CHRISTINE BREINER: Welcome back to recitation. In this video, I'd like us to see how geometric methods can help us understand the flux of a vector field across a curve.

So in particular, what we're going to do is try and use geometric methods to compute the flux of four different vector fields F across curve C. So I've labeled them for later purposes. I've labeled them F1, F2, F3, and F4, and they are as follows.

F sub 1 is the vector field that is a scalar function of only the radius times the vector x comma y. And the curve I'm interested in in this part is the unit circle. The second one, part b, is the vector field that is g of r. Again, where g is a scalar function and r is the radius-- so it depends only on the radius-- times the vector minus y, x. And again the C will be the unit circle.

The third and fourth ones, we use a different C, but they will be the same there. I'll point that out. So in the third one, the vector field is 3 times the vector (1, 1). And C in this case will be the segment connecting (0, 0) to (1, 1). So it's a piece of the line y equals x. And then in part d, F will be 3 times the vector (-1, 1). And C is, again, this segment from (0, 0) to (1, 1).

So again, what I'd like you to do is rather than trying to parametrize the curve and do the entire calculation, I'd like to see you try and understand the relationship between each of these factor fields F and the normal to the curve that they're on. And see if you can figure out the flux based on that relationship.

So why don't you pause the video, give that a shot, and then when you're ready to see how I did it, you can bring the video back up.

OK, welcome back. Again, I'm going to try and use my geometric intuition to understand what the flux is for each of these four vector fields along these four curves. So what I'd like to do is if I'm going to try and understand geometrically what's happening, it's always good to draw a picture. So I'm going to draw a picture that I'll use for a and b, and then I'll draw another picture later that I'll use for c and d.

So notice again in a, my curve is the unit circle. And I have a somewhat explicit understanding of what the vector fields are. So I'm going to draw the unit circle and see if I can figure out where F1 and F2 are. It's not a perfect unit circle, but it looks sort of like the unit circle. So this is going to be my unit circle.

And what I want to point out first is that I was not trying to trick you, but just to help you notice, that in a and b, they both depended on this radial function. But the radius on the unit circle is fixed at 1. So in both of these vector fields, it will simply be g of 1. So that will be a constant value. So all that's giving me is some scalar multiple of whatever the length of this one is. So it's this direction times this scalar g of 1.

One other thing I want to point out is that both of these vectors, F1 and F2, if I ignore the g part, and I look at just the x comma y and the minus y comma x, is these parts have unit length. And you should be able to see that right away, because x and y are on the unit circle. And so the length of the vector whose tail is at the origin and head is on the edge of the unit circle has length 1. And then you can easily see that this vector and this vector have the same length, because their individual components are the same absolute value.

We already understand a few things about F1 and F2 right away. That all the length is coming from this g, and g is fixed all the way around the unit circle at g of 1. Now let's figure out what the flux is for these two things.

So if you notice first, F sub 1 is the vector (x, y) times the scalar g of 1. So if I come over here, I want to point out-- I'll do this part in white first-- that if I'm at this point-- this is the point (x, y)-- the vector (x, y) is equal to this vector. So if I think about putting that at this point-- I'm going to draw it here-- I get something that looks like this.

So this is F sub 1, probably. I'm going to have to make one comment about that. But notice, this should look like it's all in one direction. So this is the vector (x, y). If I slide it so its tail is here, it's again in the same direction, and now I've just scaled it by g of 1. Now, this is assuming, obviously, that g of 1 is positive. So we're going to assume that throughout this problem. I'll mention what happens when g is negative at the end.

So this is my vector F sub 1. Let's think about what is the normal to this curve. The normal to this curve, actually at each point on the circle, points in exactly the same direction as F sub 1. Because if I'm parametrizing in this direction-- I'll draw one down here so we can see what it looks like-- the normal-- actually, let me come from there-- the normal is going to look something like this. OK.

So it would be connecting from the origin to that point on the circle, and keep going out in that direction. That's the normal direction. So F sub 1, if g is a positive function, it points in exactly the same direction as the normal. If g is a negative function, it points in exactly the opposite direction. So F1 would be flipped exactly around if g was a negative function.

So if I want to compute the flux for part a-- I'll do it down here-- if I want to compute the flux, remember, I'm taking the integral along the curve of F dotted with n ds. Well, F dotted with n is constant. And that's the main point that's going to make this easier. At each point-- I guess I should say F1-- F1 dotted with n is always equal to the length of F1 times the length of n times cosine of the angle between them. With a very quick calculation, you can see that winds up being g of 1.

So the only reason I don't have to worry about absolute value, is if g is positive, I'm pointing in the same direction. If g is negative, I'm pointing in the opposite direction. And so the cosine theta is minus 1 instead of plus 1. You might want to check that for yourself, but this is just g of one. So that's a constant.

So this is actually equal to g of 1 times the integral over C of ds. Now, what is this? If I integrate this, I should pick up exactly the length of the curve. OK. Because this is the derivative of arc length, so when I integrate this, I get arc length. But it's a unit circle, so the arc length is just the circumference of the unit circle. So that's 2 pi times g of 1.

OK. And that's all you get. That's it. So we didn't actually have to parametrize anything. We just had to understand F1 relating to the normal. So I didn't draw the normal here, but if I take this normal and I spin around to here, the normal is in the same direction as F sub 1.

So now let's look at F sub 2. And let's do this first by pointing something out about the relationship between F sub 2 and F sub 1. Notice that if I take F sub 1 and I dot it with F sub 2, I get 0. Right? Because ignoring even the scalar part, I get x times minus y, plus y times x, which gives me 0. If the scalars come along for the ride, I still get 0. So F sub 1 and F sub 2 are orthogonal.

And in fact-- you can do this for yourself-- but if I come over and draw the picture, F sub 2 is going to be F sub 1 rotated by 90 degrees. Something like this. This is my F sub 2. Again, if g of 1 was negative, F sub 1 would be this direction, and F sub 2 would then be around here.

But ultimately, it's not going to matter in this case whether g is positive or negative, because notice what happens. If I want to integrate F sub 2 dotted with the normal, notice the normal is in the same direction as F sub 1, so F sub 2 dotted with the normal is 0. So if I'm going to integrate the function 0 all along the curve, I shouldn't be surprised that my answer to part b is 0.

So there was even less work in part b. Because I immediately had that F sub 2 is really in the direction of the tangent to the curve. And so I have something in the direction of the tangent dotted with something in the direction of the normal-- in fact, the normal-- so I get 0.

All right. So now I'm going to draw a picture for c and d, and then we're going to use that one. And I have to make sure I come on this side. Sorry about that. So here is (0, 0). And here is (1, 1).

Actually, you know what? I'm going to make it a little longer. I might need more room. So (1, 1) I'll make a little further up. OK? (0, 0) and (1, 1) and that's my curve. There's (1, 1). OK.

Now, if I parametrize it in this direction, then I can draw my normal. Because it's a line segment, my normal is constant in its length and direction. So at any given point, it's exactly equal to this vector, up to the right scaling. And it should be something like (1, -1) divided by square root 2. That's my normal. If you want it precisely, that's what it is. You don't actually need it to solve this problem, though. But that's what it is if you want it precisely.

Now let's look at what F3 is, and then we'll look at what F4 actually is. So let's look at F sub 3. F sub 3 was the vector 3 times (1, 1).

So if we come back to our picture, at any point on this curve, if I go in the (1, 1) direction, I stay parallel to this curve. I don't want to draw the whole thing, because it would take up the entire curve. It's longer than the curve itself. But F sub 3, at any given point, points in this direction. So this is F sub 3, but not as

long as it actually is. But only the direction is going to matter in this case. So F sub 3 is pointing in this direction.

So if I want to compute the flux, I dot it with the normal. But look at what happens. The normal is orthogonal to F sub 3 at every point. And so F sub 3 dotted with the normal is 0.

And so again, for exactly the same reason as part b, in part c, the flux is 0. So again, it's exactly the same that the vector field I was looking at and I wanted to compute the flux for, is actually tangent to the curve at every point. And so when I dot it to the normal to the curve at every point, I get 0. And so computing the flux is 0.

So now, I have one more to do, and that one is part d. And in this case, F sub 4-- let me just remind you-- is 3 times the vector (-1, 1).

And so if we go back to our picture here, F sub 4, if I compare it to the normal, in fact, what I get is very long. This is probably not quite long enough. But that's at least the direction of F sub 4. And so F sub 4 is exactly the opposite direction to the normal.

So if I want to compute the flux of F sub 4 along this curve, all I have to understand is F sub 4 dotted with the normal and the length of the curve. This is exactly the same type of solution as part a. So let's notice this.

First, F sub 4, the length is 3 root 2. You can compute that pretty quickly. The length of n is just 1. It's a normal. That's why this stuff didn't really matter what exactly it was. It's good to know what direction it points in.

So F sub 4 dotted with n is exactly 3 root 2 times cosine of the angle between n and F sub 4. The angle is pi. You notice they differ by 180 degrees, right? So it's cosine pi, which is minus 1.

So what I do in part d, is I'm integrating along the curve the constant negative 3 square root 2 ds. This is exactly F sub 4 dotted with the normal. And so as before, this is going to equal to negative 3 root 2 times the arc length. Because the integral over the curve ds is going to be the arc length.

And the arc length is very easy to see. You've gone over 1 and up 1. Right? So Pythagorean theorem, understanding right triangles, however you want to do it, the length of this curve is root 2. So this works out to be negative 3 root 2 root 2, which is just negative 6.

So let me just remind you, there were four problems here. There are two sets of problems, where in each case, you have one similar to the other. So let me point this out one more time, and just sort of step back.

We had a circle, and then in the next part we had a line segment. But in the circle, one of the problems had the vector in the direction of the normal, and you wanted to compute the flux for that. And in the other, the vector was tangent to the curve, and so it was orthogonal to the normal. And you wanted to compute the flux for that. Obviously, when you're tangent to the curve and then orthogonal to the normal, you get 0. And that was the case for b and for c.

When you're normal to the curve and of constant length-- which was the case actually for both a and d-- then all you have to do is find F dotted with the normal, and then find the arc length, and multiply them together. So that was the real strategy we had to use for a and for d.

So hopefully, this helps you see how the geometric quantities are interacting to understand the flux of a vector field across a curve. And that's where I'll stop.

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