Second derivative test

1. Find and classify all the critical points of

$$w = (x^3 + 1)(y^3 + 1).$$

Answer: Taking the first partials and setting them to 0:

$$w_x = 3x^2(y^3 + 1) = 0 \quad \text{and} \quad w_y = 3y^2(x^3 + 1) = 0.$$

The first equation implies $x = 0$ or $y = -1$. We use these one at a time in the second equation.
If $x = 0$ then $w_y = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$ is a critical point.
If $y = -1$ then $w_y = 0 \Rightarrow x^3 + 1 = 0 \Rightarrow x = -1 \Rightarrow (-1, -1)$ is a critical point.
The critical points are $(0, 0)$ and $(-1, -1)$.

Taking second partials:

$$w_{xx} = 6x(y^3 + 1), \quad w_{xy} = 9x^2y^2, \quad w_{yy} = 6y(x^3 + 1).$$

We analyze each critical point in turn.
At (-1,-1): $A = w_{xx}(-1, -1) = 0$, $B = w_{xy}(-1, -1) = 9$, $C = w_{yy}(-1, -1) = 0$. Therefore $AC - B^2 = -81 < 0$, which implies the critical point is a saddle.

At (0,0): $A = w_{xx}(0, 0) = 0$, $B = w_{xy}(0, 0) = 0$, $C = z_{yy}(-1, 2) = 0$. Therefore $AC - B^2 = 0$. The second derivative test fails.