JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about two-dimensional flux and the normal form of Green's theorem, so normal here meaning perpendicular. So I want to give you a problem about that. So what I'd like you to do is to verify Green's theorem in normal form for this particular field, the field F equals xi hat plus yj hat and the curve C that consists of the upper half of the unit circle and the x-axis interval (-1, 1).

So first of all, let me say what I mean by this C. So C, it's the usual unit circle, circle of radius 1 centered at the origin, so just its top half, And the x-axis interval (-1, 1), I mean, the line segment that connects the points (-1, 0) and (1, 0), so the diameter of that semicircle. So that's the curve C and the field F.

What I mean by verify Green's theorem is I'd like you to compute both the double integral that appears in Green's theorem and the line integral that appears in Green's theorem and check that they're really equal to each other. So that'll confirm Green's theorem in this particular instance and hopefully help give us a feel for how it works a little bit. So why don't you pause the video, have a go, compute both of those integrals, come back and we can work them out together.

Hopefully you had some luck with this. Let's have a go at it. So what Green's theorem tells you is that the flux across a curve, which we usually compute as a line integral, is also equal to an integral of the divergence over the region bounded by that curve. So here the curve has to be a closed curve so that it bounds a region of the plane.

So in particular, let's draw the picture here so we know what we're talking about. So the curve C, so we've got the segment from -1 to 1 along the x-axis, and then we have the top half of the unit circle. That's the curve C. And I didn't specify an orientation but in any context like this, when you don't specify an orientation, what you mean is positively oriented. So it's a positively oriented curve. So that's our curve C.

And so the region that it bounds is this half of the circle, the semicircle. I keep saying half of the semicircle. Sorry about that. This upper half of the circle, of the disc, in fact. That region of the plane. OK, so what Green's theorem tells us is that when we compute the surface integral-- sorry, the double

integral-- of the divergence of F over this region, that should be the same as what we do if we compute F dot n around the boundary of the curve. And now we're going to check it.

So let's do the double integral first. The double integral in this case, so it's the double integral over-- let me call that region R-- so it's the double integral over R of the divergence of F dA. So what is the divergence of F? Well, here's F. It's xi hat plus yj hat. So it's divergence is the partial of x with respect to x, plus the partial of y with respect to y. So that's 1 plus 1. So the divergence of F is just 2, in this case. So it's equal to the double integral over the semicircle of 2 dA.

And of course when you integrate a constant over a region, what you get is just that constant times the area of the region. So dA here is the area of the semicircle, it's half of a circle of radius 1. So circle of radius 1 has area pi. So this is 2 times 1/2 pi. So this is pi. OK. So that's the double integral that we get from Green's theorem in normal form. And what Green's theorem says is that this is equal to a particular line integral.

So what is the line integral? Well, it's the integral around C of F dot n. So let's write down what it is, the line integral part now. So it's an integral. It's a closed curve. So it's in and around C of F dot n ds. So that is the line integral that we're looking to compute. So how do we compute this? Well, usually we compute it by using the coordinates of F. So this is equal to, and we know that this is always equal to, the integral around C of-- and let me make sure I'm getting this right before I screw anything up. Yes. OK, I am. Good. M dy minus N dx, where M and N are the coordinates of F-- or the components of F. M and N are the components of F. So M is the first component and N is the second component.

So in our case, F has this fairly simple form. So in our case this is equal to the integral around C of-- so M, the first component of F is x. So this is xdy minus the second component is ydx. So this is the line integral we're interested in computing. But, of course, this curve is not easy to parameterize as a single go. So we want to split it into two pieces. So let's look.

So the first piece we want to split it into is that line segment. So let's call that maybe-- well, I'm not even going to bother giving them names. We want to split it into the integral over the line segment plus the integral over the semicircle. The boundary of that-- yes, the semicircle. So this is equal to-- so it's the integral over the line segment. So let's see. So that integral-- well, OK, I will give them names. I will give them names. I take it back. We'll call the line segment C1 and we'll call the semicircle C2. So it's equal to the integral over c1.

OK, well, what is the integral over c1? What are x and dy and y and dx in this case? So in this case, well, x is what we're integrating. But dy, we're on this line segment, y isn't changing. y is constant. So dy is just 0. So it's 0 minus-- OK, and now on this line segment y is 0 also. So it's 0dx. So the first integral, the integral over C1, is the integral of 0. Plus we have to integrate over C2. Of OK, so x dy minus y dx. All right. So this one's just going to be 0, that's easy.

So now we just have to work with this second one. OK, so for the second one, we're integrating over the semicircle so we want to parameterize it. And we're going to use our usual parameterization. x equals cosine t, y equals sine t. And in this case, we just want to do this semicircle. So we just want to go from (1, 0) all the way around to (-1, 0). So that is from going from 0 to pi. So this is equal to-- so this first integral is just 0. It's the integral of 0 and that's 0. So it's the integral-- OK, so t is going to go from 0 to pi. So now x dy. So x is cosine t. y is sine t. So dy is sine t dt. Sorry, it's cosine t dt. y is sine t. dy is cosine t dt. So it's cosine t times cosine t dt minus. All right, now y is sine t again. And x is cosine t. So dx is minus sine t dt. So this is times minus sine t dt.

OK, well what happens here? So this becomes cosine squared t dt minus minus sine squared t. Minus minus is plus. So it's cosine squared t dt plus sine squared t dt. But, of course, cosine squared plus sine squared is just 1. So we can write this even more simply as the integral from 0 to pi of 1 dt or just dt, and the integral dt is just t. So this is t between 0 and pi, which is pi. Whew! OK. Pi. Good. And what did we get before? We also got pi. Great.

So we have successfully verified Green's theorem in normal form in this particular instance. So let's just recap again what we did. We had this field F and this curve C. This closed curve C that bounded some region. And so what we've done is we computed the double integral over the region of d of F dA. So that's what we did first. Double integral over the region bounded by the curve.

And then second we computed the line integral around the boundary of F dot n ds. So that was what-- this is always a useful form in which to write this F dot n dA. OK, and then we substituted and computed it. And Green's theorem tells us that the two intervals have to be equal to each other. And indeed, for this particular F and this particular C, we verified that in this case they both give us pi. And, of course, Green's theorem tells us that that would have been true, that they would have come out the same, regardless of what the choice of F and the choice of C that we made were. So I'll stop there.

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