Do We Need the Quotient Rule?

The quotient rule can be difficult to memorize, and some students are more comfortable with negative exponents than they are with fractions. In this exercise we learn how we can use the chain and product rules together in place of the quotient rule.

a) Use the quotient rule to find the derivative of $\dfrac{x^3}{x+1}$.

b) Use the product and chain rules to find the derivative of $x^3 \cdot (x+1)^{-1}$. Note that $x^3 \cdot (x+1)^{-1} = \dfrac{x^3}{x+1}$.

c) Use the chain and product rules (and not the quotient rule) to show that the derivative of $u(x)(v(x))^{-1}$ equals $\dfrac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$.

Solution

a) Use the quotient rule to find the derivative of $\dfrac{x^3}{x+1}$.

The quotient rule tells us that:

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}.$$

Here $u(x) = x^3$, $u'(x) = 3x^2$, $v(x) = x+1$ and $v'(x) = 1$. Therefore,

$$\begin{aligned}
\left(\frac{x^3}{x+1}\right)' &= \frac{3x^2(x+1) - x^3 \cdot 1}{(x+1)^2} \\
&= \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} \\
&= \frac{2x^3 + 3x^2}{(x+1)^2}
\end{aligned}$$

b) Use the product and chain rules to find the derivative of $x^3 \cdot (x+1)^{-1}$. (Note that $x^3 \cdot (x+1)^{-1} = \dfrac{x^3}{x+1}$.)

The product rule tells us that $(u(x) \cdot v(x))' = u'(x)v(x) + u(x)v'(x)$. Once again $u(x) = x^3$ and $u'(x) = 3x^2$, but now $v(x) = (x+1)^{-1}$. We apply the chain rule to find $v'(x) = -1 \cdot (x+1)^{-2} \cdot 1 = -(x+1)^{-2}$.

Now we can apply the product rule to find that:

$$\begin{aligned}
(x^3 \cdot (x+1)^{-1})' &= 3x^2(x+1)^{-1} + x^3(-(x+1)^{-2}) \\
&= 3x^2(x+1)^{-1} - x^3(x+1)^{-2}
\end{aligned}$$

Although $x^3 \cdot (x+1)^{-1} = \dfrac{x^3}{x+1}$, the derivatives of the two expressions look very different. In fact, they are algebraically equivalent; different-looking but equivalent answers are a common occurrence in calculus. We can show the equivalence of the two answers by applying some basic algebra:

$$\begin{aligned}
3x^2(x+1)^{-1} - x^3(x+1)^{-2} &= 3x^2(x+1)^{-1} \cdot ((x+1)^{-1} \cdot (x+1)^{+1}) - x^3(x+1)^{-2} \\
&\qquad \text{create a common factor of } (x+1)^{-2} \\
&= 3x^2(x+1)^{-2} \cdot (x+1) - x^3(x+1)^{-2} \\
&\qquad \text{factor out } (x+1)^{-2} \\
&= (3x^2(x+1) - x^3)(x+1)^{-2} \\
&\qquad \text{eliminate the negative exponent.} \\
&= \frac{3x^2(x+1) - x^3}{(x+1)^2} \\
&= \frac{2x^3 + 3x^2}{(x+1)^2}
\end{aligned}$$

c) Use the chain and product rules (and not the quotient rule) to show that the derivative of $f(x)(g(x))^{-1}$ equals $\dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

We can use the previous solution as an outline for this one. Here $u(x) = f(x)$, $u'(x) = f'(x)$, $v(x) = (g(x))^{-1}$ and, by the chain rule, $v'(x) = -(g(x))^{-2}g'(x)$. Hence,

$$\begin{aligned}
(f(x)(g(x))^{-1})' &= f'(x)(g(x))^{-1} + f(x)(-(g(x))^{-2}g'(x)) \\
&= f'(x)(g(x))^{-1} - f(x)(g(x))^{-2}g'(x) \\
&= f'(x)(g(x))^{-2}g(x) - f(x)g'(x)(g(x))^{-2} \\
&= [f'(x)g(x) - f(x)g'(x)](g(x))^{-2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}$$

Note that if time permits, you can use this alternate method of differentiation to check your work on problems involving the chain rule.