FOURTH HOMEWORK

Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

1. (10 pts) (i) Where is the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & x \text{ is rational} \\ 1 & x \text{ is irrational,} \end{cases}$$

continuous?

(ii) Where is the function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$g(x) = \begin{cases} x & x \text{ is rational} \\ 2x & x \text{ is irrational,} \end{cases}$$

continuous?

2. (10 pts) If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is any function, then

$$f(P) = (f_1(P), f_2(P), \dots, f_m(P)) = f_1(P)\hat{e}_1 + f_2(P)\hat{e}_2 + \cdots + f_m(P)\hat{e}_m,$$

where $f_1\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, $f_2\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, $\dots$, $f_n\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ are functions.

(i) Show that if f is continuous, then so are $f_1, f_2, \dots, f_m$.

(ii) Show that if $f_1, f_2, \dots, f_m$ are continuous, then so is f.

3. (10 pts) Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function given $f(x,y) = |xy|$.

(a) Show directly that f is differentiable at $(0,0)$.

(b) Show that the partial derivatives are not continuous in any neighbourhood of the origin.

4. (10 pts) Find a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = 3x^2y^2 - xy\sin(xy) + \cos(xy) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2x^3y - x^2\sin(xy) + 3y^2.$$

5. (10 pts) (2.3.21).
6. (10 pts) (2.3.25).
7. (10 pts) (2.3.30).
8. (10 pts) (2.3.31).
9. (10 pts) (2.3.33)
10. (10 pts) (2.3.51).

Just for fun: (i) Let $a \geq 0$ be a real number and let x be a real number. How does one define a^x? You may use the fact that if

$$a_1 \leq a_2 \leq a_3 \dots,$$

is an increasing sequence of numbers which are bounded from above (that is, there is a real number M such that $a_i \leq M$), then the limit $\lim_{n\to\infty} a_n$ exists.
(ii) Let $a \geq 0$ be a real number. Show that the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = a^x$ is continuous.
(iii) Let $a_1, a_2, \dots, a_n$ be non-negative real numbers and let $x_1, x_2, \dots, x_n$ be non-negative real numbers whose sum is one. Show that

$$\prod_{i=1}^{n} a_i^{x_i} \leq \sum_{i=1}^{n} x_i a_i.$$