PROFESSOR: Hi. Welcome back to recitation.

I have here a little bit of a strange problem for you. So let me just tell it to you, and then I'll give you some time to work on it.

So I want to define a function, g of x, and I want to define it piecewise. So when x is positive, I just want g of x to be 1 over x. But when x is negative, I want g of x to be 1 of x plus 2. So I've got a little graph here of the function. So you've got, you know, when x is positive, it's just your usual y equals 1 over x. But when x is negative, I've taken, I've shifted it up by 2. So this is a perfectly good function. It's not defined at 0. OK?

So what I would like you to do is to compute the derivative of this function, wherever it's defined. And you'll notice, when you get there, that you'll have some.. you'll get some answer, and maybe you'll notice something a little weird about that answer. So if you notice something weird about it, what I want you to do is try and explain why this is true. And if you don't notice something weird, then, you know, come back and we'll talk about it together.

So why don't you pause the video, go do that computation, and think about what, if there's something strange going on here. And then come back and we can talk about it together.

Welcome back. Hopefully you had some fun working on this problem and thinking about it.

So let's do the first part, which is just the computational part. Let's have a go at it.

So, because this function is defined piecewise, when we compute a derivative, we can just compute the derivative on the different pieces. So the function isn't defined at 0, so of course, it doesn't have a derivative at 0. But then we can compute a derivative when x is positive, and we can compute a derivative when x is negative.

So when x is bigger than 0, g prime of x, well, that's just d over dx of 1 over x. So that's something we're familiar with. Its minus 1 over x squared. So that's for x positive.

When x is less than 0, g prime of x is d over dx of 1 over x plus 2, because that's what g of x is. And, OK, and so this is, well, the plus 2 gets killed, and so then we have the derivative of 1 over x. That's minus 1 over x squared. So one thing you've noticed is that this is minus 1 over x squared here, and it's minus 1 over x squared here. So although we defined this piecewise, we could, we can summarize this by saying, so the derivative is minus 1 over x squared always, so for all x not equal to-- you know, it doesn't have a derivative at x equals 0. It's not defined at 0, it can't have a derivative there. So, but we don't need the piecewise definition, anymore.

So that was kind of interesting, that we can summarize the derivative of this piecewise function in a non-piecewise way. Now, the thing is, we've learned what the anti-derivative of this function is. So we know that the anti-derivative of minus 1 over x squared dx is 1 over x plus a constant. So we know that the functions whose derivative is minus 1 over x squared are of the form, 1 over x plus a constant.

The thing is, this function g that we just talked about, this function g isn't of that form. Right? You don't get this function by taking the function 1 over x and just shifting it up or down. You-- something weird happens. You've shifted it up on one piece and not on the other piece. And yet, it's still true that the derivative of g is equal to minus 1 over x squared.

So this is a little bit of a head-scratcher. And I wanted to talk about why this happens. And the thing is that there's a sort of theoretical reason for this, which is that you remember that the reason that we know that anti-derivatives have this form, a function plus a constant, is because we know that constants are the functions with derivative 0.

And so we were able to apply the mean value theorem in order to show that if two functions have the same derivative, then they differ by each other, differ from each other by a constant. If two functions have the same derivative, they differ by a constant. And we used, as a really crucial step in that proof, the mean value theorem.

Now the thing is, the mean value theorem has, as one of its assumptions, as one of its hypotheses, that the functions that you're working with are continuous and differentiable in some interval. OK? So what's happened here is that the functions that we're talking about, the function 1 over x and the function minus 1 over x squared, those functions are continuous and differentiable on certain intervals.

So if we look-- if we go back to this picture here we see that this function g of x, just like the function 1 over x, it's continuous and differentiable for positive x, it's continuous and differentiable for negative x, but at 0, there's a discontinuity. So there's no interval that crosses 0 on which this function is continuous or differentiable. As a result, the mean value theorem can't tell us anything about intervals that cross 0.

So if the mean value theorem doesn't tell us anything in that case, it means the conclusion isn't true and we get a situation-- sorry. I should rephrase that. It means the conclusion doesn't have to be true. Our proof doesn't work in a case where we have a discontinuity.

And what happens, in fact, is right what we have here, which is that when you have a function that has a discontinuity and you look at its anti-derivatives, what you can do is that, in addition to shifting the whole thing up and down, you can shift the pieces on either side of the discontinuity separately. Just like in this case we can shift the piece to the left of 0 separately from the piece to the right of 0 and get a function whose derivative is still what we started with. So this function g of x, we get by shifting part of 1 over x up, and it gives us a function whose derivative is still minus 1 over x squared.

So this is true anytime you have a function with a discontinuity. So one consequence of this-- I'm going to go back over here and just write down one special case of this-- is that to say, we say that the anti-derivative of 1 over x dx is equal to ln of the absolute value of x plus c. What this really means is that when x is positive, we have a single kind of anti-derivative, and they're of the form, ln x plus a constant.

And when x is negative, we have a single anti-derivative, that's-- or single family of anti-derivatives, of the form ln of minus x-- remember, absolutely value of x is minus x when x is negative-- plus c. But if we consider x to be positive and negative at the same time, the two constants don't necessarily have to agree. You can have the same situation that you had before where one side can shift up and down independently of the other, because there's that discontinuity at 0 there.

So this is just something to keep in mind. It also means you have to be careful with certain substitutions. You don't want to do substitutions that have discontinuities. If you do substitutions that have discontinuities, you might accidentally introduce a discontinuity and bad things can happen that I won't go into now. You can make-- end up with statements that don't make any sense by making a substitution where the function that you're substituting has a discontinuity in it. So you-- or another way of saying it is you have to restrict to some interval on which it really is continuous. And then on each of those intervals it makes sense, but bad things could happen when you cross those discontinuities.

So this is a little bit theoretical, but I think it's a nice thing to be aware of, a nice thing to keep in mind when you're working with some of these expressions.

So I'll end there.