Implicit Differentiation and the Chain Rule

The chain rule tells us that:

$$\frac{d}{dx}(f \circ g) = \frac{df}{dg}\frac{dg}{dx}.$$

While implicitly differentiating an expression like $x + y^2$ we use the chain rule as follows:

$$\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy}\frac{dy}{dx} = 2yy'.$$

Why can we treat y as a function of x in this way?

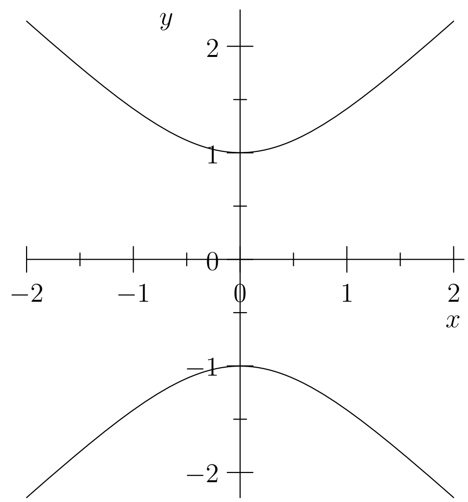

Figure 1: The hyperbola $y^2 - x^2 = 1$.

Consider the equation $y^2 - x^2 = 1$, which describes the hyperbola shown in Figure 1. We cannot write y as a function of x, but if we start with a point (x, y) on the graph and then change its x coordinate by sliding the point along the graph its y coordinate will be constrained to change as well. The change in y is *implied* by the change in x and the constraint $y^2 - x^2 = 1$. Thus, it makes sense to think about $y' = \frac{dy}{dx}$, the rate of change of y with respect to x.

Given that $y^2 - x^2 = 1$:

a) Use implicit differentiation to find y'.

b) Check your work by using Figure 1 to estimate the slope of the tangent line to the hyperbola when $y = -1$ and when $x = 1$.

c) Check your work for $y > 0$ by solving for y and using the direct method to take the derivative.

Solution

a) Use implicit differentiation to find y'.

We differentiate both sides of the equation $y^2 - x^2 = 1$ to get:

$$\begin{aligned} 2yy' - 2x &= 0 \\ 2yy' &= 2x \\ y' &= \frac{2x}{2y} \\ y' &= \frac{x}{y} \end{aligned}$$

b) Check your work by using Figure 1 to estimate the slope of the tangent line to the hyperbola when $y = -1$ and when $x = 1$.

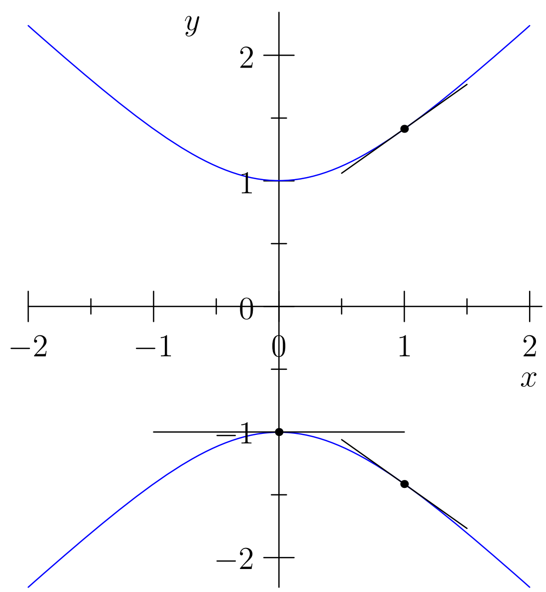

Figure 2: Tangent lines to the hyperbola $y^2 - x^2 = 1$.

From the graph, we see that when $y = -1$, $x = 0$ and so $y' = x/y = 0$. This agrees with the fact that at the point $(-1, 0)$ the tangent line to the graph is horizontal.

When $x = 1$, $y^2 = 2$ so $y = \pm\sqrt{2}$. At the point $(1, \sqrt{2})$ the slope of the tangent line is $y' = x/y = 1/\sqrt{2} \approx 2/3$ and at $(1, -\sqrt{2})$ the slope of the tangent line is approximately $-2/3$.

c) Check your work for $y > 0$ by solving for y and using the direct method to take the derivative.

We start by solving for y:

$$\begin{aligned} y^2 - x^2 &= 1 \\ y^2 &= 1 + x^2 \\ y &= \pm\sqrt{1+x^2} \end{aligned}$$

We're told to consider the case $y > 0$, in which $y = \sqrt{1+x^2} = (1+x^2)^{1/2}$.

$$\begin{aligned} \frac{d}{dx}(1+x^2)^{1/2} &= \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x \\ &= \frac{x}{\sqrt{1+x^2}} \\ &= \frac{x}{y} \end{aligned}$$

We conclude that for $y > 0$, differentiating directly gives the same result as implicit differentiation.