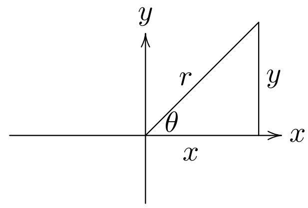
Integration in polar coordinates

Polar Coordinates

Polar coordinates are a different way of describing points in the plane. The polar coordinates (r, θ) are related to the usual *rectangular coordinates* (x, y) by by

$$x = r\cos\theta, \quad y = r\sin\theta$$

The figure below shows the standard polar triangle relating x, y, r and θ.



Because cos and sin are periodic, different (r, θ) can represent the same point in the plane. The table below shows this for a few points.

(x, y)	$(1, 0)$	$(0, 1)$	$(2, 0)$	$(1, 1)$	$(-1, 1)$	$(-1, -1)$	$(0, 0)$
(r, θ)	$(1, 0)$	$(1, \pi/2)$	$(2, 0)$	$(\sqrt{2}, \pi/4)$	$(\sqrt{2}, 3\pi/4)$	$(\sqrt{2}, 5\pi/4)$	$(0, \pi/2)$
(r, θ)	$(1, 2\pi)$			$(\sqrt{2}, 9\pi/4)$		$(-\sqrt{2}, \pi/4)$	$(0, -7.2)$
(r, θ)	$(1, 4\pi)$						

In fact, you can add any multiple of 2π to θ and the polar coordinates will still represent the same point.

Because θ is not uniquely specified it's a little trickier going from rectangular to polar coordinates. The equations are easily deduced from the standard polar triangle.

$$r = \sqrt{x^2 + y^2}, \qquad "\theta = \tan^{-1}(y/x)".$$

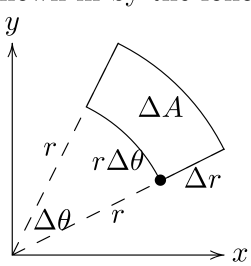
We use quotes around $\tan^{-1}$ to indicate it is not a single valued function.

The area element in polar coordinates

In polar coordinates the area element is given by

$$dA = r\,dr\,d\theta.$$

The geometric justification for this is shown in by the following figure.

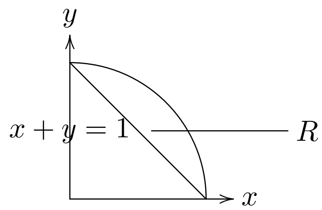


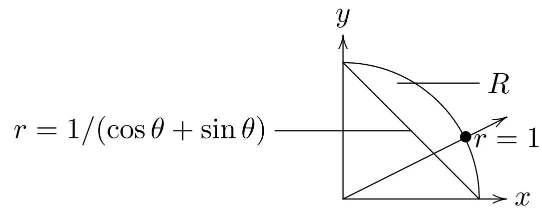
The small curvy rectangle has sides Δr and $r\Delta\theta$, thus its area satisfies $\Delta A \approx (\Delta r)(r\,\Delta\theta)$. As usual, in the limit this becomes $dA = r\,dr\,d\theta$.

Double integrals in polar coordinates

The area element is one piece of a double integral, the other piece is the limits of integration which describe the region being integrated over.

Finding procedure for finding the limits in polar coordinates is the same as for rectangular coordinates. Suppose we want to evaluate $\int\int_R dr\,d\theta$ over the region R shown.





(The integrand, including the r that usually goes with $r\,dr\,d\theta$, is irrelevant here, and therefore omitted.)
As usual, we integrate first with respect to r. Therefore, we

1. Hold θ fixed, and let r increase (since we are integrating with respect to r). As the point moves, it traces out a ray going out from the origin.

2. Integrate from the r-value where the ray enters R to the r-value where it leaves. This gives the limits on r.

3. Integrate from the lowest value of θ for which the corresponding ray intersects R to the highest value of θ.

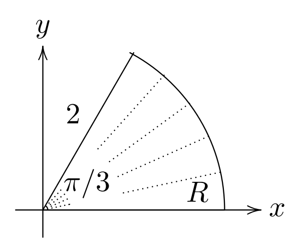
To follow this procedure, we need the equation of the line in polar coordinates. We have

$$x + y = 1 \quad \to \quad r\cos\theta + \mathbf{r}\sin\theta = 1, \quad \text{or} \quad r = \frac{1}{\cos\theta + \sin\theta}.$$

This is the r value where the ray enters the region; it leaves where $r = 1$. The rays which intersect R lie between $\theta = 0$ and $\theta = \pi/2$. Thus the double iterated integral in polar coordinates has the limits

$$\int_0^{\pi/2}\int_{1/(\cos\theta+\sin\theta)}^{1} dr\,d\theta.$$

Example: Find the mass of the region R shown if it has density $\delta(x,y) = xy$ (in units of mass/unit area)
In polar coordinates: $\delta = r^2\cos\theta\sin\theta$.
Limits of integration: (radial lines sweep out R):
inner (fix θ): $0 < r < 2$, outer: $0 < \theta < \pi/3$.



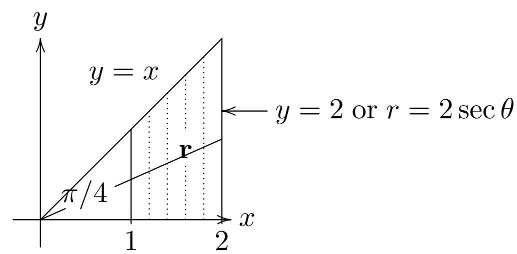
$\Rightarrow$ Mass $M = \int\int_R \delta(x,y)\,dA = \int_{\theta=0}^{\pi/3}\int_{r=0}^{2} r^2\cos\theta\sin\theta\,r\,d\theta\,dr$

Inner: $\int_0^2 r^3\cos\theta\sin\theta\,dr = \frac{r^4}{4}\cos\theta\sin\theta\Big|_0^2 = 4\cos\theta\sin\theta$

Outer: $M = \int_0^{\pi/3} 4\cos\theta\sin\theta\,d\theta = 2\sin^2\theta\Big|_0^{\pi/3} = \frac{3}{2}.$

Example: Let $I = \int_1^2 \int_0^x \frac{1}{(x^2+y^2)^{3/2}}\, dy\, dx$. Compute I using polar coordinates.

Answer: Here are the steps we take.
Draw the region.



Find limits in polar coordinates:
Inner (fix θ): $\sec\theta < r < 2\sec\theta$, outer: $0 < \theta < \pi/4$.

$$\Rightarrow \; I = \int_{\theta=0}^{\pi/4} \int_{r=\sec\theta}^{2\sec\theta} \frac{1}{r^3} r\, dr\, d\theta.$$

Compute the integral:

Inner: $\displaystyle \int_{\sec\theta}^{2\sec\theta} \frac{1}{r^2}\, dr = -\frac{1}{r}\Big|_{\sec\theta}^{2\sec\theta} = \frac{1}{2}\cos\theta.$

Outer: $\displaystyle I = \int_0^{\pi/4} \frac{1}{2}\cos\theta\, d\theta = \frac{1}{2}\sin\theta\Big|_0^{\pi/4} = \frac{\sqrt{2}}{4}.$

Example: Find the volume of the region above the xy-plane and below the graph of $z = 1 - x^2 - y^2$.

You should draw a picture of this.

In polar coordinates we have $z = 1 - r^2$ and we want the volume under the graph and above the inside of the unit disk.

$$\Rightarrow \; \text{volume } V = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r dr\, d\theta.$$

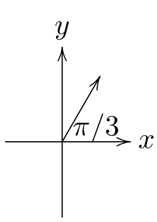
Inner integral: $\displaystyle \int_0^1 (1 - r^2)\, r dr = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$

Outer integral: $\displaystyle V = \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}.$

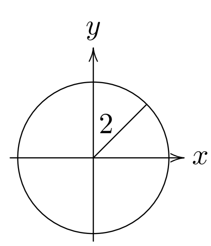
Gallery of polar graphs ($r = f(\theta)$)

A point P is on the graph **if any** representation of P satisfies the equation.

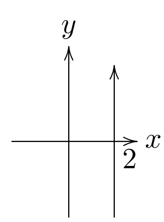
Examples:



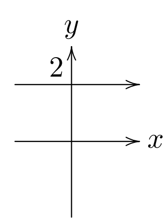
Ray: $\theta = \pi/3$



Circle centered on 0: $r = 2$

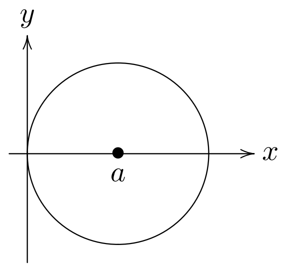


Vertical line $x = 2 \Leftrightarrow r = 2\sec\theta$.

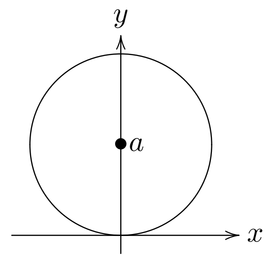


Horizontal line $y = 2 \Leftrightarrow r = 2/\sin\theta$.

Example: Show the graph of $r = 2a\cos\theta$ is a circle of radius a centered at $(a, 0)$.
Some simple algebra gives $r^2 = 2ar\cos\theta = 2ax \quad \Rightarrow \; x^2+y^2 = 2ax \quad \Rightarrow \; (x-a)^2+y^2 = a^2$.
This is a circle or radius a centered at $(a, 0)$.
Note: we can determine from the graph that the range of theta is $-\pi/2 \le \theta \le \pi/2$.



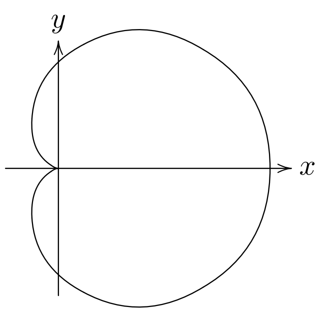
$r = 2a\cos\theta$
$-\pi/2 \le \theta \le \pi/2$.



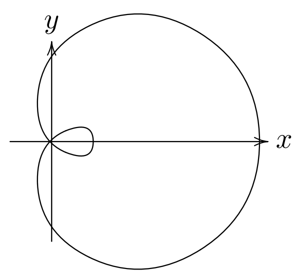
$r = 2a\sin\theta$
$0 \le \theta \le \pi$.

Warning: We can use negative values of r for plotting. You should never use it in integration. In integration it is better to make use of symmetry and only integrate over regions where r is positive.

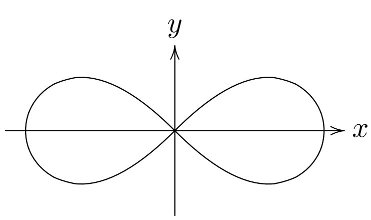
Here are a few more curves.



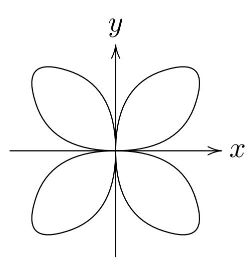
Cardiod: $r = a(1 + \cos\theta)$



Limaçon: $r = a(1 + b\cos\theta)$ $(b > 1)$



Lemniscate: $r^2 = 2a^2\cos 2\theta$



Four leaved rose: $r = a\sin 2\theta$