CHRISTINE BREINER: Welcome back to recitation. In this video, I'd like us to work on the following problem. We're going to let F be the vector field that's defined by r to the n, times the quantity x*i plus y*j. And r in this case is x squared plus y squared to the 1/2, as it usually is. The square root of x squared plus y squared.

And then I'd like us to do the following. Use extended Green's theorem to show that F is conservative for all integers n. And then find a potential function.

So there are two parts. The first part is that you want to show that F is conservative. And then once you know it's conservative, you can find a potential function. So why don't you take a little while to work on that. And then when you're feeling good about your answer, bring the video back up, and I'll show you what I did.

OK, welcome back. So again, what was the point of this video? We want to do two things. We want to work on two problems.

The first is to show that this vector field F I've given you is conservative. And then we want to find a potential function. And we want to be able to show it's conservative for all integers n.

And what I want to point out is that for certain integer values of n, we're going to run into some problems with differentiability at the origin. OK? So we're going to try and deal with all of it at once, and simultaneously deal with all of the integers, by allowing ourselves to show that F is conservative even if we don't include the origin in our region. OK.

So I want to point out a few things first. And the first thing I want to point out is if we denote F as we usually do in two dimensions as [M, N], then the curl of F is going to be N sub x minus M sub y. OK.

I actually calculated these earlier, but I want to point out that M sub y is actually equal to n times r to the n minus 2, times x*y. Let me make sure I wrote that correctly. Yes. But that is also exactly equal to N sub x.

And so what does that give us? Since N sub x minus M sub y is the curl of F, when we have this vector [M, N], we know that the curl of F is equal to 0 by this work. OK, now if our vector field was defined on a simply-connected region, then that's enough to show that F is conservative. OK? We just use Green's theorem right away. Right?

But the problem is that we are not necessarily on a simply-connected region because we could have problems at the origin. And so I'm going to deal with this in a slightly different way. To show that F is conservative, what do I want to show?

I want to show that when I take the line integral F dot dr over any closed loop that I get 0. That's ultimately what I'm trying to show. So there are fundamentally two types of curves that I'm concerned with. Two closed curves in R^2 that I'm concerned with, and I'm going to draw a picture of those two types of curves.

So in R^2, I'm going to have curves that miss the origin-- some curve like this, which I'll call C_1. And then I'm going to have curves that go around the origin, and I'll call this C_2. OK? Fundamentally, there's a difference between this curve and this curve, because this curve contains the region where F is defined and differentiable, right? Every point on the interior of this curve, F is defined and differentiable and therefore, I can apply regular old Green's theorem here. OK?

So I know by Green's theorem, the integral over the closed curve C_1 of F dot dr is equal to 0, and that's simply because the curl of F is equal to 0. Right? We can immediately use Green's theorem because we know that the integral over this loop C_1 is equal to the integral over this region of the curl of F. That's just Green's theorem. So I can apply Green's theorem here. Now the problem here is I can't apply Green's theorem because this origin is a trouble spot. Right? I'm not necessarily differentiable there, so I have to be a little more careful. OK, and so what I do is I'm going to explain why, immediately, I can get the integral over C2 is actually also 0.

And what I'm going to do is I'm actually going to draw, hopefully, a circle that contains C_2. So I'm going to draw a circle. It's a lot of curves, but this is supposed to look like a circle. Sorry about that. It's a little big on the low side, but it's a circle. OK. This is a circle. And I'm going to call this C_3.

Now, I can tell you right away that the integral over the curve C_3 of F dot dr is equal to 0, and let me explain why. OK? F is a normal vector field relative to a circle. Let's look at this again.

It's radial, and that's why we know this. F is a radial vector field. It's really the vector field [x, y] times a scalar, depending on the radius. So if I look at this picture right here, then F is going to-- let me draw it in color-- F is going to, at any given point, be in the radial direction.

But that is exactly normal to the tangent direction of this curve. So this is the direction F points, and this is the direction the tangent vector points to the curve. But remember, F dot dr is the same as F dotted with the tangent vector ds. OK?

And so that is why for this circle, it's immediately obvious that F dot dr is equal to 0. Because at any given point on this circle, I'm taking a vector field, I'm dotting it with a vector field that's orthogonal to it, so I get 0, and when I integrate 0 I get 0. OK? So that's why this is 0.

And now where the extended version of Green's theorem comes in, is the fact that, if I look in this region, F is defined and differentiable. Right? F is defined and differentiable in this entire region that I've just shaded. Which is the region between my circle and my curve C_2. And what that tells me is that because this one is 0-- when I integrate along this curve it's 0-- the integral along this curve also has to be 0, right?

That's what you actually have seen already when you talked about the extended version of Green's theorem. You can compare the integral along this curve to the integral along this curve because in the region between them, F is everywhere defined and differentiable. So you can apply Green's theorem there. It just now has two boundary components, instead of in this case where it just has one boundary component.

And so since the integral on this curve is 0, and the curl of F is 0, and F is defined and differentiable everywhere in this region, that tells you that the integral on the curve C_2 is also 0. Let me say that one more time, OK?

I'm going to label it in blue so you can see it. I'm going to call this region that's shaded R. So Green's theorem says that the double integral in R of the curl of F is equal to the integral around this curve. And then I come in and I go around this direction and I come back out, and that gives me the entire integral of the curl of F on this region. Right? The curl of F is 0 everywhere in this region, so that integral is 0.

And so the sum of the integral on C_3 minus the integral on C_2 has to be 0. Since this one is 0, that one is 0. So you've seen this before. I just want to remind you about where that's coming from. All right.

So now we have to do one other thing, and that's we have to find a potential function. OK, so let's talk about how to find a potential function. I'm going to do this by one of the methods we saw in lecture. have--

I'm in R^2, and I'm going to start at a certain point and I'm going to integrate up to (x_1, y_1) from this certain point. And then I'm going to figure out what the function is that way. So what I'm going to do-- again, I'll write it this way-- I'm going to figure out f of (x_1, y_1) by integrating along a certain curve, F dot dr.

Now I can't do exactly what I did previously, because for certain values of n, I run into trouble with integrating F from the origin. So what I'm going to do is instead of integrating from the origin, I'm going to integrate from the point (1, 1). OK?

So I'm going to start at the point 1 comma 1, and I'm going to integrate in the y-direction, and then I'm going to integrate in the x-direction. So this will be my first curve and this will be my second curve. And I will land at x_1 comma y_1. So again, this is one of the strategies we've seen previously.

This is the idea that I'm going to integrate in the y-direction, from y equals 1 to y equals y_1. So this will be the point 1 comma y_1, so x is fixed there. And I'm going to integrate in the x-direction, from x equals 1 to x equals x_1, when y is equal to y_1. So let's break this down.

And let me remind you, also, the integral along this curve C of F dot dr should be P*dx plus Q*dy. Right? And so I'm going to look at what P*dx is and what Q*dy is on C_1 and on C_2. All right. So let's do that.

OK, so I have to remind myself what P and Q actually are in order to do this. So let me write that down, because this will be helpful: [P, Q]. P is r to the n, x, and Q is r to the n, y. All right? So that's what we're dealing with here. I'm going to come back to this picture, and then I'm going to come back and forth a little bit at this point.

So if I want to integrate P*dx plus Q*dy on the curve C_1, what I need to observe first is that x is fixed, so dx is 0. So I'm actually just going to integrate Q*dy. All right.

So the first integral along C_1 is just a parameterization in y. So it's the integral from 0 to y_1 of Q evaluated at x equal 1, and y going from 1 to y_1.

SPEAKER 1: 1 to y_1.

CHRISTINE BREINER: y going from 1 to y1. OK? Sorry. Yes. y going from 1 to y_1. Sorry about that. Right? I was avoiding the origin, so I'd better not put a 0 down there, because that's where I was running into problems. OK.

So Q is r to the n, y. So I have to remember what r is. r is x squared plus y squared to the 1/2. So in this case, Q is-- x is 1, and then I square it and I get 1, and then I have y squared, and then to the n over 2-- so this is my r to the n part along the curve C_1-- and then I multiply by y, and then I take dy.

So there are a lot of pieces here, so let me just make sure we understand what's happening. I am interested in this entire thing, P*dx plus Q*dy along the curve C_1. dx is 0 along that curve. x is 1. And y is going from 1 to y_1.

So if I come back over here, I see I'm only interested in the Q*dy part. y is going from 1 to y1. And then this is r to the n, when x is 1 and y is y. And this is the y part. So this is exactly Q*dy on the curve C_1.

Now let's look at what happens on the curve C_2. So if I come back over here again, I want to have P*dx plus Q*dy on the curve C_2. Notice y is fixed at y1 there, so dy is 0. And so I'm only interested in the P*dx part. Everything is going to be in terms of x. And let's see if we can do the same kind of thing.

I'm going to be integrating from 1 to x_1. Now r is going to be of the form x plus y_1 squared, to the n over 2. And then-- P has an x here and not a y-- times x dx. So again, P is r to the n times x, so this is r to the n times x exactly on the curve C_2. Because on C_2, y is fixed at y_1, so that's why I actually substituted in a y_1 here. It's the same reason I substituted in a 1 here for x, because x was fixed at 1 on the curve C_1.

So now I have to integrate these two things. I'm going to just write down what you get in both cases, because it's really single-variable calculus at this point in both cases. The easiest way to do this, probably, in my mind, is to do a u-substitution.

Oops, I made a mistake. This should be an x squared. I apologize. This should be an x squared, because this is supposed to be a radius, right? It's x squared plus whatever y is squared, to the n over 2. So if you didn't see the squared here, and you got nervous, you were correct. There should be a squared here.

So anyway, I'm going to go back to what I was saying previously. To integrate these things, the easiest thing to do is to take what is inside the parentheses and set it equal to u, and then do a u-substitution from there. So again, I'm not going to actually do that for you, but I'm going to tell you what you get.

Now, there are two different situations. And the situations follow when n is any integer except negative 2, and then when n is negative 2. And the reason is because when n is negative 2, this exponent is a minus 1. So when you integrate, you end up with a natural log.

So let me just point out the two things that you get in each case, and then we'll evaluate and see what the solutions are in each case. So I'm just going to, at this point, write down what I got, because this is your single-variable calculus.

OK, so what I got when n was not equal to minus 2, you get the following thing. You get 1 plus y squared, evaluated at n plus 2, over 2, over n plus 2. And this is evaluated from 1 to y_1.

And then this one you get a similar thing there, but now the y_1 is fixed here. So you get an x squared plus y_1 squared, to the n plus 2, over 2, over n plus 2, evaluated from 1 to x_1. So here, the y_1 is fixed and it's the x-values that are changing, and here the y-values are changing.

So when n is not equal to 2, I get exactly this quantity when I integrate these two terms. And so now, let's see what happens. OK? Exactly what happens is the following.

Notice that when I put in y_1 here, I get a 1 plus y_1 squared, to the n plus 2 over 2, over n plus 2. Right? I'm not going to write it down, because I'm going to show you it gets killed off immediately.

Where does it get killed off? It gets killed off when I evaluate this one at 1. OK? So the upper bound here is the same as the lower bound here. When I put in a 1 here, I get 1 plus y_1 squared to the n plus 2 over 2 over n plus 2. It's a lot of n's and 2's. But the point is that when I evaluate this one at y_1 and I evaluate this one at 1, I get exactly the same thing, but the signs are opposite and so they subtract off. In the final answer, I'm not going to see this upper bound and I'm not going to see this lower bound, because they're going to subtract off.

And what I'm actually left with is just two terms. And those two terms I'm going to write up here. Those two terms are going to be x_1 squared plus y_1 squared to the n plus 2, over 2, over n plus 2. Minus, 1 plus 1-- which is just 2-- to the n plus 2, over 2, over n plus 2.

What it this really? This is just r to the n plus 2, over n plus 2, plus a constant. Because this is just a constant for any n. And notice n is not equal to minus 2-- negative 2. That was the place we were going to run into trouble otherwise.

And so when n is not equal to negative 2-- when you do all the integration-- you should arrive at this as your potential function. OK? And again, what I did was I evaluated to make it simpler on ourselves so we didn't have to write everything out.

I noticed that if I evaluate this at the two bounds, and evaluate this at the two bounds, and I add them together, that the evaluation here plus the evaluation here are the same numerically but opposite in sign, and so they subtract off. And then I just have to evaluate at this one and this one.

So that's n not equal to negative 2. Now let's do the n equal to negative 2 case.

OK, so now I'm integrating this exact same thing in the n equal to negative 2 case. And I'll just write down again what I get by the substitution. And what I get is natural log of 1 plus y squared, over 2, evaluated from 1 to y_1. Plus, natural log of x squared plus y_1 squared, over 2, evaluated from 1 to x_1. Let me make sure I have that right. Yes.

And the same kind of thing is going to happen that happened before, in terms of when I put y_1 in here, and I put 1 in here, I get the same thing but with an opposite sign. Here it's a positive. It's natural log 1 plus y_1 squared over 2. And here it's natural log 1 plus y_1 squared over 2, but because it's the lower bound, it's a negative sign. So whatever I get here and what I get here subtract off.

And then in the end, I wind up getting just the following two terms. I get x_1 squared plus y_1 squared over 2, minus natural log of 2 over 2. So this term comes from evaluating this at x_1. And this term comes from evaluating this one at y equal 1. And if you notice, what is this? This is exactly natural log of r plus a constant.

So let me step to the other side so we can see it clearly. So this is natural log of r squared, but by log rules, that's really 2 times natural log of r, so it divides by 2 and I'm just left with natural log of r, and this is just a constant. And so my potential function in that case is exactly natural log of r plus a constant.

All right, this was a long problem, so I'm just going to remind us where we came from and what we were doing. So let's go back to the beginning.

So what we did initially, was we had this vector field F. It was a radial vector field. r to the n times x*i plus y*j. And we wanted to first show that it was conservative for any integer value of n, and then to find its potential function. And obviously we do it in that order, because if it's not conservative, we're not going to find a potential function.

In this case, what I observed first was that the curl of F was 0. And so in places where I had a closed curve that didn't contain the origin, I knew that the integral all around that closed curve was 0 just by Green's theorem.

But if I had a closed curve that contained the origin, because F is not differentiable for all the n-values there, I have to be a little careful. It's actually even 0, right? When x is 0 and y is 0, I'm going to get something 0 there.

So I need to figure out a way to determine the line integral on C_2. Right? And that was my goal. For any C_2 that contains the origin, how do I figure out F dot dr.

And so I just compared it to what I get when I take F dot dr around a circle. Because I know that I can always find a circle bigger, and then I can say I've got this region here-- in between-- on which F is defined everywhere, so I can apply Green's theorem to that inside region. And I know that the curl of F on the inside region is 0, and so the integral on C_2 and C_3 is going to agree, right? Because the integral on C_3 I showed was 0 just geometrically. And then the integral on C_2 then has to be 0. All right? And so that was just when you were using the extended version of Green's theorem.

And then to find a potential function, we came over here. And we had to avoid the origin because of the differentiability problem at the origin. So we started-- instead of where we usually start, which is from (0, 0)-- we started from the point (1, 1). And we just determined the potential function going from the point (1, 1) to the point (x_1, y_1) along a curve that went straight up, so x was fixed, and then along the curve that went straight over, so y was fixed.

And so then we were able to break up this thing where I'm integrating over C P*dx plus Q*dy into two separate pieces, and each of them was fairly simple to write down. So let's look at what they were.

This first one was where we were moving up. And there was no dx. x was just fixed at 1. And y was going from 1 to y_1. Right?

And so x is fixed at 1, so I put a 1 there. And y is going from 1 to y_1. So I evaluate Q*dy on that curve. And then the next one was P*dx on the curve where I'm moving straight across.

Right? dy is 0 there, so I just pick up the P*dx. And my y-value was fixed at y_1, and x was varying from 1 to x_1.

And so then I just had to be a little bit careful. I didn't show you exactly how you integrate, but using a substitution trick-- single-variable calculus-- shouldn't be too bad for you at this point.

We distinguished between when n was not equal to negative 2 and when n was equal to negative 2. In the case n not equal to negative 2, we determined the integral, we simplified, and we got to a place where the potential function was exactly equal to r to the n plus 2 over n plus 2, plus some constant.

Then in the case where n was equal to negative 2, when you do the substitution, you get a different integral. And in that case, you get the natural log. And so again, we just had the natural log. We have these different functions. We're evaluating the natural log of these different functions. We have the bounds. We simplify everything, and we get exactly to the place where you have natural log of r plus a constant. And so we found our potential function in the case n is equal to negative 2, and then any other n-value.

So, a very long problem. I hope you got something out of it. And this is where I will stop.