Professor: In today's lecture I want to develop several more formulas that will allow us to reach our goal of differentiating everything. So these are derivative formulas, and they come in two flavors. The first kind is specific, so some specific function we're giving the derivative of. And that would be, for example, x^n or (1/x) . Those are the ones that we did a couple of lectures ago. And then there are general formulas, and the general ones don't actually give you a formula for a specific function but tell you something like, if you take two functions and add them together, their derivative is the sum of the derivatives. Or if you multiply by a constant, for example, so c times u, the derivative of that is c times u' where c is constant.

All right, so these kinds of formulas are very useful, both the specific and the general kind. For example, we need both kinds for polynomials. And more generally, pretty much any set of formulas that we give you, will give you a few functions to start out with and then you'll be able to generate lots more by these general formulas. So today, we wanna concentrate on the trig functions, and so we'll start out with some specific formulas. And they're going to be the formulas for the derivative of the sine function and the cosine function.

So that's what we'll spend the first part of the lecture on, and at the same time I hope to get you very used to dealing with trig functions, although that's something that you should think of as a gradual process.

Alright, so in order to calculate these, I'm gonna start over here and just start the calculation. So here we go. Let's check what happens with the sine function. So, I take sin (x + delta x), I subtract sin x and I divide by delta x. Right, so this is the difference quotient and eventually I'm gonna have to take the limit as delta x goes to 0. And there's really only one thing we can do with this to simplify or change it, and that is to use the sum formula for the sine function. So, that's this. That's sin x cos delta x plus--

Oh, that's not what it is? OK, so what is it? sin x sin delta x. OK, good. Plus cosine. No? Oh, OK. So which is it? OK. Alright, let's take a vote. Is it sine, sine, or is it sine, cosine?

Audience: [INAUDIBLE]

Professor: OK, so is this going to be... cosine. All right, you better remember these formulas, alright? OK, turns out that it's sine, cosine. All right. Cosine, sine. So here we go, no gotta do x here, sin (delta x). Alright, so now there's lots of places to get confused here, and you're gonna need to make sure you get it right. Alright, so we're gonna put those in parentheses here. sin (a + b) is sin a cos b plus cos a sin b. All right, now that's what I did over here, except the letter x was a, and the letter b was delta x. Now that's just the first part. That's just this part of the expression. I still have to remember the minus sin x. That comes at the end. Minus sin x. And then, I have to remember the denominator, which is delta x. OK?

Alright, so now... The next thing we're gonna do is we're gonna try to group the terms. And the difficulty with all such arguments is the following one: any tricky limit is basically 0 / 0 when you set delta x equal to 0. If I set delta x equal to 0, this is sin x - sin x. So it's a 0 / 0 term. Here we have various things which are 0 and various things which are non-zero. We must group the terms so that a 0 stays over a 0. Otherwise, we're gonna have no hope. If we get some 1 / 0 term, we'll get something meaningless in the limit. So I claim that the right thing to do here is to notice, and I'll just point out this one thing. When delta x goes to 0, this cosine of 0 is 1. So it doesn't cancel unless we throw in this extra sine term here. So I'm going to use this common factor, and combine those terms. So this is really the only thing you're gonna have to check in this particular calculation. So we have the common factor of sin x, and that multiplies something that will cancel, which is (cos delta x - 1) / delta x. That's the first term, and now what's left, well there's a cos x that factors out, and then the other factor is (sin delta x) / (delta x).

OK, now does anyone remember from last time what this thing goes to? How many people say 1? How many people say 0? All right, it's 0. That's my favorite number, alright? 0. It's the easiest number to deal with. So this goes to 0, and that's what happens as delta x tends to 0. How about this one? This one goes to 1, my second favorite number, almost as easy to deal with as 0. And these things are picked for a reason. They're the simplest numbers to deal with. So altogether, this thing as delta x goes to 0 goes to what? I want a single person to answer, a brave volunteer. Alright, back there.

Student: Cosine

Professor: Cosine, because this factor is 0. It cancels and this factor has a 1, so it's cosine. So it's cos x.

So our conclusion over here - and I'll put it in orange - is that the derivative of the sine is the cosine. OK, now I still wanna label these very important limit facts here. This one we'll call A, and this one we're going to call B, because we haven't checked them yet. I promised you I would do that, and I'll have to do that this time. So we're relying on those things being true. Now I'm gonna do the same thing with the cosine function, except in order to do it I'm gonna have to remember the sum rule for cosine. So we're gonna do almost the same calculation here. We're gonna see that that will work out, but now you have to remember that cos (a + b) = cos cos, no it's not cosine^2, because there are two different quantities here. It's cos a cos b - sin a sin b. All right, so you'll have to be willing to call those forth at will right now.

So let's do the cosine now. So that's cos (x + delta x) - cos x divided by delta x. OK, there's the difference quotient for the cosine function. And now I'm gonna do the same thing I did before except I'm going to apply the second rule, that is the sum rule for cosine. And that's gonna give me cos x cos delta x - sin x sin delta x. And I have to remember again to subtract the cosine divided by this delta x. And now I'm going to regroup just the way I did before, and I get the common factor of cosine multiplying (cos delta x - 1) / delta x. And here I get the sin x but actually it's -sin x. And then I have (sin delta x) / delta x. All right? The only difference is this minus sign which I stuck inside there. Well that's not the only difference, but it's a crucial difference.

OK, again by A we get that this is 0 as delta x tends to 0. And this is 1. Those are the properties I called A and B. And so the result here as delta x tends to 0 is that we get negative sin x. That's the factor. So this guy is negative sin x. I'll put a little box around that too. Alright, now these formulas take a little bit of getting used to, but before I do that I'm gonna explain to you the proofs of A and B. So we'll get ourselves started by mentioning that. Maybe before I do that though, I want to show you how A and B fit into the proofs of these theorems. So, let me just make some remarks here. So this is just a remark but it's meant to help you to frame how these proofs worked. So, first of all, I want to point out that if you take the rate of change of sin x, no let's start with cosine because a little bit less obvious. If I take the rate of change of cos x, so in other words this derivative at x = 0, then by definition this is a certain limit as delta x goes to 0.

So which one is it?

Well I have to evaluate cosine at 0 + delta x, but that's just delta x. And I have to subtract cosine at 0. That's the base point, but that's just 1. And then I have to divide by delta x. And lo and behold you can see that this is exactly the limit that we had over there. This is the one that we know is 0 by what we call property A. And similarly, if I take the derivative of sin x at x=0, then that's going to be the limit as delta x goes to 0 of sin delta x / delta x. And that's because I should be subtracting sine of 0 but sine of 0 is 0.

Right?

So this is going to be 1 by our property B. And so the remark that I want to make, in addition to this, is something about the structure of these two proofs. Which is the derivatives of sine and cosine at x = 0 give all values of d/dx sin x, d/dx cos x. So that's really what this argument is showing us, is that we just need one rate of change at one place and then we work out all the rest of them.

So that's really the substance of this proof. That of course really then shows that it boils down to showing what this rate of change is in these two cases. So now there's enough suspense that we want to make sure that we know that those answers are correct.

OK, so let's demonstrate both of them. I'll start with B. I need to figure out property B. Now, we only have one alternative as to a type of proof that we can give of this kind of result, and that's because we only have one way of describing sine and cosine functions, that is geometrically. So we have to give a geometric proof. And to write down a geometric proof we are going to have to draw a picture. And the first step in the proof, really, is to replace this variable delta x which is going to 0 with another name which is suggestive of what we're gonna do which is the letter theta for an angle. OK, so let's draw a picture of what it is that we're going to do. Here is the circle. And here is the origin. And here's some little angle, well I'll draw it a little larger so it's visible. Here's theta, alright? And this is the unit circle. I won't write that down on here but that's the unit circle. And now sin theta is this vertical distance here. Maybe, I'll draw it in a different color so that we can see it all. OK so here's this distance. This distance is sin theta.

OK?

Now almost the only other thing we have to write down in this picture to have it work out is that we have to recognize that when theta is the angle, that's also the arc length of this piece of the circle when measured in radians. So this length here is also arc length theta. That little piece in there. So maybe I'll use a different color for that to indicate it. So that's orange and that's this little chunk there. So those are the two pieces.

Now in order to persuade you now that the limit is what it's supposed to be, I'm going to extend the picture just a little bit. I'm going to double it, just for my own linguistic sake and so that I can tell you a story. Alright, so that you'll remember this. So I'm going to take a theta angle below and I'll have another copy of sin theta down here. And now the total picture is really like a bow and its bow string there.

Alright?

So what we have here is a length of 2 sin theta. So maybe I'll write it this way, 2 sin theta. I just doubled it. And here I have underneath, whoops, I got it backwards. Sorry about that. Trying to be fancy with the colored chalk and I have it reversed here. So this is not 2 sin theta. 2 sin theta is the vertical. That's the green. So let's try that again. This is 2 sin theta, alright? And then in the denominator I have the arc length which is theta is the first half and so double it is 2 theta.

Alright?

So if you like, this is the bow and up here we have the bow string. And of course we can cancel the 2's. That's equal to sin theta / theta. And so now why does this tend to 1 as theta goes to 0? Well, it's because as the angle theta gets very small, this curved piece looks more and more like a straight one. Alright? And if you get very, very close here the green segment and the orange segment would just merge. They would be practically on top of each other. And they have closer and closer and closer to the same length. So that's why this is true.

I guess I'll articulate that by saying that short curves are nearly straight. Alright, so that's the principle that we're using. Or short pieces of curves, if you like, are nearly straight. So if you like, this is the principle. So short pieces of curves. Alright?

So now I also need to give you a proof of A. And that has to do with this cosine function here. This is the property A. So I'm going to do this by flipping it around, because it turns out that this numerator is a negative number. If I want to interpret it as a length, I'm gonna want a positive quantity. So I'm gonna write down 1 - cos theta here and then I'm gonna divide by theta there. Again I'm gonna make some kind of interpretation. Now this time I'm going to draw the same sort of bow and arrow arrangement, but maybe I'll exaggerate it a little bit. So here's the vertex of the sector, but we'll maybe make it a little longer.

Alright, so here it is, and here was that middle line which was the unit-- Whoops. OK, I think I'm going to have to tilt it up. OK, let's try from here. Alright, well you know on your pencil and paper it will look better than it does on my blackboard. OK, so here we are. Here's this shape. Now this angle is supposed to be theta and this angle is another theta. So here we have a length which is again theta and another length which is theta over here. That's the same as in the other picture, except we've exaggerated a bit here. And now we have this vertical line, which again I'm gonna draw in green, the bow string. But notice that as the vertex gets farther and farther away, the curved line gets closer and closer to being a vertical line. That's sort of the flip side, by expansion, of the zoom in principle. The principle that curves are nearly straight when you zoom in. If you zoom out that would mean sending this vertex way, way out somewhere. The curved line, the piece of the circle, gets more and more straight. And now let me show you where this numerator 1 - cos theta is on this picture.

So where is it? Well, this whole distance is 1. But the distance from the vertex to the green is cosine of theta. Right, because this is theta, so dropping down the perpendicular this distance back to the origin is cos theta. So this little tiny, bitty segment here is basically the gap between the curve and the vertical segment. So the gap is equal to 1 - cos theta. So now you can see that as this point gets farther away, if this got sent off to the Stata Center, you would hardly be able to tell the difference. The bow string would coincide with the bow and this little gap between the bow string and the bow would be tending to 0. And that's the statement that this tends to 0 as theta tends to 0. The scaled version of that. Yeah, question down here.

Student: Doesn't the denominator also tend to 0 though?

Professor: Ah, the question is "doesn't the denominator also tend to 0?" And the answer is yes. In my strange analogy with zooming in, what I did was I zoomed out the picture. So in other words, if you imagine you're taking this and you're putting it under a microscope over here and you're looking at something where theta is getting smaller and smaller and smaller and smaller.

Alright?

But now because I want my picture, I expanded my picture. So the ratio is the thing that's preserved. So if I make it so that this gap is tiny... Let me say this one more time. I'm afraid I've made life complicated for myself. If I simply let this theta tend in to 0, that would be the same effect as making this closer and closer in and then the vertical would approach. But I want to keep on blowing up the picture so that I can see the difference between the vertical and the curve. So that's very much like if you are on a video screen and you zoom in, zoom in, zoom in, and zoom in. So the question is what would that look like? That has the same effect as sending this point out farther and farther in that direction, to the left. And so I'm just trying to visualize it for you by leaving the theta at this scale, but actually the scale of the picture is then changing when I do that. So theta is going to 0, but I I'm rescaling so that it's of a size that we can look at it, And then imagine what's happening to it. OK, does that answer your question?

Student: My question then is that seems to prove that that limit is equal to 0/0.

Professor: It proves more than it is equal to 0/0. It's the ratio of this little short thing to this longer thing. And this is getting much, much shorter than this total length. You're absolutely right that we're comparing two quantities which are going to 0, but one of them is much smaller than the other. In the other case we compared two quantities which were both going to 0 and they both end up being about equal in length. Here the previous one was this green one. Here it's this little tiny bit here and it's way shorter than the 2 theta distance. Yeah, another question.

Student: cos theta - 1 over cos theta is the same as 1- cos theta over theta?

Professor: cos theta - 1 over...

Student: [INAUDIBLE]

Professor: So here, what I wrote is (cos delta x - 1) / delta x, OK, and I claimed that it goes to 0. Here, I wrote minus that, that is I replaced delta x by theta. But then I wrote this thing. So (cos theta - 1) minus 1 is the negative of this.

Alright?

And if I show that this goes to 0, it's the same as showing the other one goes to 0. Another question?

Student: [INAUDIBLE]

Professor: So the question is, what about this business about arc length. So the word arc length, that orange shape is an arc. And we're just talking about the length of that arc, and so we're calling it arc length. That's what the word arc length means, it just means the length of the arc.

Student: [INAUDIBLE]

Professor: Why is this length theta? Ah, OK so this is a very important point, and in fact it's the very next point that I wanted to make. Namely, notice that in this calculation it was very important that we used length. And that means that the way that we're measuring theta, is in what are known as radians. Right, so that applies to both B and A, it's a scale change in A and doesn't really matter but in B it's very important. The only way that this orange length is comparable to this green length, the vertical is comparable to the arc, is if we measure them in terms of the same notion of length. If we measure them in degrees, for example, it would be completely wrong. We divide up the angles into 360 degrees, and that's the wrong unit of measure. The correct measure is the length along the unit circle, which is what radians are. And so this is only true if we use radians.

So again, a little warning here, that this is in radians. Now here x is in radians. The formulas are just wrong if you use other units. Ah yeah?

Student: [INAUDIBLE].

Professor: OK so the second question is why is this crazy length here 1. And the reason is that the relationship between this picture up here and this picture down here, is that I'm drawing a different shape. Namely, what I'm really imagining here is a much, much smaller theta. OK? And then I'm blowing that up in scale. So this scale of this picture down here is very different from the scale of the picture up there. And if the angle is very, very, very small then one has to be very, very long in order for me to finish the circle. So, in other words, this length is 1 because that's what I'm insisting on. So, I'm claiming that that's how I define this circle, to be of unit radius. Another question?

Student: [INAUDIBLE] the ratio between 1 - cos theta and theta will get closer and closer to 1. I don't understand [INAUDIBLE].

Professor: OK, so the question is it's hard to visualize this fact here. So let me, let me take you through a couple of steps, because I think probably other people are also having trouble with this visualization. The first part of the visualization I'm gonna try to demonstrate on this picture up here. The first part of the visualization is that I should think of a beak of a bird closing down, getting narrower and narrower. So in other words, the angle theta has to be getting smaller and smaller and smaller. OK, that's the first step. So that's the process that we're talking about. Now, in order to draw that, once theta gets incredibly narrow, in order to depict that I have to blow the whole picture back up in order be able to see it. Otherwise it just disappears on me. In fact in the limit theta = 0, it's meaningless. It's just a flat line. That's the whole problem with these tricky limits. They're meaningless right at the zero-zero level. It's only just a little away that they're actually useful, that you get useful geometric information out of them.

So we're just a little away. So that's what this picture down below in part A is meant to be. It's supposed to be that theta is open a tiny crack, just a little bit. And the smallest I can draw it on the board for you to visualize it is using the whole length of the blackboard here for that. So I've opened a little tiny bit and by the time we get to the other end of the blackboard, of course it's fairly wide. But this angle theta is a very small angle.

Alright? So I'm trying to imagine what happens as this collapses. Now, when I imagine that I have to imagine a geometric interpretation of both the numerator and the denominator of this quantity here. And just see what happens. Now I claimed the numerator is this little tiny bit over here and the denominator is actually half of this whole length here. But the factor of 2 doesn't matter when you're seeing whether something tends to 0 or not. Alright? And I claimed that if you stare at this, it's clear that this is much shorter than that vertical curve there. And I'm claiming, so this is what you have to imagine, is this as it gets smaller and smaller and smaller still that has the same effect of this thing going way, way way, farther away and this vertical curve getting closer and closer and closer to the green. And so that the gap between them gets tiny and goes to 0. Alright? So not only does it go to 0, that's not enough for us, but it also goes to 0 faster than this theta goes to 0. And I hope the evidence is pretty strong here because it's so tiny already at this stage.

Alright. We are going to move forward and you'll have to ponder these things some other time. So I'm gonna give you an even harder thing to visualize now so be prepared. OK, so now, the next thing that I'd like to do is to give you a second proof. Because it really is important, I think, to understand this particular fact more thoroughly and also to get a lot of practice with sines and cosines. So I'm gonna give you a geometric proof of the formula for sine here, for the derivative of sine. So here we go. This is a geometric proof of this fact. This is for all theta. So far we only did it for theta = 0 and now we're going to do it for all theta. So this is a different proof, but it uses exactly the same principles.

Right? So, I want do this by drawing another picture, and the picture is going to describe y, which is sin theta, which is if you like the vertical position of some circular motion. So I'm imagining that something is going around in a circle. Some particle is going around in a circle. And so here's the circle, here the origin. This is the unit distance. And right now it happens to be at this location P. Maybe we'll put P a little over here. And here's the angle theta. And now we're going to move it. We're going to vary theta and we're interested in the rate of change of y. So y is the height of P but we're gonna move it to another location. We'll move it along the circle to Q. Right? So here it is. Here's the thing. So how far did we move it? Well we moved it by an angle delta theta. So we started theta, theta is going to be fixed in this argument, and we're going to move a little bit delta theta. And now we're just gonna try to figure out how far the thing moved. Well, in order to do that we've got to keep track of the height, the vertical displacement here. So we're going to draw this right angle here, this is the position R. And then this distance here is the change in y. Alright? So the picture is we have something moving around a unit circle. A point moving around a unit circle. It starts at P, it moves to Q. It moves from angle theta to angle theta plus delta theta. And the issue is how much does y move? And the formula for y is sin theta. So that's telling us the rate of change of sin theta.

Alright, well so let's just try to think a little bit about what this is. So, first of all, I've already said this and I'm going to repeat it here. Delta y is PR. It's going from P and going straight up to R. That's how far y moves. That's the change in y That's what I said up in the right hand corner there. Oops. I said PR but I wrote PQ. Alright, that's not a good idea. Alright. So delta Y is PR. And now I want to draw the diagram again one time. So here's Q, here's R, and here's P, and here's my triangle.

And now what I'd like to do is draw this curve here which is a piece of the arc of the circle. But really what I want to keep in mind is something that I did also in all these other arguments. Which is, maybe I should have called this orange, that I'm gonna think of the straight line between. So it's the straight line approximation to the curve that we're always interested in. So the straight line is much simpler, because then we just have a triangle here. And in fact it's a right triangle. Right, so we have the geometry of a right triangle which is going to now let us do all of our calculations. OK, so now the key step is this same principle that we already used which is that short pieces of curves are nearly straight. So that means that this piece of the circular arc here from P to Q is practically the same as the straight segment from P to Q. So, that's this principle that - well, let's put it over here - Is that PQ is practically the same as the straight segment from P to Q.

So how are we going to use that?

We want to use that quantitatively in the following way. What we want to notice is that the distance from P to Q is approximately delta theta. Right? Because the arc length along that curve, the length of the curve is delta theta. So the length of the green which is PQ is almost delta theta. So this is essentially delta theta, this distance here. Now the second step, which is a little trickier, is that we have to work out what this angle is. So our goal, and I'm gonna put it one step below because I'm gonna put the geometric reasoning in between, is I need to figure out what the angle QPR is. If I can figure out what this angle is, then I'll be able to figure out what this vertical distance is because I'll know the hypotenuse and I'll know the angle so I'll be able to figure out what the side of the triangle is.

So now let me show you why that's possible to do. So in order to do that first of all I'm gonna trade the boards and show you where the line PQ is. So the line PQ is here. That's the whole thing. And the key point about this line that I need you to realize is that it's practically perpendicular, it's almost perpendicular, to this ray here. Alright? It's not quite because the distance between P to Q is non-zero. So it isn't quite, but in the limit it's going to be perpendicular. Exactly perpendicular. The tangent line to the circle. So the key thing that I'm going to use is that PQ is almost perpendicular to OP. Alright? The ray from the origin is basically perpendicular to that green line. And then the second thing I'm going to use is something that's obvious which is that PR is vertical. OK? So those are the two pieces of geometry that I need to see. And now notice what's happening upstairs on the picture here in the upper right. What I have is the angle theta is the angle between the horizontal and OP. That's angle theta. If I rotate it by ninety degree, the horizontal becomes vertical. It becomes PR and the other thing rotated by 90 degrees becomes the green line. So the angle that I'm talking about I get by taking this guy and rotating it by 90 degrees. It's the same angle. So that means that this angle here is essentially theta. That's what this angle is. Let me repeat that one more time.

We started out with an angle that looks like this, which is the horizontal-- that's the origin straight out horizontally. That's the thing labeled 1. That distance there. That's my right arm which is down here. My left arm is pointing up and it's going from the origin to the point P. So here's the horizontal and the angle between them is theta. And now, what I claim is that if I rotate by 90 degrees up, like this, without changing anything - so that was what I did - the horizontal will become a vertical. That's PR. That's going up, PR. And if I rotate OP 90 degrees, that's exactly PQ.

OK?

So let me draw it on there one time. Let's do it with some arrows here. So I started out with this and then, we'll label this as orange, OK so red to orange, and then I rotate by 90 degrees and the red becomes this starting from P and the orange rotates around 90 degrees and becomes this thing here. Alright? So this angle here is the same as the other one which I've just drawn. Different vertices for the angles.

OK?

Well I didn't say that all arguments were supposed to be easy. Alright, so I claim that the conclusion is that this angle is approximately theta. And now we can finish our calculation, because we have something with the hypotenuse being delta theta and the angle being theta and so this segment here PR is approximately the hypotenuse length times the cosine of the angle. And that is exactly what we wanted. If we divide, we divide by delta theta, we get (delta y) / (delta theta) is approximately cos theta. And that's the same thing as... So what we want in the limit is exactly the delta theta going to 0 of (delta y) / (delta theta) is equal to cos theta. So we get an approximation on a scale that we can visualize and in the limit the formula is exact.

OK, so that is a geometric argument for the same result. Namely that the derivative of sine is cosine. Yeah?

Student: [INAUDIBLE].

Professor: You will have to do some kind of geometric proofs sometimes. When you'll really need this is probably in 18.02. So you'll need to make reasoning like this. This is, for example, the way that you actually develop the theory of arc length. Dealing with delta x's and delta y's is a common tool. Alright, I have one more thing that I want to talk about today, which is some general rules. We took a little bit more time than I expected with this. So what I'm gonna do is just tell you the rules and we'll discuss them in a few days. So let me tell you the general rules. So these were the specific ones and here are some general ones. So the first one is called the product rule. And what it says is that if you take the product of two functions and differentiate them, you get the derivative of one times the other plus the other times the derivative of the one. Now the way that you should remember this, and the way that I'll carry out the proof, is that you should think of it is you change one at a time. And this is a very useful way of thinking about differentiation when you have things which depend on more than one function. So this is a general procedure. The second formula that I wanted to mention is called the quotient rule and that says the following. That (u / v) prime has a formula as well. And the formula is (u'v - uv') / v^2. So this is our second formula. Let me just mention, both of them are extremely valuable and you'll use them all the time. This one of course only works when v is not 0.

Alright, so because we're out of time we're not gonna prove these today but we'll prove these next time and you're definitely going to be responsible for these kinds of proofs.