SOLUTIONS TO 18.01 EXERCISES

Unit 3. Integration

3A. Differentials, indefinite integration

3A-1 a) $7x^6dx$. ($d(\sin 1) = 0$ because $\sin 1$ is a constant.)

b) $(1/2)x^{-1/2}dx$

c) $(10x^9 - 8)dx$

d) $(3e^{3x}\sin x + e^{3x}\cos x)dx$

e) $(1/2\sqrt{x})dx + (1/2\sqrt{y})dy = 0$ implies

$$dy = -\frac{1/2\sqrt{x}dx}{1/2\sqrt{y}} = -\frac{\sqrt{y}}{\sqrt{x}}dx = -\frac{1-\sqrt{x}}{\sqrt{x}}dx = \left(1 - \frac{1}{\sqrt{x}}\right)dx$$

3A-2 a) $(2/5)x^5 + x^3 + x^2/2 + 8x + c$

b) $(2/3)x^{3/2} + 2x^{1/2} + c$

c) Method 1 (slow way) Substitute: $u = 8 + 9x$, $du = 9dx$. Therefore

$$\int \sqrt{8+9x}dx = \int u^{1/2}(1/9)du = (1/9)(2/3)u^{3/2} + c = (2/27)(8+9x)^{3/2} + c$$

Method 2 (guess and check): It's often faster to guess the form of the antiderivative and work out the constant factor afterwards:

$$\text{Guess } (8+9x)^{3/2}; \quad \frac{d}{dx}(8+9x)^{3/2} = (3/2)(9)(8+9x)^{1/2} = \frac{27}{2}(8+9x)^{1/2}.$$

So multiply the guess by $\frac{2}{27}$ to make the derivative come out right; the answer is then

$$\frac{2}{27}(8+9x)^{3/2} + c$$

d) Method 1 (slow way) Use the substitution: $u = 1 - 12x^4$, $du = -48x^3dx$.

$$\int x^3(1-12x^4)^{1/8}dx = \int u^{1/8}(-1/48)du = -\frac{1}{48}(8/9)u^{9/8}+c = -\frac{1}{54}(1-12x^4)^{9/8}+c$$

Method 2 (guess and check): guess $(1-12x^4)^{9/8}$;

$$\frac{d}{dx}(1-12x^4)^{9/8} = \frac{9}{8}(-48x^3)(1-12x^4)^{1/8} = -54(1-12x^4)^{1/8}.$$

So multiply the guess by $-\frac{1}{54}$ to make the derivative come out right, getting the previous answer.

e)

$$\int \frac{x}{\sqrt{8-2x^2}}dx$$

$$= -\frac{\sqrt{8-2x^2}}{2} + c$$

The next four questions you should try to do (by Method 2) in your head. Write down the correct form of the solution and correct the factor in front.

f) $(1/7)e^{7x} + c$

g) $(7/5)e^{x^5} + c$

h) $2e^{\sqrt{x}} + c$

i) $(1/3)\ln(3x+2)+c$. For comparison, let's see how much slower substitution is:

$$u = 3x + 2, \quad du = 3dx, \quad \text{so}$$

$$\int \frac{dx}{3x+2} = \int \frac{(1/3)du}{u} = (1/3)\ln u + c = (1/3)\ln(3x+2) + c$$

j)

$$\int \frac{x+5}{x}dx = \int \left(1 + \frac{5}{x}\right) dx = x + 5\ln x + c$$

k)

$$\int \frac{x}{x+5} dx = \int \left(1 - \frac{5}{x+5}\right) dx = x - 5\ln(x+5) + c$$

In Unit 5 this sort of algebraic trick will be explained in detail as part of a general method. What underlies the algebra in both (j) and (k) is the algorithm of long division for polynomials.

l) $u = \ln x$, $du = dx/x$, so

$$\int \frac{\ln x}{x} dx = \int u du = (1/2)u^2 + c = (1/2)(\ln x)^2 + c$$

m) $u = \ln x$, $du = dx/x$.

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u + c = \ln(\ln x) + c$$

3A-3 a) $-(1/5)\cos(5x) + c$

b) $(1/2)\sin^2 x + c$, coming from the substitution $u = \sin x$ or $-(1/2)\cos^2 x + c$, coming from the substitution $u = \cos x$. The two functions $(1/2)\sin^2 x$ and $-(1/2)\cos^2 x$ are not the same. Nevertheless the two answers given are the same. Why? (See 1J-1(m).)

c) $-(1/3)\cos^3 x + c$

d) $-(1/2)(\sin x)^{-2} + c = -(1/2)\csc^2 x + c$

e) $5\tan(x/5) + c$

f) $(1/7)\tan^7 x + c$.

g) $u = \sec x$, $du = \sec x \tan x dx$,

$$\int \sec^9 x \tan x dx \int (\sec x)^8 \sec x \tan x dx = (1/9)\sec^9 x + c$$

3B. Definite Integrals

3B-1 a) $1 + 4 + 9 + 16 = 30$ b) $2 + 4 + 8 + 16 + 32 + 64 = 126$

c) $-1 + 4 - 9 + 16 - 25 = -15$ d) $1 + 1/2 + 1/3 + 1/4 = 25/12$

3B-2 a) $\sum_{n=1}^{6}(-1)^{n+1}(2n+1)$ b) $\sum_{k=1}^{n}1/k^2$ c) $\sum_{k=1}^{n}\sin(kx/n)$

3B-3 a) upper sum = right sum = $(1/4)[(1/4)^3+(2/4)^3+(3/4)^3+(4/4)^3] = 15/128$

lower sum = left sum = $(1/4)[0^3 + (1/4)^3 + (2/4)^3 + (3/4)^3] = 7/128$

b) left sum = $(-1)^2+0^2+1^2+2^2 = 6$; right sum = $0^2+1^2+2^2+3^2 = 14$;

upper sum = $(-1)^2 + 1^2 + 2^2 + 3^2 = 15$; lower sum = $0^2 + 0^2 + 1^2 + 2^2 = 5$.

c) left sum = $(\pi/2)[\sin 0+\sin(\pi/2)+\sin(\pi)+\sin(3\pi/2)] = (\pi/2)[0+1+0-1] = 0$;

right sum = $(\pi/2)[\sin(\pi/2)+\sin(\pi)+\sin(3\pi/2)+\sin(2\pi)] = (\pi/2)[1+0-1+0] = 0$;

upper sum = $(\pi/2)[\sin(\pi/2)+\sin(\pi/2)+\sin(\pi)+\sin(2\pi)] = (\pi/2)[1+1+0+0] = \pi$;

lower sum = $(\pi/2)[\sin(0)+\sin(\pi)+\sin(3\pi/2)+\sin(3\pi/2)] = (\pi/2)[0+0-1-1] = -\pi$.

3B-4 Both x^2 and x^3 are increasing functions on $0 \le x \le b$, so the upper sum is the right sum and the lower sum is the left sum. The difference between the right and left Riemann sums is

$$(b/n)[f(x_1 + \cdots + f(x_n)] - (b/n)[f(x_0 + \cdots + f(x_{n-1})] = (b/n)[f(x_n) - f(x_0)]$$

In both cases $x_n = b$ and $x_0 = 0$, so the formula is

$$(b/n)(f(b) - f(0))$$

a) $(b/n)(b^2 - 0) = b^3/n$. Yes, this tends to zero as $n \to \infty$.

b) $(b/n)(b^3 - 0) = b^4/n$. Yes, this tends to zero as $n \to \infty$.

3B-5 The expression is the right Riemann sum for the integral

$$\int_0^1 \sin(bx)dx = -(1/b)\cos(bx)|_0^1 = (1 - \cos b)/b$$

so this is the limit.

3C. Fundamental theorem of calculus

3C-1

$$\int_3^6 (x-2)^{-1/2}dx = 2(x-2)^{1/2}\Big|_3^6 = 2[(4)^{1/2} - 1^{1/2}] = 2$$

3C-2 a) $(2/3)(1/3)(3x+5)^{3/2}\Big|_0^2 = (2/9)(11^{3/2} - 5^{3/2})$

b) If $n \neq -1$, then

$$(1/(n+1))(1/3)(3x+5)^{n+1}\Big|_0^2 = (1/3(n+1))((11^{n+1} - 5^{n+1})$$

If $n = -1$, then the answer is $(1/3)\ln(11/5)$.

c) $(1/2)(\cos x)^{-2}\Big|_{3\pi/4}^{\pi} = (1/2)[(-1)^{-2} - (-1/\sqrt{2})^{-2}] = -1/2$

3C-3 a) $(1/2)\ln(x^2+1)\Big|_1^2 = (1/2)[\ln 5 - \ln 2] = (1/2)\ln(5/2)$

b) $(1/2)\ln(x^2+b^2)\Big|_b^{2b} = (1/2)[\ln(5b^2) - \ln(2b^2)] = (1/2)\ln(5/2)$

3C-4 As $b \to \infty$,

$$\int_1^b x^{-10}dx = -(1/9)x^{-9}\Big|_1^b = -(1/9)(b^{-9} - 1) \to -(1/9)(0-1) = 1/9.$$

This integral is the area of the infinite region between the curve $y = x^{-10}$ and the x-axis for $x > 0$.

3C-5 a) $\displaystyle\int_0^{\pi} \sin x dx = -\cos x|_0^{\pi} = -(\cos\pi - \cos 0) = 2$

b) $\displaystyle\int_0^{\pi/a} \sin(ax)dx = -(1/a)\cos(ax)|_0^{\pi/a} = -(1/a)(\cos\pi - \cos 0) = 2/a$

3C-6 a) $x^2 - 4 = 0$ implies $x = \pm 2$. So the area is

$$\int_{-2}^{2}(x^2-4)dx = 2\int_0^2 (x^2-4)dx = \frac{x^3}{3} - 4x\Big|_0^2 = \frac{8}{3} - 4\cdot 2 = -16/3$$

(We changed to the interval $(0,2)$ and doubled the integral because x^2-4 is even.) Notice that the integral gave the wrong answer! It's negative. This is because the graph $y = x^2 - 4$ is concave up and is below the x-axis in the interval $-2 < x < 2$. So the correct answer is $16/3$.

b) Following part (a), $x^2 - a = 0$ implies $x = \pm\sqrt{a}$. The area is

$$\int_{-\sqrt{a}}^{\sqrt{a}}(a-x^2)dx = 2\int_0^{\sqrt{a}}(a-x^2)dx = 2ax - \frac{x^3}{3}\Big|_0^{\sqrt{a}} = 2\big(a^{3/2} - \frac{a^{3/2}}{3}\big) = \frac{4}{3}a^{3/2}$$

3D. Second fundamental theorem

3D-1 Differentiate both sides;

left side $L(x)$: $L'(x) = \dfrac{d}{dx}\displaystyle\int_0^x \frac{dt}{\sqrt{a^2+x^2}} = \frac{1}{\sqrt{a^2+x^2}}$, by FT2;

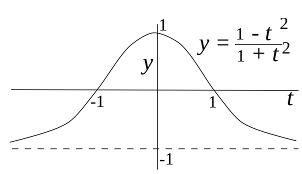
right side $R(x)$: $R'(x) = \dfrac{d}{dx}(\ln(x+\sqrt{a^2+x^2}) - \ln a) = \dfrac{1+\frac{x}{\sqrt{a^2+x^2}}}{x+\sqrt{a^2+x^2}} = \dfrac{1}{\sqrt{a^2+x^2}}$

Since $L'(x) = R'(x)$, we have $L(x) = R(x) + C$ for some constant $C = L(x) - R(x)$. The constant C may be evaluated by assigning a value to x; the most convenient choice is $x = 0$, which gives

$L(0) = \displaystyle\int_0^0 = 0$; $R(0) = \ln(0+\sqrt{0+a^2}) - \ln a = 0$; therefore $C = 0$ and $L(x) = R(x)$.

b) Put $x = c$; the equation becomes $0 = \ln(c+\sqrt{c^2+a^2})$; solve this for c by first exponentiating both sides: $1 = c + \sqrt{c^2+a^2}$; then subtract c and square both sides; after some algebra one gets $c = \frac{1}{2}(1-a^2)$.

3D-3 Sketch $y = \dfrac{1-t^2}{1+t^2}$ first, as shown at the right.

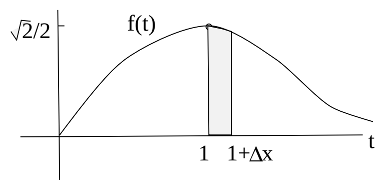


3D-4 a) $\displaystyle\int_0^x \sin(t^3)dt$, by the FT2. b) $\displaystyle\int_0^x \sin(t^3)dt + 2$ c) $\displaystyle\int_1^x \sin(t^3)dt - 1$

3D-5 This problem reviews the idea of the proof of the FT2.

a) $f(t) = \dfrac{t}{\sqrt{1+t^4}}$

$$\frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt \ = \frac{\text{shaded area}}{\text{width}} \approx \text{height} \quad .$$



$$\lim_{\Delta x\to 0}\ \frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt \ = \ \lim_{\Delta x\to 0}\ \frac{\text{shaded area}}{\text{width}} = \text{height} = f(1) = \frac{1}{\sqrt{2}}.$$

b) By definition of derivative,

$$F'(1) = \lim_{\Delta x\to 0}\frac{F(1+\Delta x)-F(1)}{\Delta x} \ = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt;$$

by FT2, $F'(1) = f(1) = \dfrac{1}{\sqrt{2}}$.

3D-6 a) If $F_1(x) = \displaystyle\int_{a_1}^{x} dt$ and $F_2(x) = \displaystyle\int_{a_2}^{x} dt$, then $F_1(x) = x - a_1$ and $F_2(x) = x - a_2$. Thus $F_1(x) - F_2(x) = a_2 - a_1$, a constant.

b) By the FT2, $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$; therefore $F_1 = F_2 + C$, for some constant C.

3D-7 a) Using the FT2 and the chain rule, as in the Notes,

$$\frac{d}{dx}\int_0^{x^2}\sqrt{u}\sin u du = \sqrt{x^2}\sin(x^2)\cdot\frac{d(x^2)}{dx} = 2x^2\sin(x^2)$$

b) $= \dfrac{1}{\sqrt{1-\sin^2 x}}\cdot\cos x = 1.$ (So $\displaystyle\int_0^{\sin x}\frac{dt}{1-t^2} = x$)

c) $\dfrac{d}{dx}\displaystyle\int_x^{x^2}\tan u du = \tan(x^2)\cdot 2x - \tan x$

3D-8 a) Differentiate both sides using FT2, and substitute $x = \pi/2$: $f(\pi/2) = 4$.

b) Substitute $x = 2u$ and follow the method of part (a); put $u = \pi$, get finally $f(\pi/2) = 4 - 4\pi$.

3E. Change of Variables; Estimating Integrals

3E-1 $L(\frac{1}{a}) = \int_1^{1/a} \frac{dt}{t}$. Put $t = \frac{1}{u}$, $dt = -\frac{1}{u^2}du$. Then

$$\frac{dt}{t} = -\frac{u}{u^2}du \implies L(\frac{1}{a}) = \int_1^{1/a} \frac{dt}{t} = -\int_1^a \frac{du}{u} = -L(a)$$

3E-2 a) We want $-t^2 = -u^2/2$, so $u = t\sqrt{2}$, $du = \sqrt{2}dt$.

$$\frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2} du = \frac{1}{\sqrt{2\pi}} \int_0^{x/\sqrt{2}} e^{-t^2}\sqrt{2}dt = \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-t^2} dt$$

$$\implies E(x) = \frac{1}{\sqrt{\pi}} F(x/\sqrt{2}) \quad \text{and} \quad \lim_{x\to\infty} E(x) = \frac{1}{\pi} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{2}$$

b) The integrand is even, so

$$\frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-u^2/2} du = \frac{2}{\sqrt{2\pi}} \int_0^N e^{-u^2/2} du = 2E(N) \longrightarrow 1 \quad \text{as } N \to \infty$$

$$\lim_{x\to-\infty} E(x) = -1/2 \quad \text{because } E(x) \text{ is odd.}$$

$\frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2} du = E(b) - E(a)$ by FT1 or by "interval addition" Notes PI (3).

Commentary: The answer is consistent with the limit,

$$\frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-u^2/2} du = E(N) - E(-N) = 2E(N) \longrightarrow 1 \text{ as } N \to \infty$$

3E-3 a) Using $u = \ln x$, $du = \frac{dx}{x}$, $\int_1^e \frac{\sqrt{\ln x}}{x} dx = \int_0^1 \sqrt{u} du = \frac{2}{3} u^{3/2} \Big|_0^1 = \frac{2}{3}$.

b) Using $u = \cos x$, $du = -\sin x$,

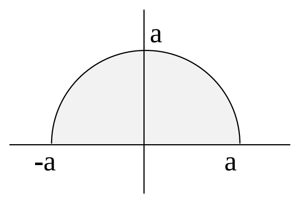
$$\int_0^\pi \frac{\sin x}{(2+\cos x)^3} dx = \int_1^{-1} \frac{-du}{(2+u)^3} = \frac{1}{2(2+u)^2}\Big|_1^{-1} = \frac{1}{2}(\frac{1}{1^2} - \frac{1}{3^2}) = \frac{4}{9} ..$$

c) Using $x = \sin u$, $dx = \cos u du$, $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\cos u} du = u\Big|_0^{\pi/2} = \frac{\pi}{2}$

.

3E-4 Substitute $x = t/a$; then $x = \pm 1 \;\Rightarrow\; t = \pm a$. We then have

$\frac{\pi}{2} = \int_{-1}^{1}\sqrt{1-x^2}dx = \int_{-a}^{a}\sqrt{1-\frac{t^2}{a^2}}\frac{dt}{a} = \frac{1}{a^2}\int_{-a}^{a}\sqrt{a^2-t^2}dt$. Multiplying by a^2 gives the value $\pi a^2/2$ for the integral, which checks, since the integral represents the area of the semicircle.



3E-5 One can use informal reasoning based on areas (as in Ex. 5, Notes FT), but it is better to use change of variable.

a) Goal: $F(-x) = -F(x)$. Let $t = -u$, $dt = -du$, then

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^{x} f(-u)(-du)$$

Since f is even ($f(-u) = f(u)$), $F(-x) = -\int_0^x f(u)du = -F(x)$.

b) Goal: $F(-x) = F(x)$. Let $t = -u$, $dt = -du$, then

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^{x} f(-u)(-du)$$

Since f is odd (($f(-u) = -f(u)$), $F(-x) = \int_0^x f(u)du = F(x)$.

3E-6 a) $x^3 < x$ on (0,1) $\;\Rightarrow\; \frac{1}{1+x^3} > \frac{1}{1+x}$ on (0,1); therefore

$$\int_0^1 \frac{dx}{1+x^3} > \int_0^1 \frac{dx}{1+x} = \ln(1+x)\Big|_0^1 = \ln 2 = .69$$

b) $0 < \sin x < 1$ on $(0,\pi) \;\Rightarrow\; \sin^2 x < \sin x$ on $(0,\pi)$; therefore

$$\int_0^{\pi} \sin^2 x dx < \int_0^{\pi} \sin x dx = -\cos x\Big|_0^{\pi} = -(-1-1) = 2.$$

c) $\int_{10}^{20} \sqrt{x^2+1}dx > \int_{10}^{20} \sqrt{x^2}dx = \frac{x^2}{2}\Big|_{10}^{20} = \frac{1}{2}(400-100) = 150$

3E-7 $\left|\int_1^N \frac{\sin x}{x^2}dx\right| \le \int_1^N \frac{|\sin x|}{x^2}dx \le \int_1^N \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^N = -\frac{1}{N}+1 < 1.$

3F. Differential Equations: Separation of Variables. Applications

3F-1 a) $y = (1/10)(2x+5)^5 + c$

b) $(y+1)dy = dx \implies \int (y+1)dy = \int dx \implies (1/2)(y+1)^2 = x + c$. You can leave this in implicit form or solve for y: $y = -1 \pm \sqrt{2x+a}$ for any constant a $(a = 2c)$

c) $y^{1/2}dy = 3dx \implies (2/3)y^{3/2} = 3x + c \implies y = (9x/2 + a)^{2/3}$, with $a = (3/2)c$.

d) $y^{-2}dy = xdx \implies -y^{-1} = x^2/2 + c \implies y = -1/(x^2/2 + c)$

3F-2 a) Answer: $3e^{16}$.

$$y^{-1}dy = 4xdx \implies \ln y = 2x^2 + c$$

$$y(1) = 3 \implies \ln 3 = 2 + c \implies c = \ln 3 - 2.$$

Therefore

$$\ln y = 2x^2 + (\ln 3 - 2)$$

At $x = 3$, $y = e^{18+\ln 3-2} = 3e^{16}$

b) Answer: $y = 11/2 + 3\sqrt{2}$.

$$(y+1)^{-1/2}dy = dx \implies 2(y+1)^{1/2} = x + c$$

$$y(0) = 1 \implies 2(1+1)^{1/2} = c \implies c = 2\sqrt{2}$$

At $x = 3$,

$$2(y+1)^{1/2} = 3 + 2\sqrt{2} \implies y + 1 = (3/2 + \sqrt{2})^2 = 13/2 + 3\sqrt{2}$$

Thus, $y = 11/2 + 3\sqrt{2}$.

c) Answer: $y = \sqrt{550/3}$

$$ydy = x^2dx \implies y^2/2 = (1/3)x^3 + c$$

$$y(0) = 10 \implies c = 10^2/2 = 50$$

Therefore, at $x = 5$,

$$y^2/2 = (1/3)5^3 + 50 \implies y = \sqrt{550/3}$$

d) Answer: $y = (2/3)(e^{24} - 1)$

$$(3y+2)^{-1}dy = dx \implies (1/3)\ln(3y+2) = x + c$$
$$y(0) = 0 \implies (1/3)\ln 2 = c$$

Therefore, at $x = 8$,

$(1/3)\ln(3y+2) = 8 + (1/3)\ln 2 \implies \ln(3y+2) = 24 + \ln 2 \implies (3y+2) = 2e^{24}$

Therefore, $y = (2e^{24} - 2)/3$

e) Answer: $y = -\ln 4$ at $x = 0$. Defined for $-\infty < x < 4$.

$$e^{-y}dy = dx \implies -e^{-y} = x + c$$
$$y(3) = 0 \implies -e^0 = 3 + c \implies c = -4$$

Therefore,

$$y = -\ln(4-x), \quad y(0) = -\ln 4$$

The solution y is defined only if $x < 4$.

3F-3 a) Answers: $y(1/2) = 2$, $y(-1) = 1/2$, $y(1)$ is undefined.

$$y^{-2}dy = dx \implies -y^{-1} = x + c$$
$$y(0) = 1 \implies -1 = 0 + c \implies c = -1$$

Therefore, $-1/y = x - 1$ and

$$y = \frac{1}{1-x}$$

The values are $y(1/2) = 2$, $y(-1) = -1/2$ and y is undefined at $x = 1$.

b) Although the formula for y makes sense at $x = 3/2$, $(y(3/2) = 1/(1-3/2) = -2)$, it is not consistent with the rate of change interpretation of the differential equation. The function is defined, continuous and differentiable for $-\infty < x < 1$. But at $x = 1$, y and dy/dx are undefined. Since $y = 1/(1-x)$ is the only solution to the differential equation in the interval $(0, 1)$ that satisfies the initial condition $y(0) = 1$, it is impossible to define a function that has the initial condition $y(0) = 1$ and also satisfies the differential equation in any longer interval containing $x = 1$.

To ask what happens to y after $x = 1$, say at $x = 3/2$, is something like asking what happened to a rocket ship after it fell into a black hole. There is no obvious reason why one has to choose the formula $y = 1/(1-x)$ after the "explosion." For example, one could define $y = 1/(2-x)$ for $1 \le x < 2$. In fact, any formula $y = 1/(c-x)$ for $c \ge 1$ satisfies the differential equation at every point $x > 1$.

3F-4 a) If the surrounding air is cooler $(T_e - T < 0)$, then the object will cool, so $dT/dt < 0$. Thus $k > 0$.

b) Separate variables and integrate.

$$(T - T_e)^{-1}dT = -kdt \implies \ln|T - T_e| = -kt + c$$

Exponentiating,

$$T - T_e = \pm e^c e^{-kt} = Ae^{-kt}$$

The initial condition $T(0) = T_0$ implies $A = T_0 - T_e$. Thus

$$T = T_e + (T_0 - T_e)e^{-kt}$$

c) Since $k > 0$, $e^{-kt} \to 0$ as $t \to \infty$. Therefore,

$$T = T_e + (T_0 - T_e)e^{-kt} \longrightarrow T_e \text{ as } t \to \infty$$

d)

$$T - T_e = (T_0 - T_e)e^{-kt}$$

The data are $T_0 = 680$, $T_e = 40$ and $T(8) = 200$. Therefore,

$$200 - 40 = (680 - 40)e^{-8k} \implies e^{-8k} = 160/640 = 1/4 \implies -8k = -\ln 4.$$

The number of hours t that it takes to cool to 50° satisfies the equation

$$50 - 40 = (640)e^{-kt} \implies e^{-kt} = 1/64 \implies -kt = -3\ln 4.$$

To solve the two equations on the right above simultaneously for t, it is easiest just to divide the bottom equation by the top equation, which gives

$$\frac{t}{8} = 3, \quad t = 24.$$

e)

$$T - T_e = (T_0 - T_e)e^{-kt}$$

The data at $t = 1$ and $t = 2$ are

$$800 - T_e = (1000 - T_e)e^{-k} \quad \text{and} \quad 700 - T_e = (1000 - T_e)e^{-2k}$$

Eliminating e^{-k} from these two equations gives

$$\begin{aligned}
\frac{700 - T_e}{1000 - T_e} &= \left(\frac{800 - T_e}{1000 - T_e}\right)^2 \\
(800 - T_e)^2 &= (1000 - T_e)(700 - T_e) \\
800^2 - 1600T_e + T_e^2 &= (1000)(700) - 1700T_e + T_e^2 \\
100T_e &= (1000)(700) - 800^2 \\
T_e &= 7000 - 6400 = 600
\end{aligned}$$

f) To confirm the differential equation:

$$y'(t) = T'(t - t_0) = k(T_e - T(t - t_0)) = k(T_e - y(t))$$

The formula for y is

$$y(t) = T(t - t_0) = T_e + (T_0 - T_e)e^{-k(t-t_0)} = a + (y(t_0) - a)e^{-c(t-t_0)}$$

with $k = c$, $T_e = a$ and $T_0 = T(0) = y(t_0)$.

3F-6 $y = \cos^3 u - 3\cos u$, $x = \sin^4 u$

$dy = (3\cos^2 u \cdot (-\sin u) + 3\sin u)du,\ dx = 4\sin^3 u \cos u du$

$$\frac{dy}{dx} = \frac{3\sin u(1-\cos^2 u)}{4\sin^3 u \cos u} = \frac{3}{4\cos u}$$

3F-7 a) $y' = -xy;\ y(0) = 1$

$$\frac{dy}{y} = -xdx \implies \ln y = -\frac{1}{2}x^2 + c$$

To find c, put $x = 0$, $y = 1$: $\ln 1 = 0 + c \implies c = 0$.

$$\implies \ln y = -\frac{1}{2}x^2 \implies y = e^{-x^2/2}$$

b) $\cos x \sin y dy = \sin x dx;\ y(0) = 0$

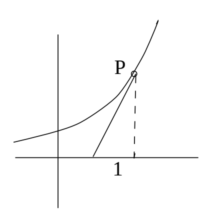
$$\sin y dy = \frac{\sin x}{\cos x}dx \implies -\cos y = -\ln(\cos x) + c$$

Find c: put $x = 0$, $y = 0$: $-\cos 0 = -\ln(\cos 0) + c \implies c = -1$

$$\implies \cos y = \ln(\cos x) + 1$$

3F-8 a) From the triangle, $y' = \text{slope tangent} = \dfrac{y}{1}$

$$\implies \frac{dy}{y} = dx \implies \ln y = x + c_1 \implies y = e^{x+c_1} = Ae^x \ \ (A = e^{c_1})$$

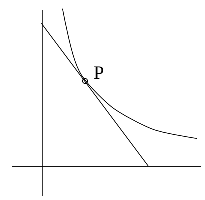


b) If P bisects tangent, then P_0 bisects OQ (by euclidean geometry)

So $P_0Q = x$ (since $OP_0 = x$).

Slope tangent $= y' = \dfrac{-y}{x} \Longrightarrow \dfrac{dy}{y} = -\dfrac{dx}{x}$

$\Longrightarrow \ln y = -\ln x + c_1$



Exponentiate: $y = \dfrac{1}{x} \cdot e^{c_1} = \dfrac{c}{x}, c > 0$

Ans: The hyperbolas $y = \dfrac{c}{x}, c > 0$

3G. Numerical Integration

3G-1 Left Riemann sum: $(\Delta x)(y_0 + y_1 + y_2 + y_3)$

Trapezoidal rule: $(\Delta x)((1/2)y_0 + y_1 + y_2 + y_3 + (1/2)y_4)$

Simpson's rule: $(\Delta x/3)(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$

a) $\Delta x = 1/4$ and

$$y_0 = 0,\ y_1 = 1/2,\ y_2 = 1/\sqrt{2},\ y_3 = \sqrt{3}/2,\ y_4 = 1.$$

Left Riemann sum: $(1/4)(0 + 1/2 + 1/\sqrt{2} + \sqrt{3}/2) \approx .518$

Trapezoidal rule: $(1/4)((1/2) \cdot 0 + 1/2 + 1/\sqrt{2} + \sqrt{3}/2 + (1/2)1) \approx .643$

Simpson's rule: $(1/12)(1 \cdot 0 + 4(1/2) + 2(1/\sqrt{2}) + 4(\sqrt{3}/2) + 1) \approx .657$

as compared to the exact answer $.6666\ldots$

b) $\Delta x = \pi/4$

$$y_0 = 0,\ y_1 = 1/\sqrt{2},\ y_2 = 1,\ y_3 = 1/\sqrt{2},\ y_4 = 0.$$

Left Riemann sum: $(\pi/4)(0 + 1/\sqrt{2} + 1 + 1/\sqrt{2}) \approx 1.896$

Trapezoidal rule: $(\pi/4)((1/2)\cdot 0 + 1/\sqrt{2} + 1 + 1/\sqrt{2} + (1/2)\cdot 0) \approx 1.896$ (same as Riemann sum)

Simpson's rule: $(\pi/12)(1\cdot 0 + 4(1/\sqrt{2}) + 2(1) + 4(1/\sqrt{2}) + 1\cdot 0) \approx 2.005$

as compared to the exact answer 2

c) $\Delta x = 1/4$

$$y_0 = 1,\ y_1 = 16/17,\ y_2 = 4/5,\ y_3 = 16/25,\ y_4 = 1/2.$$

Left Riemann sum: $(1/4)(1 + 16/17 + 4/5 + 16/25) \approx .845$

Trapezoidal rule: $(1/4)((1/2)\cdot 1 + 16/17 + 4/5 + 16/25 + (1/2)(1/2)) \approx .8128$

Simpson's rule: $(1/12)(1\cdot 1 + 4(16/17) + 2(4/5) + 4(16/25) + 1(1/2)) \approx .785392$

as compared to the exact answer $\pi/4 \approx .785398$

(Multiplying the Simpson's rule answer by 4 gives a passable approximation to π, of 3.14157, accurate to about 2×10^{-5}.)

d) $\Delta x = 1/4$

$$y_0 = 1,\ y_1 = 4/5,\ y_2 = 2/3,\ y_3 = 4/7,\ y_4 = 1/2.$$

Left Riemann sum: $(1/4)(1 + 4/5 + 2/3 + 4/7) \approx .76$

Trapezoidal rule: $(1/4)((1/2)\cdot 1 + 4/5 + 2/3 + 4/7(1/2)(1/2)) \approx .697$

Simpson's rule: $(1/12)(1\cdot 1 + 4(4/5) + 2(2/3) + 4(4/7) + 1(1/2)) \approx .69325$

Compared with the exact answer $\ln 2 \approx .69315$, Simpson's rule is accurate to about 10^{-4}.

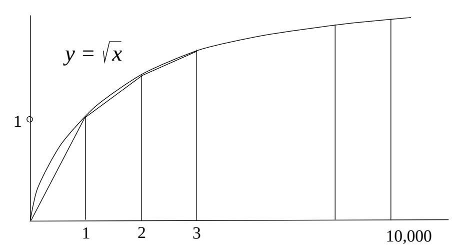
3G-2 We have $\int_0^b x^3dx = \frac{b^4}{4}$. Using Simpson's rule with two subintervals, $\Delta x = b/2$, so that we get the same answer as above:[1]

$$S(x^3) = \frac{b}{6}(0 + 4(b/2)^3 + b^3) = \frac{b}{6}\left(\frac{3}{2}b^3\right) = \frac{b^4}{4}.$$

3G-3 The sum

$$S = \sqrt{1} + \sqrt{2} + ... + \sqrt{10,000}$$

is related to the trapezoidal estimate of $\int_0^{10^4} \sqrt{x}dx$:



(1) $$\int_0^{10^4} \sqrt{x}dx \approx \frac{1}{2}\sqrt{0} + \sqrt{1} + ... + \frac{1}{2}\sqrt{10^4} = S - \frac{1}{2}\sqrt{10^4}$$

But

$$\int_0^{10^4} \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_0^{10^4} = \frac{2}{3} \cdot 10^6$$

From (1),

(2) $$\frac{2}{3} \cdot 10^6 \approx S - 50$$

Hence

(3) $$S \approx 666,717$$

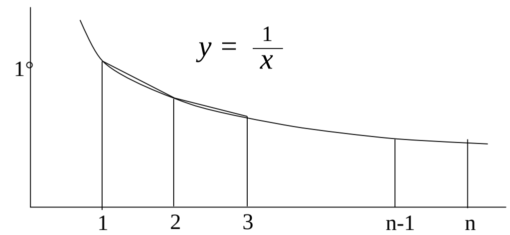
In (1), we have >, as in the picture. Hence in (2), we have >, so in (3), we have <, Too high.

3G-4 As in Problem 3 above, let

$$S = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}$$

Then by trapezoidal rule,

[1]The fact that Simpson's rule is exact on cubic polynomials is very significant to its effectiveness as a numerical approximation. It implies that the approximation converges at a rate proportional to the the fourth derivative of the function times $(\Delta x)^4$, which is fast enough for many practical purposes.



$$\int_1^n \frac{dx}{x} \approx \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2}\cdot\frac{1}{n} = S - \frac{1}{2} - \frac{1}{2n}$$

Since $\displaystyle\int_1^n \frac{dx}{x} = \ln n$, we have $S \approx \ln n + \frac{1}{2} + \frac{1}{2n}$. (Estimate is too low.)

3G-5 Referring to the two pictures above, one can see that if $f(x)$ is concave down on $[a, b]$, the trapezoidal rule gives too low an estimate; if $f(x)$ is concave up, the trapezoidal rule gives too high an estimate..