Integral of a Power Series

We can multiply, add and differentiate power series. Can we integrate them? Yes; as you’d expect, integration of power series is very similar to integration of polynomials. We’ll use integration to find a power series expansion for:

\[ \ln(1 + x) = \int_0^x \frac{dt}{1 + t} \quad (x > -1). \]

We know that:

\[ \frac{1}{1 + t} = 1 - t + t^2 - t^3 + \cdots \]

So:

\[
\ln(1 + x) = \int_0^x \left(1 - t + t^2 - t^3 + \cdots\right)dt \\
= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right]_0^x \\
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\]

Because we began with a power series whose radius of convergence was 1, the radius of convergence of the result will also be 1. This reflects the fact that \( \ln(1 + x) \) is undefined for \( x \leq -1 \).

**Question:** If you only use positive values of \( x \) is there still a radius of convergence?

**Answer:** Yes. If \( x > 1 \) then the numerators \( x, x^2, x^3, x^4 \) and so on are increasing exponentially. The denominators \( 1, 2, 3, 4 \ldots \) only grow linearly. So as \( n \) goes to infinity, \( \frac{x^n}{n} \) will also go to infinity. If the terms of a series go to infinity then the series diverges.

Euler used this kind of power series expansion to calculate natural logarithms much more efficiently than was previously possible.