Integration by Change of Variables

Use a change of variables to compute the following integrals. Change both the variable and the limits of substitution.

a) $\displaystyle\int_0^4 \sqrt{3x+4}\,dx$

b) $\displaystyle\int_1^3 \frac{x}{x^2+1}\,dx$

c) $\displaystyle\int_0^{\pi/2} \sin^5 x \cos x\,dx$

Solution

a) $\displaystyle\int_0^4 \sqrt{3x+4}\,dx$

The integrand of $\displaystyle\int_0^4 \sqrt{3x+4}\,dx$ consists of a (relatively complicated) square root function with a simple expression inside it. Expressions used as input to complicated functions are good candidates for substitution; if we let $u = 3x+4$ then $du = 3x\,dx$ and $dx = \frac{1}{3}\,du$. When $x = 0$, $u = 4$ and when $x = 4$, $u = 16$.

We now know enough to rewrite the integrand and limits of integration in terms of u, after which we can complete the integral.

$$\begin{aligned}
\int_{x=0}^{x=4} \sqrt{3x+4}\,dx &= \int_{u=4}^{u=16} \sqrt{u}\frac{1}{3}\,du \\
&= \frac{1}{3}\int_4^{16} u^{1/2}\,du \\
&= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_4^{16} \\
&= \frac{2}{9}\left(16^{3/2} - 4^{3/2}\right) \\
&= \frac{2}{9}(64-8) \\
&= \frac{112}{9} \approx 12
\end{aligned}$$

We can see from a rough graph of the function that the value of the integral must be between 8 and 16, so this is a reasonable answer.

b) $\displaystyle\int_1^3 \frac{x}{x^2+1}\,dx$

In this example, the most complicated operation is division by x^2+1. We'll try the substitution $u = x^2+1$:

$$u = x^2+1, \quad du = 2x\,dx, \quad x\,dx = \frac{1}{2}\,du.$$

When $x = 1$, $u = 2$. When $x = 3$, $u = 10$.

$$\begin{aligned}
\int_1^3 \frac{x}{x^2+1}\,dx &= \int_{x=1}^{x=3} \frac{1}{x^2+1} \cdot x\,dx \\
&= \int_{u=2}^{u=10} \frac{1}{u}\frac{1}{2}\,du \\
&= \frac{1}{2}\ln(u)\big|_{u=2}^{u=10} \\
&= \frac{1}{2}\left(\ln(10) - \ln(2)\right) \\
&= \frac{1}{2}\ln\left(\frac{10}{2}\right) = \ln(\sqrt{5}) \approx 1
\end{aligned}$$

This answer is not easy to check, but by using an electronic graphing tool we can confirm that it is approximately correct.

c) $\displaystyle\int_0^{\pi/2} \sin^5 x \cos x\,dx$

Although the sine and cosine functions may be the most complicated part of this problem, performing the substitution $u = x$ does not help us integrate this expression. We shift our focus to the next most complicated operation – exponentiation.

Let $u = \sin(x)$. Then $du = \cos(x)\,dx$ and when $x = 0$, $u = 0$; when $x = \pi/2$, $u = 1$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^5 x \cos x\,dx &= \int_{x=0}^{x=\pi/2} (\sin x)^5 \cdot \cos x\,dx \\
&= \int_0^1 u^5\,du \\
&= \frac{1}{6}u^6\Big|_0^1 \\
&= \left(\frac{1}{6}\cdot 1^6 - \frac{1}{6}\cdot 0^6\right) \\
&= \frac{1}{6}
\end{aligned}$$

Again the answer is hard to check without graphing. It seems reasonable given the fact that $0 \leq \sin x \leq 1$ on the interval in question.