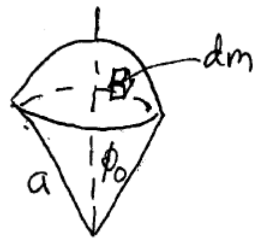
18.02 Problem Set 11, Part II Solutions

1. We put the center of the sphere at the origin O as usual, and take the "North Pole" $N = (0, 0, a)$ as the fixed point. Let P be an arbitrary point on the surface of the the sphere S, and D the straight-line distance from N to P. Then D is the length of a side of the triangle Δ ONP. The other two sides $\overline{ON}$ and $\overline{OP}$ both have length a and the angle between them is ϕ in spherical coordinates, so the Law of Cosines gives $D = a\sqrt{2}\,(1-\cos\phi)^{\frac{1}{2}}$. Then $\bar{D} = \frac{1}{\text{SA}} \iint_S D\,dS$. We'll use the formula SA $= 4\pi a^2$ for the surface area of the sphere. The integral $\iint_S D\,dS = \int_0^{2\pi} \int_0^{\pi} a\sqrt{2}\,(1-\cos\phi)^{\frac{1}{2}}\; a^2 \sin\phi\,d\phi\,d\theta$
$= 2\pi\sqrt{2}\,a^3 \int_0^{\pi} (1-\cos\phi)^{\frac{1}{2}} \sin\phi\,d\phi = 2\pi\sqrt{2}\,a^3\,\frac{2}{3}\,(1-\cos\phi)^{\frac{3}{2}}\,\Big|_0^{\pi} = \frac{16\pi a^3}{3}$.
Dividing this by SA $= 4\pi a^2$, we get $\bar{D} = \frac{4a}{3}$.
(As a check: $\bar{D}$ clearly scales by a, i.e. $\bar{D} = Ka$ for some constant K. $D = a$ when $\phi = \frac{\pi}{3}$, or at 30 degrees North latitude. Since there are more points on S below this latitude, we should have $K > 1$. But $D_{\max} = 2a$ (when P is the South Pole), so we also must have $K < 2$. So $K = \frac{4}{3}$ is at least in the correct range.

2. Limits (in spherical coordinates): ρ from 0 to a, ϕ from 0 to ϕ_0, θ from 0 to 2π.
If dm is located at (x, y, z) then the force due to dm is

$$d\mathbf{F} = G\frac{\langle x, y, z\rangle}{\rho^3}\,dm = G\frac{\langle x, y, z\rangle}{\rho^3}\,\delta\,dV = G\frac{\langle x, y, z\rangle}{\rho^3}\,dV.$$

Let the total force $\mathbf{F} = \langle a, b, c\rangle$. By symmetry $a = b = 0$.

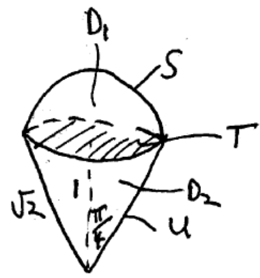


We compute $c = \iiint_D G\frac{z}{\rho^3}\,dV = G\int_0^{2\pi}\int_0^{\phi_0}\int_0^a \frac{\rho\cos\phi}{\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$= G\int_0^{2\pi}\int_0^{\phi_0}\int_0^a \cos\phi\sin\phi\,d\rho\,d\phi\,d\theta..$$

Inner integral: $a\cos\phi\sin\phi$. Middle integral: $a\sin^2\phi_0/2$. Outer integral: $G\pi a\sin^2\phi_0$.

$\Rightarrow$ $\boxed{\mathbf{F} = \langle 0,\ 0,\ G\pi a \sin^2\phi_0\rangle.}$

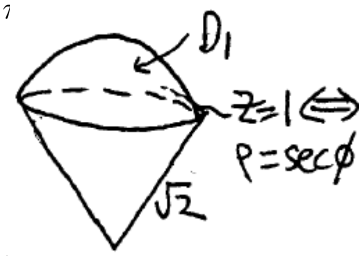


3. a) T = disk of radius 1 at height $z = 1$, $\mathbf{n} = \mathbf{k}$.

On T: $\mathbf{F}\cdot\mathbf{n} = 1 \Rightarrow$ flux $= \iint_T \mathbf{F}\cdot\mathbf{n}\,dS = \iint_T dS =$ area $= \boxed{\pi.}$

b) See the picture. Let D_1 be the volume bounded by S and T.

Let D_2 be the volume bounded by T and U.
Remember we are consistently using upward normals and upward flux.
The divergence theorem gives
flux through S - flux through $T = \iiint_{D_1} \operatorname{div}\mathbf{F}\, dV = \iiint_{D_1} dV = \text{volume}(D_1)$.
$\Rightarrow$ flux through S = volume(D_1) + flux through T = volume(D_1) + π



Likewise, flux through T - flux through U = volume(D_1).
$\Rightarrow$ flux through $U = \pi$ - volume(D_2).
Computing volumes:
D_2: Volume(D_2) $\frac{1}{3}$ base $\times$ height $= \boxed{\frac{\pi}{3}.}$
D_1: We do this at the end in two different ways. The answer is volume$(D_1) = 2\pi\left(\frac{2\sqrt{2}}{2} - \frac{5}{6}\right) = \frac{4\pi\sqrt{2}}{3} - \frac{5\pi}{3}$.

Thus we have, flux through S = volume$(D_1) + \pi = \boxed{\frac{4\pi\sqrt{2}}{3} - \frac{2\pi}{3}}$.

Flux through $U = \pi$ - volume$(D_2) = \boxed{\frac{2\pi}{3}.}$
As promised we compute volume(D_1) two different ways.

Method 1: volume$(D_1) = \int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{\sqrt{2}} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$.
(The ρ limits are from $z = 1 \Leftrightarrow \rho = \sec\phi$.)

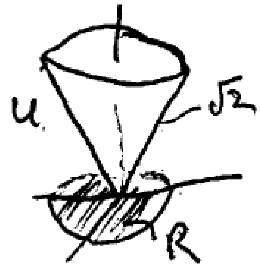
Inner integral: $\left.\frac{\rho^3}{3}\sin\phi\right|_{\sec\phi}^{\sqrt{2}} = \frac{2\sqrt{2}}{3}\sin\phi - \frac{\cos^{-3}\phi}{3}\sin\phi$.

Middle integral: $\left.-\frac{2\sqrt{2}}{3}\cos\phi - \frac{\cos^{-2}}{6}\right|_0^{\pi/4} = \frac{2\sqrt{2}}{3}(1 - \frac{1}{\sqrt{2}}) + \frac{1}{6}(1-2) = \frac{2\sqrt{2}}{3} - \frac{5}{6}$.

Outer integral: volume$(D_1) = 2\pi\left(\frac{2\sqrt{2}}{2} - \frac{5}{6}\right) = \frac{4\pi\sqrt{2}}{3} - \frac{5\pi}{3}$.

Method 2: volume(D_1) = volume$(D_1 + D_2)$ - volume(D_2).
Volume$(D_1 + D_2)$ is an easier integral than in method 1.
Volume$(D_1+D_2) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 2\pi\frac{2\sqrt{2}}{3}(1-\cos(\pi/4)) = \frac{4\pi\sqrt{2}}{2} - \frac{4\pi}{3}$.
Using volume$(D_2) = \pi/3$ we get volume$(D_1) = \frac{4\pi\sqrt{2}}{3} - \frac{5\pi}{3}$ (same as method 1)

c) U is given by $z = \sqrt{x^2+y^2} = r$.



$\Rightarrow$ $\mathbf{n}\,dS = \langle -z_x, -z_y, 1\rangle\,dx\,dy$ $\Rightarrow$ $\mathbf{F}\cdot\mathbf{n}\,dS = z\,dx\,dy$

$$\Rightarrow \text{ flux} = \iint_R z\,dx\,dy = \boxed{\int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \frac{2\pi}{3}.}$$

4. a)Use $\dfrac{\partial\rho}{\partial x} = \dfrac{x}{\rho}$, etc. $\Rightarrow$ $\dfrac{\partial}{\partial x}\rho^{-1} = -\dfrac{x}{\rho^3}$ etc. $\Rightarrow$ $\boxed{\mathbf{F} = \boldsymbol{\nabla} f = -\left\langle \frac{x}{\rho^3}, \frac{y}{\rho^3}, \frac{z}{\rho^3}\right\rangle}$.

Now use $\dfrac{\partial}{\partial x} x\rho^{-3} = \rho^{-3} - 3x^2\rho^{-5}$ (and similarly for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$).

$\Rightarrow$ $\text{div}\mathbf{F} = (-\rho^{-3}+3x^2\rho^{-5}) + (-\rho^{-3}+3y^2\rho^{-5}) + (-\rho^{-3}+3z^2\rho^{-5}) = -3\rho^{-3} + 3\rho^2\cdot\rho^{-5} = 0.$ QE

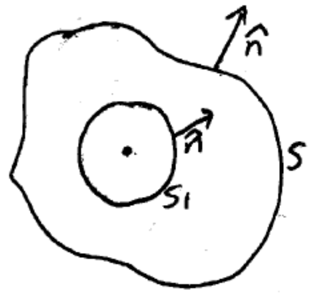
b) The divergence theorem does not apply because $\mathbf{F}$ is not defined at 0.

On S: $\mathbf{n} = \dfrac{\langle x,y,z\rangle}{a}$, $\mathbf{F} = \dfrac{\langle -x.-y,-z\rangle}{a^3}$.

$\Rightarrow$ $\mathbf{F}\cdot\mathbf{n} = -\dfrac{1}{a^2}$ $\Rightarrow$ flux $= -\dfrac{1}{a^2}\cdot$area $= -\dfrac{1}{a^2}4\pi a^2 = \boxed{-4\pi.}$

c)Let S be *any* closed surface around 0. Let S_1 be a small sphere centered at 0 and completely insided S. Use *outward* normals for both surfaces (and be careful with signs).

D is the volume between S and S_1.



From part (a) we know div$\mathbf{F} = 0$, so the divergence theorem gives

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS - \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \text{div}\mathbf{F}\,dV = 0.$$

$$\Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = -4\pi. \quad \text{QED}$$

5. Use the fact that ∇f is perpendicular to the iso-surface $f = c$, so that depending on whether ∇f points inward or outward, $\nabla f\cdot\mathbf{n} = \pm\mid\nabla f\mid$, where $\mathbf{n}$ is the outward unit normal to S. Then apply the Divergence Theorem to get $\oiint_S \vec{\nabla} f\cdot\mathbf{n}\,dS = \int\int_G\int \vec{\nabla}\cdot(\vec{\nabla} f)\,dV$, where G is the interior of S.

Substituting into the RHS integral then gives

$$\pm\oiint_S \mid\nabla f\mid\,dS = \oiint_S \nabla f\cdot\mathbf{n}\,dS = \int\int_G\int \nabla^2 f\,dV.$$

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