Problems: Vector Fields in Space

Find the gravitational attraction of an upper solid half-sphere of radius a and center $(0,0,0)$ on a mass m_0 at $(0,0,0)$. Assume this half-sphere has density $\delta = z$.

Answer: Draw a picture.

We follow the steps outlined in recitation, changing only the density.

The force is $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. (This notation is unfortunately standard. The subscript indicates component, not partial derivative.) By symmetry we know $F_x = F_y = 0$.

At (x,y,z) a small volume dV has mass $dm = \delta(x,y,z)\,dV = z\,dV$. This mass dm exerts a force $\dfrac{Gm_0\,dm}{\rho^2}\dfrac{\langle x,y,z\rangle}{\rho}$ on the test mass. The z-component of this force is $\dfrac{zGm_0\,dm}{\rho^3}$, so $F_z = \displaystyle\iiint_D \frac{zGm_0\,z\,dV}{\rho^3}$.

The limits in spherical coordinates are: ρ from 0 to a, ϕ from 0 to $\pi/2$, θ from 0 to 2π. Recall that $z = \rho\cos\phi$. Then:

$$F_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a Gm_0\,\frac{z^2}{\rho^3}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a Gm_0\,\frac{\rho^2\cos^2\phi}{\rho^3}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a Gm_0\,\rho\cos^2\phi\,\sin\phi\,d\rho\,d\phi\,d\theta.$$

Inner integral: $\dfrac{Gm_0a^2}{2}\cos^2\phi\sin\phi$

Middle integral: $\dfrac{Gm_0a^2}{2}\left(\dfrac{-\cos^3\phi}{3}\right)\Bigg|_0^{\pi/2} = \dfrac{Gm_0a^2}{6}$

Outer integral: $\dfrac{Gm_0\pi a^2}{3} \Rightarrow \boxed{\mathbf{F} = \langle 0,0,Gm_0\pi a^2/3\rangle.}$

Surprisingly, this is the same as the force exerted by a half-sphere with density $\sqrt{x^2+y^2}$.