Product formula (General)

The product rule tells us how to take the derivative of the product of two functions:

$$(uv)' = u'v + uv'$$

This seems odd — that the product of the derivatives is a sum, rather than just a product of derivatives — but in a minute we'll see why this happens.

First, we'll use this rule to take the derivative of the product $x^n \sin x$ – a function we would not be able to differentiate without this rule. Here the first function, u is x^n and the second function v is $\sin x$. According to the specific rule for the derivative of x^n, the derivative u' must be nx^{n-1}. If $v = \sin x$ then $v' = \cos x$. The product rule tells us that $(uv)' = u'v + uv'$, so

$$\frac{d}{dx} x^n \sin x = nx^{n-1} \sin x + x^n \cos x.$$

By applying this rule repeatedly, we can find derivatives of more complicated products:

$$\begin{aligned} (uvw)' &= u'(vw) + u(vw)' \\ &= u'vw + u(v'w + vw') \\ &= u'vw + uv'w + uvw'. \end{aligned}$$

Now let's see why this is true:

$$\begin{aligned} (uv)' &= \lim_{\Delta x \to 0} \frac{(uv)(x + \Delta x) - (uv)(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x)}{\Delta x} \end{aligned}$$

We want our final formula to appear in terms of u, v, u' and v'. We know that $u' = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$, and we see that $u(x + \Delta x)v(x + \Delta x) - u(x)v(x)$ looks a little bit like $(u(x + \Delta x) - u(x))v(x)$. By using a little bit of algebra we can get $(u(x + \Delta x) - u(x))v(x)$ to appear in our formula; this process is described below.

First, notice that:

$$u(x + \Delta x)v(x) - u(x + \Delta x)v(x) = 0.$$

Adding zero to the numerator doesn't change the value of our expression, so:

$$(uv)' = \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x) - u(x)v(x) + u(x + \Delta x)v(x + \Delta x) - u(x + \Delta x)v(x)}{\Delta x}.$$

We then re-arrange that expression to get:

$$(uv)' = \lim_{\Delta x \to 0} \left[\left(\frac{u(x + \Delta x) - u(x)}{\Delta x} \right) v(x) + u(x + \Delta x) \left(\frac{v(x + \Delta x) - v(x)}{\Delta x} \right) \right]$$

We proved that if u and v are differentiable they must be continuous, so the limit of the sum is the sum of the limits:

$$\left[\lim_{\Delta x \to 0} \frac{u(x+\Delta x)-u(x)}{\Delta x}\right] v(x) + \lim_{\Delta x \to 0}\left(u(x+\Delta x)\left[\frac{v(x+\Delta x)-v(x)}{\Delta x}\right]\right)$$

or in other words,

$$(uv)' = u'(x)v(x) + u(x)v'(x).$$

Note: we also used the fact that:

$$\lim_{\Delta x \to 0} u(x+\Delta x) = u(x),$$

which is true because u is differentiable and therefore continuous.