JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about surface integrals, flux, and the divergence theorem. And I have a nice problem here that'll put your knowledge of those things to the test. So in this situation, I have a hemisphere. So this is a hemisphere. It's the right half of a sphere, so the part where y is positive, or non-negative, I guess. So just the surface of that hemisphere, and it has radius R and it's centered at the origin.

So I have this hemisphere and I have the field F, which is given by y times j hat. So it's just always in the y-direction. So I have this field and I have this hemisphere, and what I'd like to do is I'd like you to compute the flux of the field through this hemisphere. But as a hint, rather than doing it by computing the surface integral by parameterizing, what I'd like you to do is to use the divergence theorem to make your life a little bit easier. So why don't you pause the video, take some time to work that out, come back and we can work on it together.

Hopefully, you had some luck working on this problem. Let's see how we can go about it. So the thing that we're asked to compute-- let's give this a name, just so we can refer to it. So let's call this surface of the sphere S. All right. So we want the flux through S. And so the flux through S is the double integral over S of our field F dotted with the normal, with respect to surface area, so with respect to surface area. So I guess this was a kind of bad choice, this S. We've got S that means-- this S means the sphere, this S just means surface area. OK. Hopefully, you can keep those two sorted. This is the thing that we want.

And we'd like to compute it-- rather than by parameterizing this hemisphere, we'd like to compute it by using the divergence theorem. So the divergence theorem relates the integral over a surface to a triple integral over a region. So what we'd like to find is a solid region for which this is all or part of the boundary. Now this half hemisphere-- rather, half-sphere, this hemisphere-- doesn't actually enclose a region, but there's an obvious region that's closely related to it, which would be the solid sphere. If you just filled in that half-sphere there.

So then, for that solid sphere-- so the right half of the solid sphere, with radius R, centered at the origin, that hemisphere, that solid hemisphere, has a surface that consists of S, and it also consists of a disc centered at the origin. So it consists of-- I'm going to draw it in right here-- it's this disc. This one. That disc right there, together with our hemisphere that interests us, make up the boundary of this solid hemisphere. So I'm going to call that disc-- I'm going to give it a name-- I'm going to call it S_2.

So we want this integral and we're going to use the divergence theorem. So the way we're going to use the divergence theorem is we're going to use the divergence theorem to relate this double integral over a surface to that triple integral. But I'm going to have to get S_2 involved because I need-- for the divergence theorem to apply, I need surfaces which completely bound a solid region. So what we know by the divergence theorem is that the double integral over both S and S_2 of F dot n hat with respect to surface area is equal to the triple integral-- OK, so I guess I need another letter here. So I'm going to call this D for that whole half-hemisphere, the solid hemisphere. So D, this is the solid hemisphere.

So this D is the solid hemisphere that they bound-- of-- OK, so it's the triple integral over D of the divergence of F with respect to volume. So this is what the divergence theorem tells us and now we-- so OK. So good. So now what's going to be nice is that in this case this triple integral is going to be very easy to compute. And the surface integral over S_2 is going to be easy to compute. And so what we're going to be left with-- after we compute those two things, we'll be able to subtract and get just the surface integral over S. So lets think about doing those things. So first let's do the surface integral over S_2.

The surface integral over S_2 of F dot n with respect to surface area. Well, what is F on that disc S_2? So let's go look at our picture again. So this disc is the disc of radius R centered at the origin in the xz-plane. So it's in the xz-plane. So that means it's in the plane y equals 0. So F on that disc is just 0. It's the zero field. It's 0 everywhere on that disc. And if you take a surface integral of a zero field dot the normal vector, well, 0 dot the normal is just 0. So if we come back over here, this integrand is 0 dot the normal. So it's always 0. So we're integrating 0. So we just get 0. Definite integral of 0 is 0. Double integral of 0 is 0. Great.

So this S_2, the surface integral of F is really easy to compute. That's the first one we needed. Now we need the triple integral. So let's compute the triple integral over the solid hemisphere of div F dV. Well, what is div F? Well, you know, so div F is just the sum of the partial derivatives of the three components. So in our case if we go back and look, well, F is 0 i hat plus y j hat plus 0 k hat. So when you take the partial derivatives you get 0 plus-- the partial derivative of y with respect to y is just 1, plus-- from the third component we also get 0. So the divergence-- let's go back over here-- the divergence is just 1. It's just a constant 1. That's great.

So this triple integral is equal to-- I'm going to bring it down here-- so it's equal to the triple integral over D of 1 dV. But, of course, when you integrate 1 over a solid region what you get is just the volume of that region. So whatever this number is, it's the volume of that hemisphere. Well, OK, so what's the volume of a sphere? It's 4/3 pi R cubed. So the volume of a hemisphere is half of that. So that's 2 pi R cubed divided by 3. So that's the triple integral of the divergence over the solid region.

So what does that mean? So our integral, the integral over S of F dot n with respect to surface area is equal to this triple integral over D of div F dV minus the double integral over S2 of F dot n d surface area. And so we just saw that this is equal to 2 pi R cubed over 3 minus 0. But 0 is 0. So it's just 2 pi R cubed over 3.

So let's quickly recap. To start off with, we just had this hemisphere. And we're asked to compute a surface integral over this hemisphere of a flux of this field F. So rather than going ahead and calculating it directly, what we realized is that by the divergence theorem, we could consider this to be a difference of a triple integral minus another surface integral such that those two surfaces together bounded the region that you were triple integrating over.

So in principle, there were many possible regions that we could have chosen. Many possible solid regions that we could have chosen to do that triple integral over and to use the other half of its boundary. But there's one particularly nice one, which is just the solid hemisphere of which this hemisphere S was the boundary, or part of the boundary. And so then that gave us this other part of its boundary was this disc. So we introduced this new solid region, D, and the rest of its boundary, which was this other surface, this disc S_2.

And so then, by the divergence theorem, rather than computing the integral over our original region S we could just compute the triple integral over the solid region and the surface integral over the other half of the boundary. So sometimes that won't be helpful. Sometimes you'll have an ugly field, you'll have an ugly region, things won't work out. But in this case, it worked out really nicely. The triple integral was easy to compute, as we saw over here, because the divergence was constant. Because the divergence was just equal to 1, the triple integral just gave the volume. And the other surface integral was also easy to compute because F dot n on that surface was equal to 0. So integrating it was very easy. We just were integrating 0 and we got 0.

So in this case, these choices worked out very, very nicely. They made our life nice and simple. And so in the end, all we had to do were these two almost trivial integrals. They didn't really require any computation at all on our part. And then a little bit of subtraction, except we were subtracting 0 so even the subtraction was easy. So that was what we did, worked out very nicely, 2 pi R cubed over 3 was our answer. And I'll end there.