JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've learned a little bit about the dot product, and how the dot product can be used to compute angles between vectors, or the relationship between the dot product of vectors and the angle between them. So, here I have two questions, two related questions for you.

So the first one, you've got two vectors. i hat plus j hat plus 2k hat. And a second vector, 2i hat minus j hat plus k hat. I'm probably going to stop saying the hats in a minute, just warning you. So you've got these two different vectors, and what I'd like to know is what the angle between them is.

And the second one is that we have three points. So the P has coordinates a, 1, and minus 1. Q has coordinates 0, 1, and 1. And R has coordinates a, minus 1, and 3.

So when you've given three points, that define some angles. So what I'd like to know is for which values of a is the angle PQR a right angle? So why don't you take some time, work out these problems, come back, and we can work them out together.

Hopefully you had some luck on these problems. Let's get started. Let's start with the first one.

So we have these two vectors, i plus j plus 2k, and 2i minus j plus k. And we want to know what the measure of the angle between them is. So the thing that we have to use here, is that for vectors v and w, we know that v dot w can be written in terms of the angle between them.

So in particular, we know that for vectors v and w, if the angle between them is equal to theta, then cosine of theta is equal to v dot w, but I have to divide by the length of v and the length of w. So I have this simple formula for cosine of theta in terms of v and w. And so then if we want, we can find theta by taking an arc cosine.

So let's apply this formula in the case of problem a. So we could say that v is equal to i plus j plus 2k, and w is equal to 2i minus j plus k. And so what that means is that in our case, v dot w-- well, we just take, you know, the products of the coordinates and add them up. So in that case, this is 1 times 2 plus 1 times minus 1 plus 2 times 1. So this is 1 times 2 plus 1 times minus 1 plus 2 times 1. So that's 2 minus 1 plus 2, so that's equal to 3.

And so then we also need to know what the length of v and w are in order to apply this formula. So the length of v is equal to, well it's the square root of the sum of the squares of the coordinates-- or the square root of v dot v. So in this case, that's the square root of 1 squared plus 1 squared plus 2 squared. So that's the square root of 6. And the length of w is equal to it's the square root of 2 squared plus minus 1 squared plus 1 squared, which is also equal to the square root of 6.

So if you put these three things together, we have that cosine of theta is equal-- so I'm going to write it right up here next to this general formula just to save a little bit of space-- so in our case, this is equal to 3 divided by the square root of 6 times the square root of 6. So that's 3 divided by 6, which is equal to 1/2. So if cosine of theta is equal to 1/2, well we know that that means that theta is equal to pi over 3. OK? So the angle between these two vectors is pi over 3.

OK, great. That's part a. Now let's talk about part b.

So we have these three points and we want to know for what values is the angle that they define a right angle. For what values of this parameter. Right? Our points depend on this parameter, a.

So in order to do that, we have these three points in space. Let me draw a picture up here. So we have point P somewhere and point Q somewhere and point R somewhere, and they define this angle, and we want to know when this angle is a right angle.

Well, we don't have any vectors yet, but it's easy to get some vectors involved in this problem. We can look at the vectors QP and QR. So those two vectors are the two sides of this angle that we're interested in. And it's easy when you're given coordinates for points to find the vector between them. So you just subtract, right?

So QP. In order to get from our point Q over here-- in order to get from 0, 1, 1 to a, 1, minus 1-- well, you have to increase the first coordinate by a, the second coordinate has to stay the same, and the third coordinate has to decrease by 2. So this vector QP is equal to a, 0, minus 2. So now I'm using angle brackets, because this is a vector.

SPEAKER 1: Positive 2 or negative 2? JOEL LEWIS: Negative 2. Yes. Right. Negative 2, yeah? Because we're going from 1 to minus 1. OK, good.

And similarly QR. So to get from Q to R-- I have to go back and look at the coordinates over here-- so I increase the first coordinate by a, decrease the second coordinate by 2, increase the third coordinate by 2, to get from Q to R. So the vector QR is equal to a-- what did I say-- minus 2, 2.

So these are the vectors QP and QR, and we want to know when the angle between these two things is a right angle. We want to know when these vectors are perpendicular. OK?

So the two vectors are perpendicular exactly when their dot product is 0. Right? When the angle is 90 degrees. Cosine of 90 degrees. Cosine of pi over 2 radians is equal to 0, so those two things are the same. The dot product equals 0 and the angle is 90 degrees. So OK.

So in our case, we just have to look then at QP dot QR. So that's equal to-- well, we just take this dot product here-- so it's equal to a squared plus-- 0 times minus 2 is 0-- minus-- 2 times 2, so plus minus 2 times 2-- so that's minus 4. So that's a squared minus 4.

And so this dot product is equal to 0 exactly when a squared minus 4 is equal to 0. So a squared minus 4 is equal to 0 when a is equal to 2 or when a is equal to minus 2. So there are two values of a for which those three points form a right angle at Q.

So just to quickly summarize, we used this general formula here for the cosine of an angle in terms of the two vectors that are the sides of the angle. And with that formula, we can compute the measure of angles between two vectors as here-- or, as in the first part, in part a. Or in a special case when we're looking for a 90 degree angle, we can use the simplified form when we just need to figure out when the dot product is equal to 0. OK. So I'll end there.

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