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18.02 Multivariable Calculus, Fall 2007
Transcript – Lecture 17

Yesterday we saw how to define double integrals and how to start computing them in terms of x and y coordinates. We have defined the double integral over a region R and plane of a function f of x, y dA . You cannot hear me? Is the sound working? Can you hear me in the back now? Can we make the sound louder? Does this work? People are not hearing me in the back. Is it better? People are still saying make it louder.

Is it better? OK. Great. Thanks. That's not a reason to start chatting with your friends. Thanks. When we have a region in the x, y plane and we have a function of x and y , we are defining the double integral of f over this region by taking basically the sum of the values of a function everywhere in here times the area element. And the definition, actually, is we split the region into lots of tiny little pieces, we multiply the value of a function at the point times the area of a little piece and we sum that everywhere. And we have seen, actually, how to compute these things as iterated integrals.

First, integrating over dy and then over dx , or the other way around. One example that we did, in particular, was to compute the double integral of a quarter of a unit disk. That was the region where $x^2 + y^2 < 1$ and x and y are positive, of $1 - x^2 - y^2$ dA . Well, hopefully, I kind of convinced you that we can do it using enough trig and substitutions and so on, but it is not very pleasant. And the reason for that is that using x and y coordinates here does not seem very appropriate. In fact, we can use polar coordinates instead to compute this double integral.

Remember that polar coordinates are about replacing x and y as coordinates for a point on a plane by instead r , which is the distance from the origin to a point, and θ , which is the angle measured counterclockwise from the positive x -axis. In terms of r and θ , you have $x = r \cos \theta$, $y = r \sin \theta$. The claim is we are able, actually, to do double integrals in polar coordinates. We just have to learn how to. Just to draw a quick picture --

When we were integrating in x, y coordinates, in rectangular coordinates, we were slicing our region by gridlines that were either horizontal or vertical. And we used that to set up the iterated integral. And we said dA became $dx dy$ or $dy dx$. Now we are going to actually integrate, in terms of the polar coordinates, r and θ . Let's say we will integrate in the order with r first and then θ . That is the order that makes the most sense usually when you do polar coordinates. What does that mean? It means that we will first focus on a slice where we fix the value of θ and we will let r vary. That means we fix a direction, we fix a ray out from the origin in a certain direction.

And we will travel along this ray and see which part of it, which values of r are in our region. Here it will be actually pretty easy because r will just start at zero, and you will have to stop when you exit this quarter disk. Well, what is the equation of this

circle in polar coordinates? It is just r equals one. So, we will stop when r reaches one. But what about θ ? Well, the first ray that we might want to consider is the one that goes along the x -axis. That is when θ equals zero. And we will stop when θ reaches π over two because we don't care about the rest of the disk. We only care about the first quadrant.

We will stop at π over two. Now, there is a catch, though, which is that dA is not $dr d\theta$. Let me explain to you why. Let's say that we are slicing. What it means is we are cutting our region into little pieces that are the elementary, you know, what corresponds to a small rectangle in the x, y coordinate system, here would be actually a little piece of circle between a given radius r and r plus Δr .

And given between an angle θ and θ plus $\Delta\theta$. I need to draw, actually, a bigger picture of that because it makes it really hard to read. Let's say that I fix an angle θ and a slightly different one where I have added $\Delta\theta$ to it. And let's say that I have a radius r and I add Δr to it. Then I will have a little piece of x, y plane that is in here. And I have to figure out what is its area? What is ΔA for this guy? Well, let's see. This guy actually, you know, if my Δr and $\Delta\theta$ are small enough, it will almost look like a rectangle. It is rotated, but it is basically a rectangle.

I mean these sides, of course, are curvy, but they are short enough and it is almost straight. The area here should be this length times that length. Well, what is this length? That one is easy. It is Δr . What about that length? Well, it is not $\Delta\theta$. It is something slightly different. It is a piece of a circle of radius r corresponding to angle $\Delta\theta$, so it is $r \Delta\theta$.

So, times $r \Delta\theta$. That means now, even if we shrink things and take smaller and smaller regions, dA is going to be $r dr d\theta$. That is an important thing to remember. When you integrate in polar coordinates, you just set up your bounds in terms of r and θ , but you replace dA by $r dr d\theta$, not just $dr d\theta$. And then, of course, we have some function that we are integrating. Let's say that I call that thing f then it is the same f that I put up here.

Concretely, how do I do it here? Well, my function f was given as one minus x squared minus y squared. And I would like to switch that to polar coordinates. I want to put r and θ in there. Well, I have formulas for x and y in polar coordinates so I could just replace x squared by r squared cosine squared θ , y squared by r squared sine squared θ . And that works just fine. But maybe you can observe that this is x squared plus y squared. It is just the square of a distance from the origin, so that is just r squared. That is a useful thing. You don't strictly need it, but it is much faster if you see this right away. It saves you writing down a sine and a cosine.

Now we just end up with the integral from zero to π over two, integral from zero to one of one minus r squared $r dr d\theta$. Now, if I want to compute this integral, so let's first do the inner integral. If I integrate r minus r cubed, I will get r squared over two minus r squared over four between zero and one. And then I will integrate $d\theta$. What is this equal to? Well, for r equals one you get one-half minus one-quarter, which is going to be just one-quarter. And when you plug in zero you get zero. So, it is the integral from zero to π over two of one-quarter $d\theta$. And that just integrates to one-quarter times π over two, which is π over eight.

That is a lot easier than the way we did it yesterday. Well, here we were lucky. I mean usually you will switch to polar coordinates either because the region is easier to set up. Here it is indeed easier to set up because the bounds became very simple. We don't have that square root of one minus x squared anymore. Or because the integrand becomes much simpler. Here our function, well, it is not very complicated in x, y coordinates, but it is even simpler in r theta coordinates. Here we were very lucky. In general, there is maybe a trade off. Maybe it will be easier to set up bounds but maybe the function will become harder because it will have all these sines and cosines in it. If our function had been just x , x is very easy in x, y coordinates. Here it becomes $r \cos \theta$. That means you will have a little bit of trig to do in the integral.

Not a very big one, not a very complicated integral, but imagine it could get potentially much harder. Anyway, that is double integrals in polar coordinates. And the way you set up the bounds in general, well, in 99% of the cases you will integrate over r first. What you will do is you will look for a given theta what are the bounds of r to be in the region. What is the portion of my ray that is in the given region? And then you will put bounds for theta. But conceptually it is the same as before. Instead of slicing horizontally or vertically, we slice radially. We will do more examples in a bit. Any questions about this or the general method? Yes?

That is a very good question. Why do I measure the length inside instead of outside? Which one do I want? This one. Here I said this side is $r \Delta \theta$. I could have said, actually, $r \Delta \theta$ is the length here. Here it is slightly more, $r + \Delta r$ times $\Delta \theta$. But, if Δr is very small compared to r , then that is almost the same thing. And this is an approximation anyway. I took this one because it gives me the simpler formula. If you take the limit as Δr turns to zero then the two things become the same anyway. The length, whether you put r or $r + \Delta r$ in here, doesn't matter anymore.

If you imagine that this guy is infinitely small then, really, the lengths become the same. We will also see another proof of this formula, using changes of variables, next week. But, I mean, hopefully this is at least slightly convincing. More questions? No. OK. Let's see. We have seen how to compute double integrals. I have to tell you what they are good for as well. The definition we saw yesterday and the motivation was in terms of finding volumes, but that is not going to be our main preoccupation. Because finding volumes is fun but that is not all there is to life. I mean, you are doing single integrals.

When you do single integrals it is usually not to find the area of some region of a plane. It is for something else usually. The way we actually think of the double integral is really as summing the values of a function all around this region. We can use that to get information about maybe the region or about the average value of a function in that region and so on. Let's think about various uses of double integrals.

The first one that I will mention is actually something you thought maybe you could do with a single integral, but it is useful very often to do it as a double integral. It is to find the area of a given region R . I give you some region in the plane and you want to know just its area. In various cases, you could set this up as a single integral, but often it could be useful to set it up as a double integral. How do you express the area as a double integral? Well, the area of this region is the sum of the areas of all the little pieces. It means you want to sum one dA of the entire region. The area R is the double integral over R of a function one.

One way to think about it, if you are really still attached to the idea of double integral as a volume, what this measures is the volume below the graph of a function one. The graph of a function one is just a horizontal plane at height one. What you would be measuring is the volume of a prism with base r and height one. And the volume of that would be, of course, base times height. It would just be the area of r again. But we don't actually need to think about it that way. Really, what we are doing is summing dA over the entire region. A related thing we can do, imagine that, actually, this is some physical object. I mean, it has to be a flat object because we are just dealing with things in the plane so far.

But you have a flat metal plate or something and you would like to know its mass. Well, its mass is the sum of the masses of every single little piece. You would get that by integrating the density. The density for a flat object would be the mass per unit area. So, you can get the mass of a flat object with density. Let's use δ for density, which is the mass per unit area. Each little piece of your object will have a mass, which will be just the density, times its area for each small piece.

And you will get the total mass by summing these things. The mass will be the double integral of the density times the area element. Now, if it has constant density, if it is always the same material then, of course, you could just take the density out and you will get density times the total area if you know that it is always the same material. But if, actually, it has varying density maybe because it is some metallic thing with various metals or with varying thickness or something then you can still get the mass by integrating the density. Of course, looking at flat objects might be a little bit strange. That is because we are only doing double integrals so far. In a few weeks, we will be triple integrals. And then we will be able to do solids in space, but one thing at a time.

Another useful application is to find the average value of some quantity in a region. What does it mean to take the average value of some function f in this region r ? Well, you know what the average of a finite set of data is. For example, if I asked you to compute your average score on 18.02 problem sets, you would just take the scores, add them and divide by the number of problem sets. What if there are infinitely many things? Say I ask you to find the average temperature in this room. Well, you would have to measure the temperature everywhere.

And then add all of these together and divide by the number of data points. But, depending on how careful you are, actually, there are potentially infinitely many points to look at. The mathematical way to define the average of a continuous set of data is that you actually integrate the function over the entire set of data, and then you divide by the size of the sample, which is just the area of the region.

In fact, the average of f , the notation we will use usually for that is \bar{f} with a bar on top to tell us it is the average f . We say we will take the integral of f and we will divide by the area of the region. You can really think of it as the sum of the values of f everywhere divided by the number of points everywhere. And so that is an average where everything is, actually, equally likely. That is a uniform average where all the points on the region, all the little points of the region are equally likely. But maybe if want to do, say, an average of some solid with variable density or if you want to somehow give more importance to certain parts than to others then you can actually do a weighted average.

What is a weighted average? Well, in the case of taking the average your problem sets, if I tell you problem set one is worth twice as much as the others, then you would count twice that score in the sum and then you would count it as two, of course, when you divide. The weighted average is the sum of the values, but each weighted by a certain coefficient. And then you will divide by the sum of the weight. It is a bit the same idea as when we replace area by some mass that tells you how important a given piece. We will actually have a density. Let's call it δ again. We will see what we divide by, but what we will take is the integral of a function times the density times the area element.

Because this would correspond to the mass element telling us how to weight the various points of our region. And then we would divide by the total weight, which is the mass of a region, as defined up there. If a density is uniform then, of course, the density gets out and you can simplify and reduce to that if all the points are equally likely. Why is that important? Well, that is important for various applications. But one that you might have seen in physics, we care about maybe where is the center of mass of a given object? The center of mass is basically a point that you would say is right in the middle of the object. But, of course, if the object has a very strange shape or if somehow part of it is heavier than the rest then that takes a very different meaning.

Strictly speaking, the center of mass of a solid is the point where you would have to concentrate all the mass if you wanted it to behave equivalently from a point of view of mechanics, if you are trying to do translations of that object. If you are going to push that object that would be really where the equivalent point mass would lie. The other way to think about it, if I had a flat object then the center of mass would basically be the point where I would need to hold it so it is perfectly balanced. And, of course, I cannot do this. Well, you get the idea. And the center of mass of this eraser is somewhere in the middle. And so, in principle, that is where I would have to put my finger for it to stay. Well, it doesn't work.

But that is where the center of mass should be. I think it should be in the middle. Maybe I shouldn't call this three. I should call this 2a, because it is really a special case of the average value. How do we find the center of mass of a flat object with density δ . If you have your object in the x, y plane then its center of mass will be at positions that are actually just the coordinates of a center of mass, will just be weighted averages of x and y on the solid.

So, the center of mass will be a position that I will call \bar{x} , \bar{y} . And these are really just the averages, the average values of x and of y in the solid. Just to give you the formulas again, \bar{x} would be one over the mass times the double integral of x times density dA . And the same thing with y . \bar{y} is the weighted average of a y coordinate in your region. You see, if you take a region that is symmetric and has uniform density that will just give you the center of the region. But if the region has a strange shape or if a density is not homogeneous, if parts of it are heavier then you will get whatever the weighted average will be.

And that will be the point where this thing would be balanced if you were trying to balance it on a pole or on your finger. Any questions so far? Yes. No. Here I didn't set this up as a iterated integral yet. The function that I am integrating is x times δ where density will be given to me maybe as a function of x and y . And then I will integrate this dA . And dA could mean dx over dy , it could mean dy over dx , it could be mean r dr $d\theta$.

I will choose how to set it up depending maybe on the shape of the region. If my solid is actually just going to be round then I might want to use polar coordinates. If it is a square, I might want to use x, y coordinates. If it is more complicated, well, I will choose depending on how I feel about it. Yes? Δ is the density. In general, it is a function of x and y . If you imagine that your solid is not homogenous then its density will depend on which piece of it you are looking at. Of course, to compute this, you need to know the density. If you have a problem asking you to find the center of mass of something and you have no information about the density, assume it is uniform. Take the density to be a constant. Even take it to be a one. That is even easier. I mean it is a general fact of math. We don't care about units.

If density is constant, we might as well take it to be one. That just means our mass unit becomes the area unit. Yes? That is a good question. No, I don't think we could actually find the center of mass in polar coordinates by finding the average of R or the average of θ . For example, take a disk center at the origin, well, the center of mass should be at the origin. But the average of R is certainly not zero because R is positive everywhere. So, that doesn't work. You cannot get the polar coordinates of a center of mass just by taking the average of R and the average of θ . By the way, what is the average of θ ? If you take θ to go from zero to 2π , the average θ will be π . If you take it to go from minus π to π , the average θ will be zero. So, there is a problem there. That actually just doesn't work, so we really have to compute \bar{x} and \bar{y} .

But still we could set this up and then switch to polar coordinates to evaluate this integral. But we still would be computing the average values of x and y . We are basically re-exploring mechanics and motion of solids here. The next thing is moment of inertia. Just to remind you or in case you somehow haven't seen it in physics yet, the moment of inertia is basically to rotation of a solid where the mass is to translation. In the following sense, the mass of a solid is what makes it hard to push it. How hard it is to throw something is related to its mass. How hard it is to spin something, on the other hand, is given by its moment of inertia.

Maybe I should write this down. Mass is how hard it is to impart a translation motion to a solid. I am using fancy words today. And the moment of inertia -- The difference with a mass is that the moment of inertia is defined about some axis. You choose an axis. Then you would try to measure how hard it is to spin your object around that axis. For example, you can try to measure how hard it is to spin this sheet of paper about an axis that is in the center of it. We would try to spin it light that and see how much effort I would have to make. Well, for a sheet of paper not very much.

That would measure the same thing but it would be rotation motion about that axis. Maybe some of you know the definition but I am going to try to derive it again. I am sorry but it won't be as quite as detailed as the way you have probably seen it in physics, but I am not trying to replace your physics teachers. I am sure they are doing a great job. What is the idea for the definition to find a formula for moment of inertia? The idea is to think about kinetic energy. Kinetic energy is really when you push something or when you try to make it move and you have to put some inertia to it. Then it has kinetic energy. And then, if you have the right device, you can convert back that kinetic energy into something else.

If you try to look at the kinetic energy of a point mass, so you have something with mass m going at the velocity v , well, that will be one-half of a mass times the square

of the speed. I hope you have all seen that formula some time before. Now, let's say instead of just trying to push this mass, I am going to make it spin around something. Instead of just somewhere, maybe I will have the origin, and I am trying to make it go around the origin in a circle at a certain angular velocity. For a mass m at distance r , let's call r this distance.

And angular velocity, let's call the angular velocity ω . I think that is what physicists call it. Remember angular velocity is just the rate of the change of the angle over time. It is $d\theta/dt$, if you want. Well, what is the kinetic energy now? Well, first we have to find out what the speed is. What is the speed? Well, if we are going on a circle of radius r at angular velocity ω that means that in unit time we rotate by ω and we go by a distance of r times ω . The actual speed is the radius times angular velocity. And so the kinetic energy is one-half mv^2 , which is one-half $m r^2 \omega^2$.

And so, by similarity with that formula, the coefficient of v^2 is the mass, and here we will say the coefficient of ω^2 , so this thing is the moment of inertia. That is how we define moment of inertia. Now, that is only for a point mass. And it is kind of fun to spin just a small bowl, but maybe you would like to spin actually a larger solid and try to define this moment of inertia. Well, the moment of inertia of a solid will be just the sum of the moments of inertia of all the little pieces. What we will do is just cut our solid into little chunks and will sum this thing for each little piece.

For a solid with density δ , each little piece has mass which is the density times the amount of area. This is equal actually. And the moment of inertia of that small portion of a solid will be δm , the small mass, times r^2 , the square of a distance to the center of the axis along which I am spinning. That means if I sum these things together, well, it has moment of inertia δm times r^2 , which is r^2 times the density times δA .

And so I will be summing these things together. And so, the moment of inertia about the origin will be the double integral of r^2 times density times dA . The final formula for the moment of inertia about the origin is the double integral of a region of r^2 density dA . If you are going to do it in x, y coordinates, of course, r^2 becomes $x^2 + y^2$, it is the square of the distance from the origin. When you integrate this, that tells you how hard it is to spin that solid about the origin. The motion that we try to do --

We keep this fixed and then we just rotate around the origin. Sorry. That is a pretty bad picture, but hopefully you know what I mean. And the name we use for that is I_0 . And then the rotational kinetic energy is one-half times this moment of inertia times the square of the angular velocity. So that shows as that this replaces the mass for rotation motions. OK. What about other kinds of rotations? In particular, we have been rotating things about just a point in the plane.

What you could imagine also is instead you have your solid. What I have done so far is I have skewered it this way, and I am rotating around the axis. Instead, I could skewer it through, say, the horizontal axis. And then I could try to spin about the horizontal axis so then it would rotate in space in that direction like that. Let's say we do rotation about the x -axis. Well, the idea would still be the same. The moment of inertia for any small piece of a solid would be its mass element times the square

of a distance to the x axes because that will be the radius of a trajectory. If you take this point here, it is going to go in a circle like that centered on the x-axis.

So the radius will just be this distance here. Well, what is this distance? It is just y , or maybe absolute value of y . Distance to x-axis is absolute value of y . What we actually care about is the square of a distance, so it will just be y squared. The moment of inertia about the x-axis is going to be obtained by integrating y squared times the mass element. It is slightly strange but I have y in inertia about the x-axis. But, if you think about it, y tells me how far I am from the x-axis, so how hard it will be to spin around the x-axis.

And I could do the same about any axis that I want. Just I would have to sum the square of a distance to the axis of rotation. Maybe I should do an example. Yes? Same thing as above, distance to the x-axis, because that is what we care about. For the moment of inertia, we want the square of a distance to the axis of rotation. Let's do an example. Let's try to figure out if we have just a uniform disk how hard it is to spin it around its center. That shouldn't be very hard to figure out.

Say that we have a disk of radius a and we want to rotate it about its center. And let's say that it is of uniform density. And let's take just the density to be a one so that we don't really care about the density. What is the moment of inertia of that? Well, we have to integrate of our disk r squared times the density, which is one, times dA . What is r squared? You have here to resist the urge to say the radius is just a . We know the radius is a . No, it is not a because we are looking at rotation of any point inside this disk. And, when you are inside the disk, the distance to the origin is not a .

It is less than a . It is actually anything between zero and a . Just to point out a pitfall, r here is really a function on this disk. And we are going to integrate this function. Don't plug r equals a just yet. What coordinates do we use to compute this integral? They are probably polar coordinates, unless you want a repeat of what happened already with x and y . That will tell us we want to integrate r squared time r dr $d\theta$.

And the bounds for r , well, r will go from zero to a . No matter which direction I go from the origin, if I fixed it, r goes from zero to r equals a . The part of this ray that lives inside the disk is always from zero to a . And θ goes from, well, zero to 2π for example. And now you can compute this integral. Well, I will let you figure it out. But the inner integral becomes a to the four over four and the outer multiplies things by 2π , so you get πa to the four over two. OK. That is how hard it is to spin this disk. Now, what about instead of spinning it about the center we decided to spin it about a point on a second point.

For example, think of a Frisbee. A Frisbee has this rim so you can actually try to make it rotate around the point on the circumference by holding it near the rim and spinning it there. How much harder is that than around the center? Well, we will try to compute now the moment of inertia about this point. We have two options. One is we keep the system of coordinates centers here. But then the formula for distance to this point becomes harder. The other option, which is the one I will choose, is to change the coordinate so that this point become the origin. Let's do that.

About a point on the circumference, what I would have to do maybe is set up my region like that. I have moved the origin so that it is on the circumference of a disk,

and I will again try to find the moment of inertia of this disk about the origin. It is still, for the the double integral of $r^2 dA$. But now I want to find out how to set up the integral. I could try to use x, y coordinates and it would work.

Or I can use polar coordinates, and it works a little bit better that way. But both are doable. Let's say I do it this way. I have to figure out how to set up my bounds. What are the bounds for r ? Well, if I fix a value for θ , which means I chose an angle here, now I am shooting a ray from the origin in that direction. I enter my region at r equals zero. That hasn't changed. The question is where do I exit the region? What is that distance? Maybe you have seen it in recitation, maybe not. Let's see. Actually, I should have written down the radius of a circle is a .

So this distance here is $2a$. If you draw this segment in here, you know that here you have a right angle. You have a right triangle. The hypotenuse here has length $2a$. This angle is θ . Well, this length is $2a \cos \theta$. The polar coordinates equation of this circle passing through the origin is $r = 2a \cos \theta$. So, r will go from zero to $2a \cos \theta$. That is the distance here. Now, what are the bounds for θ ? It is not quite zero to 2π because, actually, you see in this direction, if I shoot a ray in this direction I will never meet my region.

We have to actually think a bit more. Well, the directions in which I will actually hit my circle are all the directions in the right half of a plane. I mean, of course, if I shoot very close to the axis, you might think, oh, I won't be in there. But, actually, that is not true because here the circle is tangent to the axis. No matter which direction I take, I will still have a little tiny piece.

The angle actually goes from minus $\pi/2$ to $\pi/2$. If you compute that you will get, well, the inner integral will be r^3 over four between zero and $2a \cos \theta$, which will turn out to be $4a^4 \cos^4 \theta$. And now you will integrate that for minus $\pi/2$ to $\pi/2$. And that is, again, the evil integral that we had yesterday. Either we remember the method from yesterday or we remember from yesterday that actually there are formulas in the notes to help you. On homework, you can use these formulas. In the notes at the beginning of section 3b there are formulas for these particular kinds of integrals.

And that will end up being three-halves of πa^4 . In case you wanted to know, it is three times harder to spin a Frisbee about a point on a circumference than around the center. We got three times the moment of inertia about the center. OK. That is it. Have a nice weekend.