Comparison of the Harmonic Series

We're going to compare:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

to $\displaystyle\int_1^{\infty} \frac{dx}{x}$, using Riemann sums to show that the series diverges. The same sort of reasoning is applicable to the previous two examples.

We'll use a Riemann sum to calculate the area under the graph of $y = \frac{1}{x}$ using $\Delta x = 1$. We also get to choose whether to compute the upper Riemann sum or lower; we'll do both.

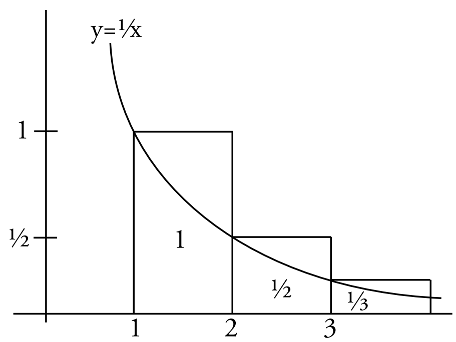

Figure 1: Upper Riemann sum; $y = \frac{1}{x}$ (not to scale).

The upper Riemann sum is the sum of the areas of the rectangles indicated in Figure 1. If we stop at the rectangle just before N we see that the area under the curve from 1 to N is $\int_0^N \frac{dx}{x}$ and that that's less than the sum of the areas of the rectangles:

$$\int_1^N \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N-1}.$$

(There are only $N - 1$ rectangles here because the distance between 1 and N is $N - 1$.) If $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$, then because there's one more term in the sum we have:

$$\int_1^N \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N-1} < S_N.$$

Thus we know that the value of the integral is less than the value of S_N. This will allow us to prove conclusively that the series diverges.

$$\int_1^N \frac{dx}{x} \quad < \quad S_N$$

$$\begin{aligned} \ln x\big|_1^N &< S_N \\ \ln N - \ln 1 &< S_N \\ \ln N - 0 &< S_N \\ \ln N &< S_N \end{aligned}$$

As N goes to infinity, $\ln N$ goes to infinity and so S_N must also go to infinity. By definition,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{N\to\infty} S_N,$$

so the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

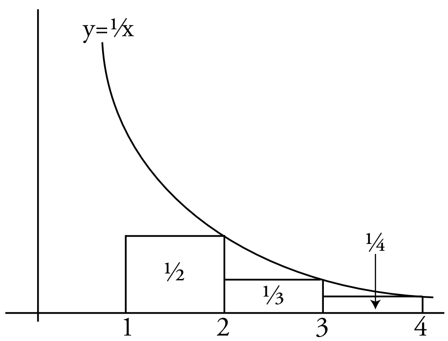

Figure 2: Lower Riemann sum; $y = \frac{1}{x}$ (not to scale).

Now we'll use the lower Riemann sum to see that S_N goes to infinity at the same rate as $\int_1^N \frac{dx}{x}$. Since the sum of the areas of the rectangles shown in Figure 2 is less than the area under the curve, we know that:

$$\int_1^N \frac{dx}{x} > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} = S_N - 1.$$

Recall that $\int_1^N \frac{dx}{x} = \ln N$, so we get:

$$\begin{aligned} S_N - 1 &< \int_1^N \frac{dx}{x} \\ S_N &< (\ln N) + 1. \end{aligned}$$

Combining this with the result from the upper Riemann sum, we conclude:

$$\ln N < S_N < (\ln N) + 1.$$

The value of S_N is hard to calculate exactly, but now we know that it's between $\ln N$ and $(\ln N) + 1$, which we can compute.