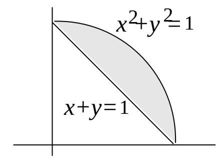
Limits in Iterated Integrals

For most students, the trickiest part of evaluating multiple integrals by iteration is to put in the limits of integration. Fortunately, a fairly uniform procedure is available which works in any coordinate system. *You must always begin by sketching the region; in what follows we'll assume you've done this.*

1. Double integrals in rectangular coordinates.



Let's illustrate this procedure on the first case that's usually taken up: double integrals in rectangular coordinates. Suppose we want to evaluate over the region R pictured the integral

$$\iint_R f(x,y)\,dy\,dx\ , \qquad R = \text{ region between } x^2+y^2=1 \quad \text{and} \quad x+y=1\ ;$$

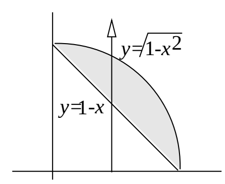
we are integrating first with respect to y. Then to put in the limits,

1. Hold x fixed, and let y increase (since we are integrating with respect to y). As the point (x,y) moves, it traces out a vertical line.

2. Integrate from the y-value where this vertical line enters the region R, to the y-value where it leaves R.

3. Then let x increase, integrating from the lowest x-value for which the vertical line intersects R, to the highest such x-value.

Carrying out this program for the region R pictured, the vertical line enters R where $y = 1-x$, and leaves where $y = \sqrt{1-x^2}$.



The vertical lines which intersect R are those between $x = 0$ and $x = 1$. Thus we get for the limits:

$$\iint_R f(x,y)\,dy\,dx \;=\; \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

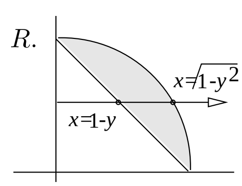
To calculate the double integral, integrating in the reverse order $\iint_R f(x,y)\,dx\,dy$,

1. Hold y fixed, let x increase (since we are integrating first with respect to x). This traces out a horizontal line.

2. Integrate from the x-value where the horizontal line enters R to the x-value where it leaves.

3. Choose the y-limits to include all of the horizontal lines which intersect R.

Following this prescription with our integral we get:



$$\iint_R f(x,y)\,dx\,dy \;=\; \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

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18.02SC Multivariable Calculus
Fall 2010

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