PROFESSOR: So, Professor Jerison is relaxing in sunny London, Ontario today and sent me in as his substitute again. I'm glad to the here and see you all again. So our agenda today: he said that he'd already talked about power series and Taylor's formula, I guess on last week right, on Friday? So I'm going to go a little further with that and show you some examples, show you some applications, and then I have this course evaluation survey that I'll hand out in the last 10 minutes or so of the class. I also have this handout that he made that says 18.01 end of term 2007. If you didn't pick this up coming in, grab it going out. People tend not to pick it up when they walk in, I see. So grab this when you're going out. There's some things missing from it. He has not decided when his office hours will be at the end of term. He will have them, just hasn't decided when. So, check the website for that information. And we're looking forward to the final exam, which is uh -- aren't we? Any questions about this technical stuff?

All right, let's talk about power series for a little bit. So I thought I should review for you what the story with power series is. OK, could I have your attention please? So, power series is a way of writing a function as a sum of integral powers of x. These a_0, a_1, and so on, are numbers. An example of a power series is a polynomial. Not to be forgotten, one type of power series is one which goes on for a finite number of terms and then ends, so that all of the other, all the higher a_i's are all 0.

This is a perfectly good example of a power series; it's a very special kind of power series. And part of what I want to tell you today is that power series behave, almost exactly like, polynomials. There's just one thing that you have to be careful about when you're using power series that isn't a concern for polynomials, and I'll show you what that is in a minute.

So, you should think of them as generalized polynomials. The one thing that you have to be careful about is that there is a number-- So one caution. There's a number which I'll call R, where R can be between 0 and it can also be infinity. It's a number between 0 and infinity, inclusive, so that when the absolute value of x is less than R. So when x is smaller than R in size, the sum converges. This sum-- that sum converges to a finite value. And when x is

bigger than R in absolute value, the sum diverges. This R is called the radius of convergence. So we'll see some examples of what the radius of convergence is in various powers series as well, and how you find it also. But, let me go on and give you a few more of the properties about power series which I think that professor Jerison talked about earlier.

So one of them is there's a radius of convergence. Here's another one. If you're inside of the radius convergence, then the function has all its derivatives, has all its derivatives, just like a polynomial does. You can differentiate it over and over again. And in terms of those derivatives, the number a_n in the power series can be expressed in terms of the value of the derivative at 0. And this is called Taylor's formula. So I'm saying that inside of this radius of convergence, the function that we're looking at, this f(x), can be written as the value of the function at 0, that's a_0, plus the value of the derivative. This bracket n means you take the derivative n times. So when n is 1, you take the derivative once at 0, divided by 1!, which is !, and multiply it by x. That's the linear term in the power series. And then the quadratic term is you take the second derivative. Remember to divide by 2!, which is 2. Multiply that by x^2 and so on out. So, in terms-- So the coefficients in the power series just record the values of the derivatives of the function at x = 0. They can be computed that way also. Let's see. I think that's the end of my summary of things that he talked about. I think he did one example, and I'll repeat that example of a power series.

This example wasn't due to David Jerison; it was due to Leonard Euler. It's the example of where the function is the exponential function e^x. So, let's see. Let's compute what-- I will just repeat for you the computation of the power series for e^x, just because it's such an important thing to do. So, in order to do that, I have to know what the derivative of e^x is, and what the second derivative of e^x is, and so on, because that comes into the Taylor formula for the coefficients. But we know what the derivative of e^x is, it's just e^x again, and it's that way all the way down. All the derivatives are e^x over and over again. So when I evaluate this at x = 0, well, the value of e^x is 1, the value of e^x is 1 at x = 0. You get a value of 1 all the way down. So all these derivatives at 0 have the value 1. And now, when I plug into this formula, I find e^x is 1 plus 1*x plus 1/2! x^2 plus 1/3! x^3, plus and so on. So all of these numbers are 1, and all you wind up with is the factorials in the denominators. That's the power series for e^x. This was a discovery of Leonhard Euler in 1740 or something. Yes, Ma'am.

AUDIENCE: When you're writing out the power series, how far do you have to write it out?

PROFESSOR: How far do you have to write the power series before it becomes well defined? Before it's a

satisfactory solution to an exam problem, I suppose, is another way to phrase the question. Until you can see what the pattern is. I can see what the pattern is. Is there anyone who's in doubt about what the next term might be? Some people would tell you that you have to write the summation convention thing. Don't believe them. If you right out enough terms to make it clear, that's good enough. OK? Is that an answer for you?

AUDIENCE: Yes, Thank you.

PROFESSOR: OK, so that's a basic example. Let's do another basic example of a power series. Oh yes, and by the way, whenever you write out a power series, you should say what the radius of convergence is. And for now, I will just to tell you that the radius of convergence of this power series is infinity; that is, this sum always converges for any value of x. I'll say a little more about that in a few minutes. Yeah?

AUDIENCE: So which functions can be written as power series?

PROFESSOR: Which functions can be written as power series? That's an excellent question. Any function that has a reasonable expression can be written as a power series. I'm not giving you a very good answer because the true answer is a little bit complicated. But any of the functions that occur in calculus like sines, cosines, tangents, they all have power series expansions, OK? We'll see more examples. Let's do another example. Here's another example. I guess this was example one. So, this example, I think, was due to Newton, not Euler. Let's find the power series expansion of this function: 1/(1+x). Well, I think that somewhere along the line, you learned about the geometric series which tells you that-- which tells you what the answer to this is, and I'll just write it out. The geometric series tells you that this function can be written as an alternating sum of powers of x. You may wonder where these minuses came from. Well, if you really think about the geometric series, as you probably remembered, there was a minus sign here, and that gets replaced by these minus signs. I think maybe Jerison talked about this also.

Anyway, here's another basic example. Remember what the graph of this function looks like when x = -1. Then there's a little problem here because the denominator becomes 0, so the graph has a pole there. It goes up to infinity at x = -1, and that's an indication that the radius of convergence is not infinity. Because if you try to converge to this infinite number by putting in x = -1, here, you'll have a big problem. In fact, you see when you put in x = -1, you keep getting 1 in every term, and it gets bigger and bigger and does not converge. In this example, the

radius of convergence is 1. OK, so, let's do a new example now. Oh, and by the way, I should say you can calculate these numbers using Taylor's formula. If you haven't seen it, check it out. Calculate the iterated derivatives of this function and plug in x = 0 and see that you get +1, -1, +1, -1, and so on. Yes sir.

AUDIENCE: For the radius of convergence I see that if you do -1 it'll blow out. If you put in 1 though, it seems like it would be fine.

PROFESSOR: The questions is I can see that there's a problem at x = -1, why is there also a problem at x = 1 where the graph is perfectly smooth and innocuous and finite. That's another excellent question. The problem is that if you go off to a radius of 1 in any direction and there's a problem, that's it. That's what the radius of convergence is. Here, what does happen if I put an x = +1? So, let's look at the partial sums. Do x = +1 in your mind here. So I'll get a partial sum 1, then 0, and then 1, and then 0, and then 1. So even though it doesn't go up to infinity, it still does not converge.

AUDIENCE: And anything in between?

PROFESSOR: Any of these other things will also fail to converge in this example. Well, that's the only two real numbers at the edge. Right? OK, let's do a different example now. How about a trig function? The sine of x. I'm going to compute the power series expansion for sin(x). and I'm going to do it using Taylor's formula. So Taylor's formula says that I have to start computing derivatives of sin(x). Sounds like it's going to be a lot of work. Let's see, the derivative of the sine is the cosine. And the derivative of the cosine, that's the second derivative of the sine, is what? Remember the minus, it's -sin(x). OK, now I want to take the third derivative of the sine, which is the derivative of sine prime prime, so it's the derivative of this. And we just decided the derivative of sine is cosine, so I get cosine, but I have this minus sign in front. And now I want to differentiate again, so the cosine becomes a minus sine, and that sign cancels with this minus sign to give me sin(x). You follow that? It's a lot of -1's canceling out there. So, all of a sudden, I'm right back where I started; these two are the same and the pattern will now repeat forever and ever. Higher and higher derivatives of sines are just plus or minus sines and cosines. Now Taylor's formula says I should now substitute x = 0 into this and see what happens, so let's do that. When x is equals to 0, the sine is 0 and the cosine is 1. The sine is 0, so minus 0 is also 0. The cosine is 1, but now there's a minus one, and now I'm back where I started, and so the pattern will repeat.

OK, so the values of the derivatives are all zeros and plus and minus ones and they go through that pattern, four-fold periodicity, over and over again. And so we can write out what sin(x) is using Taylor's formula, using this formula. So I put in the value at 0 which is 0, then I put in the derivative which is 1, multiplied by x. Then, I have the second derivative divided by 2!, but the second derivative at 0 is 0. So I'm going to drop that term out. Now I have the third derivative which is -1. And remember the 3! in the denominator. That's the coefficient of x^3. What's the fourth derivative? Well, here we are, it's on the board, it's 0. So I drop that term out go up to the fifth term, the fifth power of x. Its derivative is now 1. We've gone through the pattern, we're back at +1 as the value of the iterated derivative, so now I get 1/5! x^5. Now, you tell me, have we done enough terms to see what the pattern is? I guess the next term will be a -1/7! x^7, and so on. Let me write this out again just so we have it. x^3 / 3!-- So it's x minus x^3 / 3! plus x^5 / 5!. You guessed it, and so on. That's the power series expansion for the sine of x, OK?

And so, the sign alternate, and these denominators get very big, don't they? Exponentials grow very fast. Let me make a remark. R is infinity here. The radius of convergence of this power series again is infinity, and let me just say why. The reason is that the general term is going to be like x^(2n+1) / (2n+1)!. An odd number I can write as 2n + 1. And what I want to say is that the size of this, what happens to the size of this as n goes to infinity? So let's just think about this. For a fixed x, let's fix the number x. Look at powers of x and think about the size of this expression when n gets to be large. So let's just do that for a second. So, x^(2n+1) / (2n+1)!, I can write out like this. It's x / 1 times x / 2 -- sorry -- times x / 3, times x / (2n+1). I've multiplied x by itself 2n+1 times in the numerator, and I've multiplied the numbers 1, 2, 3, 4, and so on, by each other in the denominator, and that gives me the factorial. So I've just written this out like this. Now x is fixed, so maybe it's a million, OK? It's big, but fixed. What happens to these numbers? Well at first, they're pretty big. This is 1,000,000 / 2, this is 1,000,000 / 3. But when n gets to be-- Maybe if n is 1,000,000, then this is about 1/2. If n is a billion, then this is about 1/2,000, right? The denominators keep getting bigger and bigger, but the numerators stay the same; they're always x.

So when I take the product, if I go far enough out, I'm going to be multiplying, by very, very small numbers and more and more of them. And so no matter what x is, these numbers will converge to 0. They'll get smaller and smaller as x gets to be bigger. That's the sign that x is inside of the radius of convergence. This is the sign for you that this series converges for that value of x. And because I could do this for any x, this works. This convergence to 0 for any

fixed x. That's what tells you that you can take-- that the radius of convergence is infinity. Because in the formula, in the fact, in this property that the radius of convergence talks about, if R is equal to infinity, this is no condition on x. Every number is less than infinity in absolute value. So if this convergence to 0 of the general term works for every x, then radius of convergence is infinity. Well that was kind of fast, but I think that you've heard something about that earlier as well.

Anyway, so we've got the sine function, a new function with its own power series. It's a way of computing sin(x). If you take enough terms you'll get a good evaluation of sin(x). for any x. This tells you a lot about the function sin(x) but not everything at all. For example, from this formula, it's very hard to see that the sine of x is periodic. It's not obvious at all. Somewhere hidden away in this expression is the number pi, the half of the period. But that's not clear from the power series at all. So the power series are very good for some things, but they hide other properties of functions. Well, so I want to spend a few minutes telling you about what you can do with a power series, once you have one, to get new power series, so new power series from old. And this is also called operations on power series. So what are the things that we can do to a power series? Well one of the things you can do is multiply. So, for example, what if I want to compute a power series for x sin(x)? Well I have a power series for sin(x), I just did it. How about a power series for x? Actually, I did that here too. The function x is a very simple polynomial. It's a polynomial where that's 0, a_1 is 1, and all the other coefficients are 0. So x itself is a power series, a very simple one. sin(x) is a powers series. And what I want to encourage you to do is treat power series just like polynomials and multiply them together. We'll see other operations too. So, to compute the power series for x sin(x), of I just take this one and multiply it by x. So let's see if I can do that right. It distributes through: x^2 minus x^4 / 3! plus x^6 / 5!, and so on. And again, the radius of convergence is going to be the smaller of the two radii of convergence here. So it's R equals infinity in this case. OK, you can multiply power series together. It can be a pain if the power series are very long, but if one of them is x, it's pretty simple.

OK, that's one thing I can do. Notice something by the way. You know that even and odd functions? So, sine is an odd function, x is an odd function, the product of two odd functions is an even function. And that's reflected in the fact that all the powers that occur in the power series are even. For an odd function, like the sine, all the powers that occur are odd powers of x. That's always true. OK, we can multiply. I can also differentiate. So let's just do a case of that, and use the process of differentiation to find out what the power series for cos(x) is by

writing the cos(x) as the derivative of the sine and differentiating term by term. So, I'll take this expression for the power series of the sine and differentiate it term by term, and I'll get the power series for cosine. So, let's see. The derivative of x is one. Now, the derivative of x^3 is 3x^2, and then there's a 3! in the denominator. And the derivative of x^5 5x^4, and there's a 5! in the denominator, and so on and so on. And now some cancellation happens. So this is 1 minus, well, the 3 cancels with the last factor in this 3 factorial and leaves you with 2!. And the 5 cancels with the last factor in the 5 factorial and leaves you with a 4! in the denominator. And so there you go, there's the power series expansion for the cosine. It's got all even powers of x. They alternate, and you have factorials in the denominator.

And of course, you could derive that expression by using Taylor's formula, by the same kind of calculation you did here, taking higher and higher derivatives of the cosine. You get the same periodic pattern of derivatives and values of derivatives at x = 0. But here's a cleaner way to do it, simpler way to do it, because we already knew the derivative of the sine. When you differentiate, you keep the same radius of convergence. OK, so we can multiply, I can add too and multiply by a constant, things like that. How about integrating? That's what half of this course was about isn't it? So, let's integrate something. So, the integration I'm going to do is this one: the integral from 0 to x of dt / (1+x). What is that integral as a function? So, when I find the anti-derivative of this, I get ln(1+t), and then when I evaluate that at t = x, I get ln(1+x). And when I evaluate the natural log at 0, I get the ln 1, which is 0, so this is what you get, OK?

This is really valid, by the way, for x bigger than -1. But you don't want to think about this quite like this when x is smaller than that. Now, I'm going to try to apply power series methods here and find-- use this integral to find a power series for the natural log, and I'll do it by plugging into this expression what the power series for 1/(1+t) was. And I know what that is because I wrote it down on the board up here. Change the variable from x to t there, and so 1/(1+t) is 1 minus t plus t^2 minus t^3, and so on. So that's the thing in the inside of the integral, and now it's legal to integrate that term by term, so let's do that. I'm going to get something which I will then evaluate at x and at 0. So, when I integrate 1 I get x, and when I integrate t, I get t. I'm sorry. When I integrate t, I get t^2 / 2, and t^2 gives me t^3 / 3, and so on and so on. And then, when I put in t = x, well, I just replace all the t's by x's, and when I put in t = 0, I get 0. So this equals x. So, I've discovered that ln(1+x) is x minus x^2 / 2 plus x^3 / 3 minus x^4 / 4, and so on and so on. There's the power series expansion for ln(1+x). And because I began with a power series whose radius of convergence was just 1, I began with this power series, the radius of convergence of this is also going to be 1. Also, because this function, as I just

pointed out, this function goes bad when x becomes less than -1, so some problem happens, and that's reflected in the radius of convergence. Cool. So, you can integrate. That is the correct power series expansion for the ln(1+x), and another victory of Euler's was to use this kind of power series expansion to calculate natural logarithms in a much more efficient way than people had done before.

OK, one more property, I think. What are we at here, 3? 4. Substitute. Very appropriate for me as a substitute teacher to tell you about substitution. So I'm going to try to find the power series expansion of $e^{(-t^2)}$. OK? And the way I'll do that is by taking the power series expansion for e^x, which we have up there, and make the substitution $x = -t^2$ in the expansion for e^x. Did you have a question?

AUDIENCE: Well, it's just concerning the radius of convergence. You can't define x so that is always positive, and if so, it wouldn't have a radius of convergence, right?

PROFESSOR: Like I say, again the worry is this ln(1+x) function is perfectly well behaved for large x. Why does the power series fail to converge for large x? Well suppose that x is bigger than 1, then here you get bigger and bigger powers of x, which will grow to infinity, and they grow large faster than the numbers 2, 3, 4, 5, 6. They grow exponentially, and these just grow linearly. So, again, the general term, when x is bigger than one, the general term will go off to infinity, even though the function that you're talking about, log of net of 1 plus x is perfectly good. So the power series is not good outside of the radius of convergence. It's just a fact of life. Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: I'd rather-- talk to me after class. The question is why is it the smaller of the two radii of convergence? The basic answer is, well, you can't expect it to be bigger than that smaller one, because the power series only gives you information inside of that range about the function, so.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Well, in this case, both of the radii of convergence are infinity. x has radius of convergence infinity for sure, and sin(x) does too. So you get infinity in that case, OK? OK, let's just do this, and then I'm going to integrate this and that'll be the end of what I have time for today. So what's the power series expansion for this? The power series expansion of this is going to be a function of t, right, because the variable here is t. I get it by taking my expansion for e^x and

putting in what x is in terms of t. Whoops! And so on and so on. I just put in $-t^2$ in place of x there in the series expansion for e^x. I can work this out a little bit better. $-t^2$ is what it is. This is going to give me a t^4 and the minus squared is going to give me a plus, so I get $t^4 / 2!$. Then I get $(-t)^3$, so there'll be a minus sign and a t^6 and the denominator $3!$. So the signs are going to alternate, the powers are all even, and the denominators are these factorials. Several times as this course has gone on, the error function has made an appearance. The error function was, I guess it gets normalized by putting a 2 over the square root of pi in front, and it's the integral of $e^{(-t^2)}$ dt from 0 to x. And this normalization is here because as x gets to be large the value becomes 1. So this error function is very important in the theory of probability. And I think you calculated this fact at some point in the course. So the standard definition of the error function, you put a 2 over the square root of pi in front. Let's calculate its power series expansion. So there's a 2 over the square root of pi that hurts nobody here in the front. And now I want to integrate $e^{(-t^2)}$, and I'm going to use this power series expansion for that to see what you get. So I'm just going to write this out I think. I did it out carefully in another example over there, so I'll do it a little quicker now. Integrate this term by term, you're just integrating powers of t so it's pretty simple, so I get-- and then I'm evaluating at x and then at 0. So I get x minus $x^3 / 3$, plus $x^5 / (5*2!)$, 5 from integrating the t^4, and the $2!$ from this denominator that we already had. And then there's a $-x^7 / (7*3!)$, and plus, and so on, and you can imagine how they go on from there. I guess to get this exactly in the form that we began talking about, I should multiply through. So the coefficient of x is 2 over the square root of pi, and the coefficient of x^3 is -2 over 3 times the square root of pi, and so on. But this is a perfectly good way to write this power series expansion as well. And, this is a very good way to compute the value of the error function. It's a new function in our experience. Your calculator probably calculates it, and your calculator probably does it by this method. OK, so that's my sermon on examples of things you can do with power series. So, we're going to do the CEG thing in just a minute. Professor Jerison wanted me to make an ad for 18.02. Just in case you were thinking of not taking it next term, you really should take it. It will put a lot of things in this course into context, for one thing. It's about vector calculus and so on. So you'll learn about vectors and things like that. But it comes back and explains some things in this course that might have been a little bit strange, like these strange formulas for the product rule and the quotient rule and the sort of random formulas. Well, one of the things you learn in 18.02 is that they're all special cases of the chain rule. And just to drive that point home, he wanted me to show you this poem of his that really drives the points home forcefully, I think.