a^x and the Definition of the Derivative

Our goal is to calculate the derivative $\frac{d}{dx}a^x$. It's going to take us a while.

We start by writing down the definition of the derivative

$$\frac{d}{dx}a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

We can use the rule $a^{x_1+x_2} = a^{x_1}a^{x_2}$ to factor out a^x:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{a^x a^{\Delta x} - a^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x}\end{aligned}$$

As we're taking this limit, we're holding a and x fixed while Δx changes (approaches zero). This means that for the purposes of taking this limit, a^x is a constant. We can therefore factor the constant multiple out of the limit to get:

$$\frac{d}{dx}a^x = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

We've made a good start at finding the derivative of a^x; let's look at what we have so far. We can see from our calculations that $\frac{d}{dx}a^x$ is a^x times some multiple whose value we don't yet know. Let's call that multiple $M(a)$:

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

Using this definition of $M(a)$, we can say that $\frac{d}{dx}a^x = M(a)a^x$.