18.02SC Multivariable Calculus, Fall 2010
Transcript – Recitation 1, Coordinate Free Proofs: Centroid of a Triangle

JOEL LEWIS: Hi. Welcome to recitation. In lecture, you started learning about vectors. Now vectors are going to be really important throughout the whole of this course. And I wanted to give you one problem just to work with them in a slightly different context than what we're going to do in the future.

So this is the context of Euclidean geometry. So some of you have probably seen this problem that we're going to solve, but you probably haven't seen it solved with vectors. So let's take a look at it.

So what I'd like you to do is show that the three medians of a triangle intersect at a point, and the point is 2/3 of the way from each vertex. So let me just remind you of some terminology. So in a triangle, a median is the segment that connects one vertex to the midpoint of the opposite side. So here, this point M is exactly halfway between B and C.

So OK. So every triangle has three medians-- one from each vertex connected to the midpoint of the opposite side-- and what I'm asking you to show is that these three medians all intersect in the same point. And also, that this point divides the median into two pieces, and the big piece is twice as large as the small piece. So this is 2/3 of the median, and this is 1/3 of the median.

So, why don't you take a few minutes, work that out-- try and do it using vectors as much as possible-- pause the video, come back, and we can work on it together.

So hopefully you had some luck working on this problem. Let's get started on it. So to start, I actually want to rephrase the question a little bit. And I'll rephrase it to an equivalent question that's a little bit more clear about how we want to get started.

So another way to say this problem is that it's asking us to show-- so for each median, say this median AM here, where M is the midpoint of side BC-- there exists a point on the median that divides it into a 2:1 ratio, so the point that's 2/3 from the vertex to the midpoint of the opposite side. So for example, you know, there's a point-- so, let's call it P, say, at first-- so there's a point P such that AP is twice PM. OK?

And similarly, there's some point-- maybe called Q-- that's 2/3 of the way from B to the midpoint of this side. And there's some point that's 2/3 of the way from C to the midpoint of this side. And so an equivalent formulation of the question is to show that these three points are really the same point. That they're all in the same place.

So one way we can do that is that we can compare the position vectors of those three points. And if those three points all have the same position vector, then they're all in exactly the same position. So in order to do that we need some origin. And it happens that for this problem, it doesn't matter where the origin is, and so I'm not going to draw an origin, but I'm going to call it O.

So we're going to set up a vector coordinate system with origin O. And now I want to look at what the vector from O to P is in terms of the vectors connecting O to A, B, and C. Right? Those are the vectors that determine the vertices of the triangle. And so I want to relate the location of P to the locations of A, B, and C.

So the first thing to do is that-- well, in order to talk about where P is, I know how P is related to A and M and I know how M is related to B and C. So let's first figure out what the position vector of M is in terms of the position vectors of A, B, and C, and then we can use that to figure out the position vector of P.

So M is the midpoint of the segment BC. So I think we saw this in lecture. What this means is that the position vector OM is exactly the average of the position vectors of B and C. It's 1/2 of the quantity OB plus OC. All right?

So it's easy to express the position vector of the midpoint of a segment in terms of the position vectors of the endpoints. You just add the position vectors of the endpoints and divide by 2. So if you like, this is equivalent to the geometric fact that the diagonals of a parallelogram bisect each other. So that's the position vector of M.

Now we have to figure out what the position vector of P is. So in order to do this we can note, that in order to get from the origin to point P, well, what we have to go from the origin-- wherever it is-- to A, and then we have to go from A 2/3 of the way to M. All right?

So the vector OP is equal to OA plus 2/3 of the vector AM. Right? Because we go 2/3 of the way from A to M in order to get from A to P. This is because we've chosen P to be the point that divides segment AM into a 2:1 ratio so that AP is 2/3 of AM. OK. So good.

So now we need the vector AM. Well, we know what the position vector of A is. It's just OA. And we also know what the position vector of M is. It's OM. So that means that AM is just the difference of those two vectors. It's going to be OM minus OA.

Another way to say this is that if you add OA to both sides, you have that OA plus AM equals OM. In other words, to go from O to M, first you can go from O to A, and then go from A to M. All right. And I've just subtracted OA onto the other side here. So we can write AM in terms of OM and OA.

And we also, we have an expression for OM here in terms of OB and OC. So that means we can get an expression for AM in terms of OA, OB, and OC. So let's do that. So that's just by substituting from here into here.

So if I do that, I get that AM is equal to-- so OM is 1/2 OB plus 1/2 OC, and now I just subtract OA. All right. So that's what AM is, putting these two equations together. I get that that's AM.

And so now I need to figure out what OP is. So for OP, I just need to substitute in this new expression that I've got for AM. So I have OP is equal to, well it's equal to OA plus 2/3 of what I've written just right above-- 2/3 of AM. So that's 1/2 OB plus 1/2 OC minus OA. OK.

And so now you can multiply this 2/3 in-- you know, just distribute the scalar multiplication across the addition there-- and then we can rearrange. We'll have two terms involving OA and we can combine them. So we'll see we have a plus OA minus 2/3 OA. So that's going to be equal to 1/3 OA. And then we have, OK so 2/3 times 1/2 OB. So that's plus 1/3 OB plus 1/3 OC.

So this gives us a simple formula for the position vector of P-- that vector OP-- in terms of the position vectors of A, B, and C. So in particular, it's actually because P is the special point, it's 1/3 of their sum. Of the sum OA plus OB plus OC. OK, so that's where P is.

Now to finish the problem, I just have to show that this is the same location as the point that trisects the other medians. So how would I do that? Well, I could go back to my triangle and I could do exactly the same thing. So I could-- maybe I'll give this point a name, also. I'll call this midpoint N, say.

So I could let Q be the point that lies 2/3 of the way from B to N. And then I could write down the position vector of N in terms of OA, OB, and OC. And then I can use that to write down the position vector of Q in terms of OA, OB, and OC, and I'll get some expression. And what will happen at the end-- I hope if I'm lucky-- that expression will be equal to this expression that I found over here. OK?

So you can go through and do that, and if you do that, what you'll find is that in fact it works. So there's actually a sort of clever, shorter way of seeing that. Which is that this formula is symmetric in A, B, and C.

So that means if I just relabel the points A, B, and C, this expression for the position vector doesn't change. So rather than going through that process that I just described, you can also say, well, in order to look at, say Q, instead of P, what I need to do is I just need to switch B and A. I need to do exactly the same thing but the roles of A and B are interchanged.

Well, if the roles of A and B are interchanged, then in the resulting formula, I just have to interchange the roles of A and B, but that won't change the value of this expression. So by symmetry, the point I get really is going to be the same. If you don't like that argument, I invite you to go through this computation again in the case of the other medians. In either case, what you'll find is that the points that trisect the three medians all have position vector 1/3 OA plus 1/3 OB plus 1/3 OC, but that means they're the same point. So what we've shown then, is that the points that trisect the three medians-- that trisect-- that divide them into 2:1 ratios from the vertex to the midpoint of the opposite side, that those three points all have the same position vector.

So in fact, they're the same point, and that's what we wanted to show, right? We wanted to show that there's one point that trisects all three of those medians. So we've shown that the three points that trisect them are actually the same. So that's the same conclusion, phrased differently. So I think I'll end there.