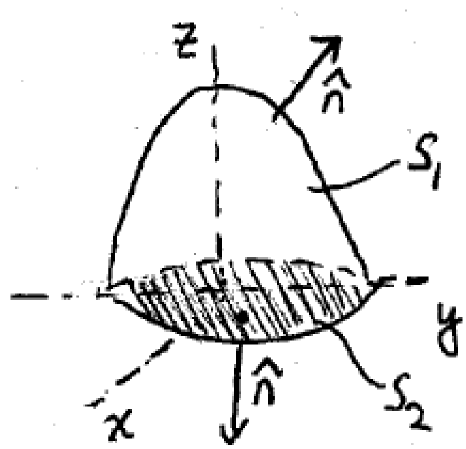
Problems: Divergence Theorem

Let S_1 be the part of the paraboloid $z = 1 - x^2 - y^2$ which is above the xy-plane and S_2 be the unit disk in the xy-plane. Use the divergence theorem to find the flux of $\mathbf{F}$ upward through S_1, where $\mathbf{F} = \langle yz, xz, xy \rangle$.

Answer: Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, where $M = yz$, $N = xz$, and $P = xy$. Then

$$\operatorname{div}\mathbf{F} = M_x + N_y + P_z = 0.$$



The divergence theorem says: flux $= \iint_{S_1+S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D 0\,dV = 0$

$\Rightarrow \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = 0 \;\Rightarrow\; \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS.$

Therefore to find what we want we only need to compute the flux through S_2.

But S_2 is in the xy-plane, so $dS = dx\,dy$, $\mathbf{n} = -\mathbf{k} \;\Rightarrow\; \mathbf{F}\cdot\mathbf{n}\,dS = -xy\,dx\,dy$ on S_2.

Since S_2 is the unit disk, symmetry gives

$$\iint_{S_2} -xy\,dx\,dy = 0 \;\Rightarrow\; \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = 0.$$