (1 - q) \cdot 1 = 1 - q$$

Back to "Bach or Stravinsky"

- So if $2q > 1 - q$, Alice's best response is Mountain, while if $2q < 1 - q$, then her best response is Lake.
- If $2q = 1 - q$ then both Mountain and Lake (and any mixture of the two) are best responses.

Thus we have for Alice:

$$B_A(q) = \begin{cases} \{0\} & \text{if } q < \frac{1}{3} \\ \{p : 0 \leq p \leq 1\} & \text{if } q = \frac{1}{3} \\ \{1\} & \text{if } q > \frac{1}{3}. \end{cases} \tag{1}$$

Back to “Bach or Stravinsky”

Now consider Bob. Suppose that Alice plays Mountain with probability p. Like for Alice, Bob's best response will depend on Alice's choice of p. Thus we need Bob's expected utilities:

$$EU_B(M) = 1 \cdot p + 0 \cdot (1 - p) = p$$
$$EU_B(L) = 0 \cdot p + (1 - p) \cdot 2 = 2 - 2p$$

Thus we have for Bob:

$$B_B(p) = \begin{cases} \{0\} & \text{if } p < \frac{2}{3} \\ \{q : 0 \leq q \leq 1\} & \text{if } p = \frac{2}{3} \\ \{1\} & \text{if } p > \frac{2}{3}. \end{cases}$$

Back to “Bach or Stravinsky”

These best response functions overlap at $p = \frac{2}{3}$ and $q = \frac{1}{3}$.

We write this MSNE as (Alice: Mountain, Lake; Bob: Mountain, Lake)

Thus, BoS has three equilibria:
$(\mathit{Mountain}, \mathit{Mountain}); (\mathit{Lake}, \mathit{Lake}); \left(\frac{2}{3}, \frac{1}{3}; \frac{1}{3}, \frac{2}{3}\right)$.

What Is a Public Good?

- What is a public good?
  - A public good is a good that, once provided, everyone can consume
  - Non-rival and non-excludable
  - Why does it seem so hard personally and politically to get public goods provided? Free riding.

- Examples: Clean public space, institutions, lighthouses, the search engine

The Reporting of Crime Game

- A crime is observed by a group of n people.
- Each person would like to see the crime reported and values the reporting at some utility v.
- Reporting is costly, but the cost c is smaller than the reporting value: $v > c > 0$.
- Each player can choose to call or not call.
- If the crime is reported and they don't call they get a payoff v.
- If it is not reported they get a payoff 0, and if they report the crime, they get a payoff $v - c$.

Pure Strategy Nash Equilibria

- There are many (n) pure strategy Nash equilibria to this game (what are they?)
- All are asymmetric, where one person calls and all the others don't.
- One might then ask, how do the players coordinate on such an equilibrium?
- Since there is nothing about our game that makes coordination natural, maybe we don't like these asymmetric equilibria.

Therefore we might wonder: are there any symmetric mixed strategy Nash equilibria?

Finding a Symmetric Mixed Strategy Nash Equilibrium

For each person, the payoff to calling is fixed at $v - c > 0$.

The expected utility of not calling equals:

$$0 \cdot Pr(\text{no one calls}) + v \cdot Pr(\text{at least one person calls})$$

Setting these two equal to each other, where p is the probability that some person calls, we get:

$$\begin{aligned} & v - c = v \cdot \left(1 - (1-p)^{n-1}\right) \\ \iff \quad & \frac{c}{v} = (1-p)^{n-1} \\ \iff \quad & p = 1 - \left(\frac{c}{v}\right)^{1/(n-1)} \end{aligned}$$

Mixed Strategy Nash Equilibrium

- So there is a symmetric NE to this game where each person calls with probability $p = 1 - (\frac{c}{v})^{1/(n-1)}$.
- What happens as n increases? The police are less likely to be notified.

$$\text{Pr(no one calls)} = \text{Pr}(i \text{ doesn't call}) \cdot \text{Pr(No one else calls either)}$$

The probability any given person calls decreases in n and the probability no one else calls is just $(1-p)^{n-1} = c/v$ (see previous slide), which doesn't depend on n.

Public goods provision becomes harder as n increases (free riding). (https://en.wikipedia.org/wiki/Murder_of_Kitty_Genovese)

Defense against Terrorism

- Antiterrorism is fundamentally a problem of too many potential targets and not enough guards.
- Let's think about two players, **Terrorist** and **Government**.
- There are two targets, and each player has two resource units to allocate.
- If the terrorist allocates more resources to attacking a target than the government allocated to defending it, the terrorist has a successful attack.
- Otherwise, the attack is defeated.
- The terrorist hopes to successfully attack at least one target.
- The government successfully defends the country only if no attack is successful.

Matrix Game

Each player has 3 strategies: one resource to each target, two resources on target A, and two resources on target B.

Example

Government

		1:1	2:0	0:2
Terrorist	1:1	0,1	1,0	1,0
	2:0	1,0	0,1	1,0
	0:2	1,0	1,0	0,1

No pure strategy Nash equilibrium.

Matrix Game: Government's Expected Utility

Example

		Government		
		1:1 (q_1)	2:0 (q_2)	0:2 ($1 - q_1 - q_2$)
Terrorist	1:1 (p_1)	0,1	1,0	1,0
	2:0 (p_2)	1,0	0,1	1,0
	0:2 ($1 - p_1 - p_2$)	1,0	1,0	0,1

For a mixture $(p_1, p_2, 1 - p_1 - p_2)$, MSNE implies:

$$E[u_G(1:1)] = (p_1 * 1) + (p_2 * 0) + ((1 - p_1 - p_2) * 0) = p_1$$
$$= E[u_G(2:0)] = p_2$$
$$= E[u_G(0:2)] = 1 - p_1 - p_2$$

Solving the system of equations $p_1 = p_2 = 1 - p_1 - p_2$, $\mathbf{p} = (1/3, 1/3, 1/3)$.

Matrix Game: Terrorist's Expected Utility

Example

		Government 1:1 (q_1)	2:0 (q_2)	0:2 ($1 - q_1 - q_2$)
Terrorist	1:1 (p_1)	0,1	1,0	1,0
	2:0 (p_2)	1,0	0,1	1,0
	0:2 ($1 - p_1 - p_2$)	1,0	1,0	0,1

For a mixture $(q_1, q_2, 1 - q_1 - q_2)$, MSNE implies:

$$E[u_T(1:1)] = (q_1 * 0) + (q_2 * 1) + (1 - q_1 - q_2 * 1) = 1 - q_1$$
$$= E[u_T(2:0)] = q_1 + (1 - q_1 - q_2) = 1 - q_2$$
$$= E[u_T(0:2)] = q_1 + q_2$$

Solving the system of equations $1 - q_1 = 1 - q_2 = q_1 + q_2$, $\mathbf{q} = (1/3, 1/3, 1/3)$.