Slope of the tangent to a^x

We defined a function $M(a)$ as follows:

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

This definition allows us to say that $\frac{d}{dx} a^x = M(a) a^x$. In order to understand the derivative of a^x we must understand $M(a)$; we next look at two different ways of thinking about $M(a)$.

First, if we plug $x = 0$ in to the definition of the derivative of a^x we get:

$$\begin{aligned} \left.\frac{d}{dx} a^x\right|_{x=0} &= \left.\lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}\right|_{x=0} \\ &= \lim_{\Delta x \to 0} \frac{a^{0+\Delta x} - a^0}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} \\ &= M(a) \end{aligned}$$

(or we could simply observe that $\frac{d}{dx} a^x|_{x=0} = M(a)a^0 = M(a)$). So $M(a)$ is the value of the derivative of a^x when $x = 0$.

Remember that the derivative tells us the slope of the tangent line to the graph. So $M(a)$ can also be thought of as the slope of the graph of $y = a^x$ at $x = 0$.

Note that the shape of the graph of a^x depends on the choice of a, so for different values for a we'll get different tangent lines and different values for $M(a)$.

Because $\frac{d}{dx} a^x = M(a) a^x$, we only need to know the slope of the line tangent to the graph at $x = 0$ in order to figure out the slope of the tangent line at any point on the graph!

Remember that when we computed the derivative of the sine function we worked hard to compute the value of $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$. This value is just the derivative of $\sin x$ when $x = 0$ — check this yourself by writing down the definition of the derivative of $\sin x$ and replacing x by 0. In order to get a general formula for the derivative of the sine function we first had to know the value of its derivative when $x = 0$.

The formula for $a^{x+\Delta x}$ is simpler than the one for $\sin(x + \Delta x)$, so the first part of our calculation of $\frac{d}{dx} a^x$ was easier than the corresponding calculation for $\sin x$. But when we try to compute:

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

we get stuck. We were able to use radians and the unit circle to find $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$,

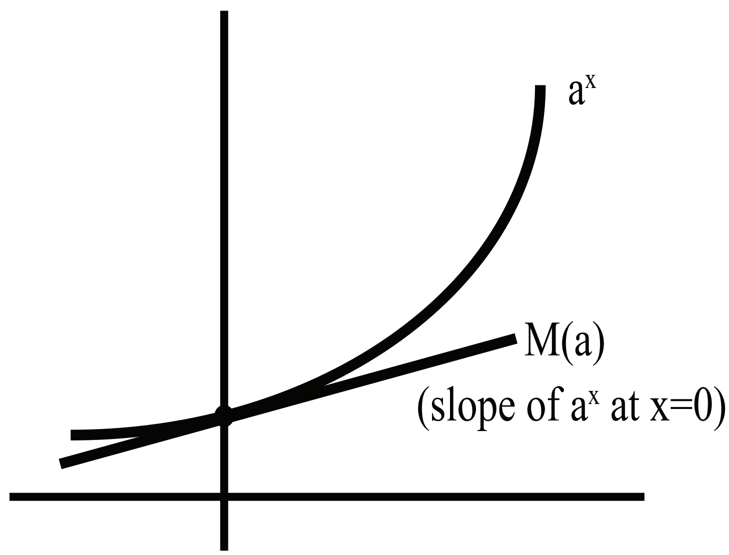

Figure 1: Geometric definition of $M(a)$

but we don't have a good way to find the exact slope of the tangent line to $y = a^x$ at $x = 0$.