18.02 Problem Set 1 - Solutions of Part B

Problem 1

a) The vertices of the tetrahedron are $P = (1,1,1)$, $Q = (1,-1,-1)$, $R = (-1,1,-1)$, $S = (-1,-1,1)$. The length of each edge is

$$|\overrightarrow{PQ}| = \sqrt{(1-1)^2 + (-1-1)^2 + (-1-1)^2} = 2\sqrt{2}.$$

They are all equal because they are diagonals of squares of sidelength 2.

b) The angle is $\theta = \pi/3$.

In fact, $\overrightarrow{PQ} = \langle 0,-2,-2\rangle$, $\overrightarrow{PR} = \langle -2,0,-2\rangle$ and the angle θ between two adjacent edges satisfy $|\overrightarrow{PQ}|\,|\overrightarrow{PR}|\cos(\theta) = \overrightarrow{PQ}\cdot\overrightarrow{PR}$.
Hence $8\cos(\theta) = 4 \quad \Rightarrow \quad \cos(\theta) = 1/2$, which implies $\theta = \pi/3$.
(One could have also observed that faces are equilateral triangles, whose internal angles sum up to π.)

c) The inner dihedral angle α is equal to $\arccos(1/3) \approx 70.5^o$.

For instance, the inner dihedral angle α between the PSQ plane and the PRS plane satisfies
$\cos(\alpha) = \dfrac{\overrightarrow{N}_1 \cdot \overrightarrow{N}_2}{|\overrightarrow{N}_1|\,|\overrightarrow{N}_2|}$ where
$$\overrightarrow{N}_1 = \overrightarrow{PQ}\times\overrightarrow{PS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & -2 \\ -2 & -2 & 0 \end{vmatrix} = \langle -4,4,-4\rangle$$
is orthogonal to the plane PSQ and
$$\overrightarrow{N}_2 = \overrightarrow{PR}\times\overrightarrow{PS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{vmatrix} = \langle -4,4,4\rangle$$
is orthogonal to the plane PRS.
Hence $\cos(\alpha) = \dfrac{\langle -4,4,-4\rangle\cdot\langle -4,4,4\rangle}{48} = 1/3$, that is $\alpha = \arccos(1/3) \approx 70.5^o$.
(Instead of the inner dihedral angle α, one could have also computed the outer dihedral angle $\pi + \alpha$.)

Problem 2

a) The equation of the plane is $\overrightarrow{AQ} \cdot \overrightarrow{N} = 0$.

The distance between P and P' is $\dfrac{|\overrightarrow{AA'} \cdot \overrightarrow{N}|}{|\overrightarrow{N}|}$.

In fact, the distance between P and P' is the length of the projection of $\overrightarrow{AA'}$ onto $\overrightarrow{N}$.

b) Let $Q = (x, y, z)$.
The equation for P is $x + 2y + z = -3$.
The equation for P' is $x + 2y + z = 3$.
The distance between P and P' is $\sqrt{6}$.

In fact, using (a), the equation for P is $[(x, y, z) - (0, -2, 1)] \cdot \langle 1, 2, 1\rangle = 0$, that is $x + 2y + z = -3$. The equation for P' is $[(x, y, z) - (1, 1, 0)] \cdot \langle 1, 2, 1\rangle = 0$, that is $x + 2y + z = 3$. The distance between P and P' is $\dfrac{\langle 1, 3, -1\rangle \cdot \langle 1, 2, 1\rangle}{\sqrt{1^2 + 2^2 + 1^2}} = \sqrt{6}$.

c) Points Q in P are characterized by $\overrightarrow{B_1Q} \cdot (\overrightarrow{B_1B_2} \times \overrightarrow{B_1'B_2'}) = 0$.
Points Q in P' are given by $\overrightarrow{B_1'Q} \cdot (\overrightarrow{B_1B_2} \times \overrightarrow{B_1'B_2'}) = 0$.

In fact a normal $\overrightarrow{N}$ to P and P' is given by $\overrightarrow{B_1B_2} \times \overrightarrow{B_1'B_2'}$.

d) Denote a general point by $Q = (x, y, z)$.
Then the equation for P is $3x - 4y - 12z = 7$.
The equation for P' is $3x - 4y - 12z = 0$.
The distance between L and L' is $7/13$.

In fact, $\overrightarrow{B_1B_2} = \langle 0, 3, -1\rangle$ and $\overrightarrow{B_1'B_2'} = \langle 4, -3, 2\rangle$. Hence, a normal vector to P and P' is $\overrightarrow{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -1 \\ 4 & -3 & 2 \end{vmatrix} = \langle 3, -4, -12\rangle$. Hence, the equation for P is given by $\langle x - 1, y + 1, z\rangle \cdot \langle 3, -4, -12\rangle = 3x - 4y - 12z - 7 = 0$. Similarly the equation for P' is $\langle x, y - 3, z + 1\rangle \cdot \langle 3, -4, -12\rangle = 3x - 4y - 12z = 0$.
The distance between the two lines L and L' is the same as the distance between

P and P' ([1]). Therefore, by part (c), this distance is

$$\frac{|\overrightarrow{B_1B_1'} \cdot \overrightarrow{N}|}{|\overrightarrow{N}|} = \frac{|\langle -1, 4, -1\rangle \cdot \langle 3, -4, -12\rangle|}{\sqrt{9+16+144}} = \frac{7}{13}.$$

Problem 3

a) $\overrightarrow{N} = \langle 3, -4, -12\rangle$ from 2(d). Note that $\overrightarrow{N}$ points from P' to P.
If Q_0 does not belong to P or P', three situations can occur:

Case 1. Q_0 is on the $\overrightarrow{N}$ side of P ($\overrightarrow{B_1Q_0} \cdot \overrightarrow{N} > 0$). In other words, Q_0 is in the half-space bounded by P which does not contain P'. You see a red hemisphere (corresponding to $\hat{u} \cdot \overrightarrow{N} < 0$) and nothing in the opposite hemisphere ($\hat{u} \cdot \overrightarrow{N} \geq 0$).

Case 2. Q_0 belongs to the portion of space bounded by P and P' ($\overrightarrow{B_1Q_0} \cdot \overrightarrow{N} < 0$ and $\overrightarrow{B_1'Q_0} \cdot \overrightarrow{N} > 0$). You see a red hemisphere ($\hat{u} \cdot \overrightarrow{N} > 0$), a blue hemisphere ($\hat{u} \cdot \overrightarrow{N} > 0$) and nothing in the separating great circle ($\hat{u} \cdot \overrightarrow{N} = 0$)

Case 3. Q_0 is on the $-\overrightarrow{N}$ side of P' ($\overrightarrow{B_1'Q_0} \cdot \overrightarrow{N} < 0$). In other words, Q_0 is in the half-space bounded by P' which does not contain P. You see a blue hemisphere (corresponding to $\hat{u} \cdot \overrightarrow{N} > 0$) and nothing in the opposite hemisphere ($\hat{u} \cdot \overrightarrow{N} \leq 0$).

[For extra credit.
If Q_0 belongs to P, then you see a blue hemisphere ($\hat{u} \cdot \overrightarrow{N} < 0$) bounded by a red great circle ($\hat{u} \cdot \overrightarrow{N} = 0$) and nothing in the opposite hemisphere ($\hat{u} \cdot \overrightarrow{N} > 0$).
If Q_0 belongs to P', then you see a red hemisphere ($\hat{u} \cdot \overrightarrow{N} > 0$) bounded by a blue great circle ($\hat{u} \cdot \overrightarrow{N} = 0$) and nothing in the opposite hemisphere ($\hat{u} \cdot \overrightarrow{N} < 0$).]

b) From Q_0 each of the lines L and L' looks like half of a great circle (an example of such great circles are longitude lines).

For instance, if $\hat{v}$ is the direction of the line L, that is $\hat{v} = \dfrac{\overrightarrow{B_1B_2}}{|\overrightarrow{B_1B_2}|}$ (or you

[1] The nearest distance between the two lines L and L' is the same as the distance between P and P'. In fact, the plane Π through L' and containing $\overrightarrow{N}$ must intersect L in exactly one point R, because L and L' are skew. Now consider the line passing through R with direction $\overrightarrow{N}$: it intersects L' in one point R'. Notice that $\overrightarrow{RR'}$ is parallel to $\overrightarrow{N}$, so $|\overrightarrow{RR'}|$ is the distance between P and P', but also L and L'.

could have chosen $\hat{v} = -\frac{\overrightarrow{B_1B_2}}{|\overrightarrow{B_1B_2}|}$), then L looks like half of a great circle with endpoints $\pm\hat{v}$.
The endpoints of each great circle do not correspond to any point on the line (they would correspond to "points at infinity" of the line).
When P is in front of P' (case 1), you see L in front of L' and inside the red hemisphere.
When Q_0 is in between P and P' (case 2) you see L and L' disjoint.
When P' is in front of P (case 3), you see L' in front of L and inside the blue hemisphere.

Let $Q_0 = (x_0, y_0, z_0)$. In coordinates:
Case 1 corresponds to $\overrightarrow{B_1Q_0} \cdot \overrightarrow{N} > 0$, that is $\langle x_0 - 1, y_0 + 1, z_0\rangle \cdot \langle 3, -4, -12\rangle = 3x_0 - 4y_0 - 12z_0 - 7 > 0$.
Case 2 corresponds to $0 < 3x_0 - 4y_0 - 12z_0 < 7$.
Case 3 corresponds to $3x_0 - 4y_0 - 12z_0 < 0$.
[Q_0 belonging P corresponds to $3x_0 - 4y_0 - 12z_0 = 7$.
Q_0 belonging to P' corresponds to $3x_0 - 4y_0 - 12z_0 = 0$.]

For $Q_0 = (1, 0, 0)$, we have $3x_0 - 4y_0 - 12z_0 = 3$ and so Q_0 is in between the two planes and the lines appear disjoint.
For $Q_0 = (5, 10, -10)$, we have $3x_0 - 4y_0 - 12z_0 = 65$ and so Q_0 sees L in front of L'.
For $Q_0 = (1/100, 0, 0)$, we have $3x_0 - 4y_0 - 12z_0 = 3/100$ and so again the lines appear disjoint.

Problem 4

a) $A_{\pi/3} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \hat{u} = \langle 1/2, \sqrt{3}/2\rangle, \quad \hat{v} = \langle -\sqrt{3}/2, 1/2\rangle.$
In fact $\hat{u} = A_{\pi/3}\hat{i} = \langle 1/2, \sqrt{3}/2\rangle$ and $\hat{v} = A_{\pi/3}\hat{j} = \langle -\sqrt{3}/2, 1/2\rangle$.

b) Multiplying

$$\begin{aligned} A_{\theta_1}A_{\theta_2} &= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} = \\ &= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix} = \\ &= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} \end{aligned}$$

The matrix A_θ performs a counterclockwise rotation of the plane of an angle θ with center 0.

c) $A_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

In fact, if $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then the matrix of cofactors of A_θ is still A_θ. The adjoint of A_θ is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and then the inverse A_θ^{-1} is obtained dividing by the determinant, which is 1 in this case.

d) The reason for $A_\theta^{-1} = A_{-\theta}$ is that the inverse operation of rotating the plane by an angle θ around the origin is rotating the plane by an angle $-\theta$ around the origin.
Comparing the two expressions for the inverse matrix we obtain that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Problem 5

a) The four matrices are
$A_1 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, $A_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, $A_3 = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ and $A_4 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

b)

$A_1 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$	$\det = 1$	rotation	
$A_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$	$\det = 1$	rotation	
$A_3 = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$	$\det = -1$	reflection	
$A_4 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$	$\det = -1$	reflection	

Pattern: $\det(A) = 1$ rotation; $\det(A) = -1$ reflection.

Problem 6

a) The system in matrix form is $AX = B$, where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

So we have

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

b) The solution is $X = \frac{1}{3}\begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 5/3 \\ -2/3 \end{pmatrix}$.

The inverse of A is $A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 2 & -3 \\ 1 & 4 & -6 \\ -1 & -1 & 3 \end{pmatrix}$.

First of all, the matrix A is invertible, because its determinant is

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 2{\cdot}2 - (-1)(2-3) = 4-1 = 3$$

which is not zero.

The matrix of minors of A is $\begin{pmatrix} 2 & -1 & -1 \\ -2 & 4 & 1 \\ -3 & 6 & 3 \end{pmatrix}$.

The matrix of cofactors is $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 4 & -1 \\ -3 & -6 & 3 \end{pmatrix}$.

The adjoint of A is $\begin{pmatrix} 2 & 2 & -3 \\ 1 & 4 & -6 \\ -1 & -1 & 3 \end{pmatrix}$.

The inverse of A is

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = \frac{1}{3}\begin{pmatrix} 2 & 2 & -3 \\ 1 & 4 & -6 \\ -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ 1/3 & 4/3 & -2 \\ -1/3 & -1/3 & 1 \end{pmatrix}.$$

The solution X of the linear system above is given by $X = A^{-1}B$, that is

$$X = \frac{1}{3}\begin{pmatrix} 2 & 2 & -3 \\ 1 & 4 & -6 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 5/3 \\ -2/3 \end{pmatrix}.$$

c) The solution is $(x, y, z) = (4/3, 5/3, -2/3)$.

By substitution, the third equation gives us $x = -2z$. So

$$\begin{aligned} -y - 4z &= 1 \\ y + z &= 1 \\ x &= -2z \end{aligned}$$

Adding the first two equations we get $-3z = 2$, so $z = -2/3$ and $x = 4/3$. Then the second equation $y = 1 - z$ gives $y = 5/3$.