17.810/17.811: Problem Set 2

Read the following instructions carefully:

- All answers must be typed or clearly written up and stapled.
- Late submission of the write-up will not be accepted.
- You are encouraged to work in groups after a solo effort has been taken first, but you should write up your answers alone and tell us who you worked with
- For analytical (proofs) questions, you should include your detailed derivation for all intermediate steps (logical arguments).

Set Up

Here is a recap of definitions, concepts and propositions that may be useful for this problem set.

Definition 1. *Nash Equilibrium: A strategy profile* $(s_1^*, \ldots, s_n^*)$ *is a (pure strategy)* ***Nash equilibrium*** *iff*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(\hat{s}_i, s_{-i}^*) \text{ for all } \hat{s}_i \in S_i \text{ and for every } i \in N.$$

That is, in a **Nash equilibrium** no one has an incentive to change their strategy, given the strategies of other players.

Definition 2. *Strictly Dominated Strategy: An strategy* $s_i \in S_i$ *is* ***strictly dominated*** *by* s_i' *for player* i *if for all* $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$.

Definition 3. *Weakly Dominated Strategy: An strategy* $s_i \in S_i$ *is* ***weakly dominated*** *by* s_i' *for player* i *if for all* $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$.

Proposition 1. *In the n-player normal-form game* $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, *if iterated elimination of strictly dominated strategies eliminates all but the strategies* $(s_1^*, ...s_n^*)$, *then these strategies are the unique Nash equilibrium of the game.*

Part I - Game Set up and Dominated Strategy

	b_1	b_2	b_3	b_4	b_5
a_1	$10, 20$	$10, 2$	$10, 4$	$4, 3$	$1, 5$
a_2	$5, 3$	$0, 4$	$2, 1$	$3, 5$	$4, -10$
a_3	$7, -1$	$-1, -1$	$3, 6$	$6, 0$	$5, 0$
a_4	$11, 10$	$-2, -1$	$3, 2$	$-1, 1$	$2, 1$
a_5	$12, 6$	$-10, 3$	$4, 7$	$-8, 6$	$6, 8$
a_6	$4, 4$	$-1, 5$	$0, 1$	$1, 2$	$0, -8$

Problem 1

Find all strictly dominated and the respective dominant strategies[1].

Problem 2

Is this game strict-dominance solvable? If so, what is the solution? If not, which strategies remain?

Problem 3

Find all pure strategy Nash equilibria, and explain why you have found all of them.

[1] Only do the first iteration for this problem

Part II - Distributive Politics[2]

Set Up

There are n legislators in the Congress, and each of them represents a constituency. Suppose the utility function of each legislator j is

$$N_j(x_j) = b(x_j) - t_j T(x_j, x_{-j}), \tag{1}$$

where x_j is the amount of public spending on public projects in district j, t_j is the share of aggregated taxation that district j is paying. $T(x_j, x_{-j})$ is the national aggregated taxation. Meanwhile, $b(x_j)$ is the benefits that district j could get directly from public spending x_j in its district.

We assume (a) the cost for investing in public projects in each district j is $c(x_j) = x_j$, (b) balanced budget, which means the cost of all public projects equals the total taxation, or formally $T(x_j, x_{-j}) = \sum_{i=1}^{n} c(x_i)$, (c) taxation is allocated across districts in proportion to population within each district, or formally, $t_j = \frac{n_j}{\sum_{i=1}^{n} n_i}$ where n_j is the population in each district j, (d) the benefits district j could obtain from public spendings x_j is $b(x_j) = x_j^{\frac{1}{2}}$, (e) legislators can only play pure strategies, and (f) public spendings cannot be negative.

Problem 1

Formulate the game into a normal-form game by specifying set of players I, set of strategies (strategy space) S_j and the pay-off function for each legislator j.

Problem 2

Suppose there is only $n = 1$ constituency district in the country, what is the optimal level of public goods provision x^*? By optimal level, I mean the level that produces largest sum of net benefits for the whole country.

Problem 3

Suppose there are $n = 2$ constituency districts in the country, what is the level of public goods provision in each district, x_1^* and x_2^* at Nash Equilibrium? What is the total amount of public spending in the country now? Is it larger than the total amount of public spending when the whole country is a single constituency?

[2] This problem is based on Weingast, Barry R., Kenneth A. Shepsle, and Christopher Johnsen. "The political economy of benefits and costs: A neoclassical approach to distributive politics." *Journal of Political Economy* 89, no. 4 (1981): 642-664. You may find it helpful to read the paper. However, the problem is self-contained, and requires no more knowledge beyond what has been covered in class.

Problem 4

Suppose there are n constituency districts in the country, what is the level of public goods provision in each district, x_j^* at Nash Equilibrium? What is the total amount of public spending in the country now? Is it larger than the total amount of public spending when the whole country is a single constituency?

Problem 5

How will a legislator changes the level of public goods provision in her district x_j^* when the population of her district decreases?

Part III - Auctions

We first consider a **second-price sealed bid auction**. There are n bidders in the auction, and each of them has a non-negative private valuation v_i, i.e. the largest amount of money she is willing to pay for the product under auction, and $v_i \geq 0$. By private, we mean that only the bidder knows about her private valuation, while neither other bidders nor the seller knows about this. Without loss of generality, we assume $v_1 > v_2 > ... > v_n$.

At the auction, each bidder submits a sealed envelope containing their bid for the product. We assume each bidder submits her bid independently, and bidders cannot collude with each other. Denote the bidding price that bidder i writes in her envelope as p_i. After each bidder has submitted her envelope, the seller opens all envelopes, and sells the product to the highest bidder, i.e. the bidder j whose bidding price p_j is the highest among all bidders, at the price of the second highest bidder. The winner has to pay for the product and cannot default. If there are multiple winners, nobody gets the product.

Problem 1

Formulate the second-price sealed bid auction as a normal-form game by specifying set of players I, set of strategies (strategy space) S_i and the pay-off function u_i.

Problem 2

Show that for each bidder i, sincere bidding, i.e. to write bidding price $p_i = v_i$, their valuation, is a weakly dominant strategy over all other pure strategies.

Problem 3

Is sincere bidding is a Nash Equilibrium of the auction? Prove or disprove your statement.

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