More Examples of Integration

Example: $\int xe^{-x^2}\,dx$

For this we guess e^{-x^2}, hoping that the chain rule will somehow provide the missing factor of x in the integral. As usual, we take the derivative to check:

$$\frac{d}{dx}e^{-x^2} = (e^{-x^2})(-2x) = -2xe^{-x^2}$$

We're off by a factor of -2, so we divide our original guess by this constant to reach the conclusion that:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + c$$

Caution: If you solve integrals by guessing and don't check your answer by taking a derivative you're likely to make mistakes.

Example: $\int \sin x \cos x\,dx$

What's a good guess?

Student: $\sin^2 x$

Let's check it!

$$\frac{d}{dx}\sin^2 x = 2\sin x\cos x.$$

So:

$$\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x + c$$

An equally acceptable answer is:

$$\int \sin x\cos x\,dx = -\frac{1}{2}\cos^2 x + c$$

This seems like a contradiction; let's check our answer:

$$\frac{d}{dx}\cos^2 x = (2\cos x)(-\sin x) = -2\sin x\cos x$$

Both answers are correct! But we just proved that integrals are unique up to a constant. What's going on?

It turns out that the difference between the two answers *is* a constant:

$$\frac{1}{2}\sin^2 x - (-\frac{1}{2}\cos^2 x) = \frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$$

So,

$$\frac{1}{2}\sin^2 x - \frac{1}{2} = \frac{1}{2}(\sin^2 x - 1) = \frac{1}{2}(-\cos^2 x) = -\frac{1}{2}\cos^2 x$$

The two answers are, in fact, equivalent. The constant c is shifted by $\frac{1}{2}$ from one answer to the other.

Example: $\displaystyle\int \frac{dx}{x \ln x}$

We will assume $x > 0$ so that $\ln x$ is defined. We don't quickly come up with a good guess, so we use the method of substitution (which is the only other method we know). The ugliest part of the integral is the natural log, so we choose:

$$u = \ln x.$$

One advantage of this choice is that taking the differential of $\ln x$ makes it simpler: $du = \frac{1}{x}\, dx$. Substitute these into the integral to get:

$$\begin{aligned} \int \frac{dx}{x \ln x} &= \int \underbrace{\frac{1}{\ln x}}_{\frac{1}{u}} \underbrace{\frac{dx}{x}}_{du} \\ &= \int \frac{1}{u} du \\ &= \ln |u| + c \\ &= \ln |\ln(x)| + c \end{aligned}$$

For this example, the method of substitution is better than guessing.