JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've begun learning about various different ways to describe planes in three-dimensional space. In particular, there are equations and ways you can translate between other characterizations. So I have here four different planes for you, described in four different ways, and what I'd like you to do is try and figure out what the equations for each of these four planes are. So let me see what they are.

So we've got the first one-- part a-- we have a plane where I'm giving you its normal vector N, which is the vector [1, 2, 3]. And I'm going to tell you that the plane is passing through the point 1, 0, minus 1.

In part b, I'm telling you that the plane passes through the origin, and also that it's parallel to two vectors. It's parallel to the vector 1, 0, minus 1, and to the vector minus 1, 2, 0.

In part c, I'm telling you that a plane passes through the points (1, 2, 0), (3, 1, 1), and (2, 0, 0).

And in part d, I'm telling you that the plane is parallel to the plane in part a, and also that it passes through the point (1, 2, 3).

So what I'd like you to do is try, for each of these four descriptions, figure out what the equation of the associated plane is. So why don't you pause the video, take a few minutes, work those all out, come back, and we can work them out together.

So hopefully you had some luck working on these problems. Let's get started. So we may as well start with the first one.

So in part a, we're given that the normal vector N is the vector i plus 2j plus 3k, or [1, 2, 3], and that it passes through the point P-- which I'm going to call P-- 1, 0, minus 1. So this is a form you learned in lecture. And so it's pretty straightforward to write down the equation here.

The thing to remember is that if a point (x, y, z) is on the plane, then we have to have that the vector N-- the normal-- is orthogonal to the vector connecting the point (x, y, z) to the point we know. So that's the vector x minus 1, y minus 0-- which is just y-- z plus 1.

So N and this vector that lies in the plane have to be orthogonal, so their dot product has to be 0. And now you just multiply this out. So in our case-- so N is [1, 2, 3], and you take the dot product with x minus 1, y, z, and you get 1 times x minus 1, plus 2 times y, plus 3 times z plus 1, equals 0. So that's the equation of the plane.

You could also rewrite this a bunch of different ways. For example, you could multiply through and collect all the constants together. So you could write this as x plus 2y plus 3z-- and then we've got a minus 1 plus 3, so that's plus 2-- equals 0. So these are two different possible forms for that equation. And you can-- you know, sometimes people write the constant over on this side instead of leaving 0 over there. All right. So several different, equivalent ways to rewrite it. All right. So there's the equation for part a.

Now let's take a look at part b. So for part b we have-- let's go just back and remind ourselves what the question was-- so we have a plane that passes through the origin. So we know a point on the plane, and we know that it's parallel to the two vectors 1, 0, minus 1, and minus 1, 2, 0. OK.

So we've got a point and we have two direction vectors. And so that definitely describes a plane for us, as long as the two directions aren't parallel, which they aren't in this case. So the question is how do we figure out what the equation for that plane is? Well, we have this nice way of figuring out equations for planes when we know a point and a normal. And we know a point, so what would be great is if we could come up with a normal direction to this plane. So OK.

So we have two vectors in the plane, and we want to find a vector that's perpendicular to the plane. Well, we have a nice tool when you're given two vectors to figure out a vector perpendicular to both of them, and that's to take the cross product. So our normal vector should be the cross product of these two vectors, or, you know, any multiple of it would do as well.

So for part b, the normal N should be the cross product of the two vectors that are in the plane, so it should be the cross product of 1, 0, minus 1, and-- what's the other one-- minus 1, 2, 0. So, all right, so we just have to compute what that cross product is. So this is a determinant whose first row is i, j, k, and whose second and third rows are the two vectors we're crossing. And OK, so we can expand this out.

So, if you like, so this is i-- the coordinate of i is going to be 0 minus minus 2, so that's 2. The coordinate of j is going to be the negative of the determinant of this minor, which is 0 minus minus 1 times minus 1. So the determinant of the minor is minus 1, so the coordinate of j is going to be plus 1.

And the coordinate of k is the determinant of this minor, which is just 2. So the normal vector in this case is the vector [2, 1, 2]. So OK.

So now we've got a normal vector and we have a point. We were given that the plane passes through the origin. So the equation-- using the same idea as in the previous question. So the origin is just [0, 0, 0]. That's a nice point to know it passes through. So the equation is just 2x plus y plus 2z is equal to 0. So this is the equation in part b. All right.

So part c, we're given that the plane passes through three points. So once again, three points. So we have a point in particular. We have three of them. And so what we need then to get to the equation is we need a normal. And we saw in part b that we could get a normal if we knew two vectors that lay in the plane.

So in this case, we have three points. So what we'd like is to find two vectors that lie in the plane, and then use those two vectors to come up with a normal to the plane. So in our case that's particularly-- well, in any case, that's not that hard. You have three points, right? So you have three points somewhere, P, Q, and R. And so if you want to know two vectors in the same plane as these three points, well, you could just take the vectors that connect one of the points to two of others, for example.

So in our case, the plane-- since the plane passes through the points (1, 2, 0), and (3, 1, 1), and-- what's the last one-- (2, 0, 0). So the plane is parallel to-- well, it doesn't matter which one we choose, so for example, we can say the vector that goes from here to here, so we take this and subtract that from it, so that would give us, for example-- 2, minus 1, 1. And we could say the vector from here to here, so we take this and subtract that from it. And that will give us 1, minus 2, 0.

So from three points we could get two vectors that are parallel to the plane. And we have a choice of a point to use. We could use, for example, the same point, (1, 2, 0), as our base point. And so then we can go back and do exactly what we did in part b. So with those two vectors, you can take their cross product, and find a normal vector to the plane. So I'm not going to do that for you. I'll leave that for you as an exercise.

Finally, in part d, we have a plane that's parallel to the plane in part a, and passes through (1, 2, 3). The point (1, 2, 3). So let me just rewrite over here, parallel to-- so the plane in part a had equation x plus 2y plus 3z plus 2 equals 0, and passing through the point (1, 2, 3). All right. So this is the information that we know about our plane in this case. We know that it's parallel to the plane with this equation, and that it passes through the point (1, 2, 3).

Well, it's parallel to this plane. Two planes are parallel exactly when they have the same normal vector. So remember that the normal vector here is always going to be encoded by these coefficients of x, y, z. In part a, this plane had normal vector [1, 2, 3], and that [1, 2, 3] shows up in the coefficient of x, coefficient of y, and coefficient of z, which are 1, 2, and 3, respectively. So to get parallel planes when you have the equation already, one thing you could do is you can just say, oh, so that just means I leave these coefficients the same, and I have to change the constant.

Another thing you could do is you could just go back to our definition and say, OK, so we know that the normal vector is [1, 2, 3]-- the vector [1, 2, 3]-- and that it passes through the point (1, 2, 3). Either of these two methods will work. So let me describe, let me show you what this second, this new method I mentioned is.

So we know that the equation of the plane has to be x plus 2y plus 3z plus something-- that's a big question mark there-- equal to 0. We know that the equation of the plane is going to have to look like this. Because it has the same normal vector, it has to be parallel to this plane. And so then we just need to figure out what goes into this box in order to make this the equation of the right plane.

Well, what else do we know? We know that it passes through the point (1, 2, 3). So when we put in 1 for x, 2 for y, and 3 for z, this equation has to be true. This point (1, 2, 3) has to be a solution to this equation. So when we put in 1, 2, and 3, we have to have that 1, plus 2 times 2, plus 3 times 3, plus that same question mark, is equal to 0. Well, this part is 1 plus 4 plus 9 is 14, so 14 plus whatever goes in here has to be equal to 0, so this better be equal to negative 14. Negative 14. So the equation for the plane in that case is exactly x plus 2y plus 3z minus 14 equals 0.

Now, if you didn't like that method, the other thing you can do-- which I said before, let me just repeat it-- is that since it's parallel to this plane, it has the same normal vector. And we knew that the normal vector to this plane was [1, 2, 3]. So you have a normal vector-- the vector [1, 2, 3]-- and you have a point-- the point (1, 2, 3)-- and so you can just use the usual process given a point and a normal vector.

So just to recap, we had four different characterizations of a plane. We had a plane given in terms of its normal vector and a point that it contains. We had a plane given in terms of a point and two vectors parallel to it. We had a plane given in terms of three points on it. And we had a plane given in terms of a point and of another plane parallel to it. So we have-- in all these different cases, we can apply different methods to compute the equation of our plane.

So in the first case, we just do this very straightforward computation that you saw in lecture here. Where you just realize that the normal vector has to be orthogonal to the vector lying in the plane. So you take their dot product and that gives you the equation right away.

In the second case, where you had two parallel vector-- or sorry, yeah, two parallel vectors to the plane, two vectors lying in the plane-- you need to come up with a normal. And you can always come up with a normal by taking a cross product of those two vectors, as long as you're careful. If you accidentally chose your two vectors parallel to each other, that wouldn't work. You'd just get 0 here, and that's no good. But, so you have to choose two non-parallel vectors in the plane in order to make this work.

In the third case, you have three points. And so with three points what you can do is you can choose two vectors connecting some of those points. And that gives you two vectors that lie in the plane, and that reduces to the case of the previous part, and then again, you can take a cross product to get a normal vector.

Finally, we did this fourth problem where we were given a plane parallel to it. And so you can read off the normal vector from the coefficients of x, y, and z in the equation. And then either use the very first method with a point and normal vector, or just realize that you just have to find the appropriate value of the constant so that this point actually lies on the plane. So I'll end there.