18.02SC Multivariable Calculus, Fall 2010
Transcript – Recitation 41, Line Integrals by Geometric Reasoning

JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about line integrals and computing them around curves and closed curves and in various different ways. So here I have some problems on line integrals for you. So in all cases I want C to be the circle of radius b. So b is some constant, some positive constant. It's the circle of radius b centered at the origin, and I want to orient it counterclockwise. And then what I'd like you to do is for each of the following vector fields F, I'd like you to compute the line integral around C of F dot dr.

So in the first case, where F is xi plus yj. In the second, where F is g(xy) times (xi + yj). So here g of xy is some scalar function. But you don't know a formula for this function. So your answer might be in terms of g, for example. You can assume it's a continuous, differentiable nice function. And then the third one, F is -yi plus xj.

Now before you start, I want to give you a little suggestion, which is often when we're given a line integral like this, the first thing you want to do is jump in and do a parameterization right away for the curve, and then you get a normal single variable integral. So what I'd like you to do for these problems is to think about the setup and think about whether you can do this without ever parameterizing C, so without ever substituting in cosine and sine or whatever. So for all three parts of this problem. So if you can use some sort of geometric reasoning to save yourself a little bit of work without ever going to the parameterization. So why don't you pause the video, spend some time, work that out, come back, and we can work it out together.

Hopefully you had some luck working on these problems. Let's get started. So let's do the first problem first. Let's think about what this vector field F looks like. This first vector field. So let me just draw a little picture over here. So here's our circle of radius b. And this vector field F given by xi plus yj, at every point (x,y), the vector F is the same as the position vector of that point. So over here the vector's like that. Over here, the vector's like that. Up here, the vector is like that. So these are just a few little values of F that I've drawn in there. And so down here, say, F is like that. So in particular, so that's just sort of, you know, if you wanted, you could draw in some more vectors, get a full vector field picture.

So the thing to observe here is that a circle is a really nice curve. So the circle has the property that the position vector at a point is orthogonal to the tangent vector to the circle. At every point on the circle, the tangent vector to the circle is perpendicular to the position vector. So that means it's perpendicular to F, because F is the same, in fact, but is parallel to the position vector.

So in Part a, you have that F dot the tangent vector to your curve is equal to zero at every point on the entire curve. All right? So your field F dot your tangent vector is always zero. So that means that the integral around C of F dot dr, well, we know that dr is T ds. So this is F dot T ds. But that's just zero. It's just an integral and the integrand is zero everywhere. And whenever you take a definite integral of something that's zero everywhere, you get zero.

So this is just zero right away. We didn't have to parameterize the curve or anything. We just had to look at this picture to sort of understand that this kind of field, it's called a radial vector field, where the vector F is always pointed directly outwards. When you integrate a radial vector field around a circle centered at the origin, you get zero, because the contribution at every point is zero.

So that's Part a. Part b is actually exactly the same. If we look back at our formula over here in Part b, we have that F is given by some function g(xy) times (xi hat + yj hat). Well, what is this g of xy doing? It's just rescaling. It's telling you every point you can scale that vector by some amount. So if we looked over at this picture, maybe over here you would scale some of these vectors to be longer, and over here they might be shorter, or you might switch them to be negative, but you don't change the direction of any vector in the field from Part a. You just change their length.

So you still have a radial vector field. And you still have the property that at every point on our curve, the tangent vector to the curve is orthogonal to the vector F. So the tangent vector is orthogonal to F, so that means you again have F dot T is equal to zero. And so F dot dr is also equal to 0 ds, and so when you integrate that, you just get zero. So that's also what happens in Part b. So Part b, I'm just going to write ditto. The exact same reasoning applies in Part b as applied in Part a. And you also get zero as your integral without having to parameterize, without having to do any tricky calculations at all.

All right. So let's now look at Part c. I'm going to draw another little picture. So in Part c, there's your curve. At the point (x,y)-- so I'm going to draw some choices of F again. So in Part c, at the point (x,y), your vector field F is -y i hat plus x j hat. Now if you draw that on the picture here, over there that's that vector. Over here, so at the point (0,1), say, that gives you the vector {-1, 0}. So that's horizontal to the left. Here are some more. There's one there, there's one there. There's another one over here and so on.

In fact, what you'll notice is that this vector F is just parallel to the tangent vector of the circle everywhere. This field is a tangential field. It's always pointing parallel to the curve. OK? It's perpendicular to the position vector. It's in the same direction as the tangent vector at every point. So this is something that you've seen before, I think. That this vector field is giving you a sort of nice rotating motion. You know, at every point it's circulating counterclockwise.

So what does that mean? Well, again, it's not exactly the same as Part a and b, but again we'll be able to compute this integral without parameterizing. Why? Because F dot T in this case-- well, so, let's see. What is the norm of F? The magnitude of F is just the square root of (x squared plus y squared). So on our circle of radius b, that means the magnitude of F is b. And the magnitude of T, the unit tangent vector, is 1, and they point in the same direction.

So when you have two vectors that point in the same direction, their dot product is just the product of their magnitudes. So that means F dot T is equal to b. This is a constant. F dot T is equal to b. So when you integrate around the circle, F dot dr, well, this is equal to the integral around a circle of F dot the tangent vector with respect to arc length. But this integrand, F dot the tangent vector, is this constant b. So you're integrating over the curve b ds. And when you integrate a constant ds, well, that just gives you the total arc length. So this is b times the total arc length. And this is a circle of radius b. So that's b times 2 pi b, which we could also write as 2 pi b squared.

So there you go. So in this third case, you have a nice tangential vector field. So that means the integrand actually works out to be constant. Because the integrand is constant, we don't ever have to parameterize the curve. We can just use the fact that we already know its arc length in order to compute this integral. Again, we could do all of these integrals if we wanted by parameterizing the circle, by x equals b cosine T, y equals b sine T, and going through and writing this as an integral from T equals 0 to 2 pi, and so on.

But these are examples of problems where it's helpful to think about what's going on first, see if you can understand the geometry of your situation. And sometimes you'll have a problem like this where you'll-- either in this class or elsewhere in your life-- where something that might seem complicated has a simple geometric explanation. And so when that does happen, it's nice when you can take advantage of it. Sometimes that won't happen and sometimes you'll have to do the parameterization and the computation. But in these cases we have these nice three examples where with a radial vector field, you get that the integrand is always zero, or with a tangential vector field, you have that the integrand is constant. All right. So, I'll stop there.