DAVID JORDAN: Hello, and welcome back to recitation. So the problem that I want to work with you now is to compute some integrals, but we want to compute them in the presence of a density function.

So the region that we're considering is very simple. It's just the unit square. So we have the origin here, we have the line x equals 1, we have the line y equals 1, and we just want to compute in this region. And so we want to use this density function to find various sort of physical characteristics of this region.

So first, we want to find its mass, and so we are going to need to recall how you get mass from density. We want to find the center of mass. That is, where is the point on which we could balance this if we cut it out? If we tried to balance it on our fingers, where is the average mass concentrated? We want to find the moment of inertia about the origin, and we want to find the moment of inertia about the x-axis. So we're going to have to remember our formulas for moments of inertia.

So why don't you pause the video and work on this for a little bit. Check back with me and I'll show you how I solved it.

Hi. Welcome back. Why don't we start by finding the mass.

So the mass is the most straightforward of these, and I find it helpful to use the language of differentials. So what I want us to do is I want us to take a little square here, and this little square has area dA. OK? And what we want to do is we want to sum up the masses of all the little squares dA here. So what we want to know is what is the little bit of mass dM which corresponds to this little bit of area dA.

And more or less by definition, this is delta dA. So delta is the ratio of area to mass. And so this little contribution of mass is just delta times the little contribution of area. OK? Now, once we write it this way, then our total mass for the entire square is just the integral over the region of all the little contributions of dM.

And so in particular, this is just the integral from x equals 0 to 1, y equals 0 to 1. We have xy-- that's our density-- and then we have dy dx. OK. And this is an integral which we can just compute. So why don't we compute this one all the way through and see what we get.

OK. So we have integral x equals 0 to 1. So we have xy, and we need to integrate that in y. So we have xy squared over 2. And then y ranges from 1 to 0, dx. So this is the integral from x equals 0 to 1 of x over 2, dx. And this is just x squared over 4 from 1 to 0. This is just 1/4. So that tells us that the total mass of this unit square is 1/4.

OK. So now, we need to do similar, we have a similar challenge for the other physical quantities. We just need to figure out what is the appropriate differential quantity, and then we just need to integrate that. For b, we need to compute the center of mass.

So remember that the center of mass involves finding the average x-coordinate and the average y-coordinate. And I wanted to remind you what the formula is for the center of mass, and remind you how I remind myself of it.

So in the formula for the center of mass, we need to take the average of x times dM divided by the integral of dM. So this is our formula for the center of mass. And I just wanted to say that the way that I remember this is by thinking about seesaws.

So if you think about it-- and if we were not doing multiple variables but a single variable-- if I had a seesaw, and I had some weights. So I had m1 and m2 and m3 and m4-- I had some weights-- and these were at positions x1 and x2 and x3 and x4. Well, the fact that the scale would be balanced would be to say that this point x here, where the fulcrum is located, is exactly the weighted average of these points. That's what's going to guarantee that there's the same amount of torque pushing this way and this way. So if we were in one variable and we just had some discrete weights, then we would want to take the average of all of these positions, and we would want to weight it with the masses. So we would want to take the sum of xi mi and divide by the sum of mi. This would be the average coordinates in this kind of toy example.

And now if you look at the formula for the center of mass, it's really the same thing, isn't it? Because integrals are just a continuous version of the sum. We have x as a function instead of xi-- as a list. And the mi's are just the little infinitesimal dM's here. And then the bottom here is just the total mass of the

system, and so is this. OK, so that's how I think about this center of mass formula. And it's actually pretty easy to compute.

So we have the integral from x equals 0 to 1, y equals 0 to 1. So now we have x times delta times dx dy. So altogether we get x squared y dy dx.

So one of those x's is because we're averaging x and the other one is from the density function. So we have this whole integral. And then we divide by this integral of the mass, but we already computed this in part a, and we found it to be 1/4. OK. So this numerator here is fairly straightforward to compute. And if you do this you'll get-- let me double check-- I believe we got 1/6.

So you should get 1/6 when you compute this integral. So we have 1/6 over 1/4, and so cancelling off, this is 2/3. OK. So that was just the x center of mass.

But now I want to make an important point, which is that this density function is symmetric in x and y. It was just x times y. It wasn't something more complicated. And so the center of mass in the x-direction is just equal to the center of mass in the y-direction, so these are both equal to 2/3.

OK. So that depended on the fact that our density was symmetric, and also on the fact that our region was symmetric about switching x and y. So we could save ourselves some trouble here. OK, very good.

So now to do c, again we need to recall what is the infinitesimal moment of inertia. So let me draw this picture again. So here's our little dA here. And we want to know the infinitesimal moment of inertia around the origin. So we tie a string to this little piece of mass, and we start spinning it, and we want to know what is our moment of inertia corresponding to that little mass.

And I'll just remind you from lecture that the formula is r squared dM. So this is r squared times xy dx dy. And so the r squared here is saying that as you get farther and farther out, your moment of inertia is getting larger and larger. And this makes sense in terms of the physical idea that you're moving a longer distance if you're farther out. So anyway. So this is our formula r squared dM.

And so that tells us that I is just the integral of dI. And so this is the integral from x goes from 0 to 1, y goes from 0 to 1. And then we have x squared plus y squared-- that's just r squared-- times xy dx dy. And so we can rewrite this as x cubed y plus xy cubed dx dy.

And this is a computation that we can do. Let me just check my notes real quick. So this is 1/4. I'll skip the computation, but this is just integrating some polynomials, so we can do that. All right.

And now finally, we want to compute the moment of inertia. So remember, d asked us to compute the moment of inertia around the x-axis. So instead of around the origin, it's around the x-axis.

So the idea here is the same. So again, dI is a factor times dM. And again, it's the radius, but now it's the radius about which we're spinning. So we're not anymore spinning around the origin as we were doing before. Now we're spinning around-- sort of out of the board-- around the x-axis here. But we still have the same formula, and now our radius is the height y. Because we're not spinning around the origin anymore, we're spinning around this rod here.

And so if you think about it, that's the radius about which we're spinning is just the height y. So this is just y squared delta. OK. And so that tells us that I --the total inertia about the x-axis-- is just the integral of dI. And so we get the integral from x equals 0 to 1, integral y equals 0 to 1. And then we just have y squared xy dy dx. And this, again, we could compute-- and let me just check my notes-- and find that it's 1/8.

So in each of these problems, the most important thing to have been able to do is to argue, what is this sort of infinitesimal contribution to the physical quantity that you want to compute? And eventually, you want to express it in terms of the quantity dA, because dA is what we actually can integrate. And so all the other physical quantities that we need to study are going to be an integral of some infinitesimal element, and that infinitesimal element is going to be some coefficient times dA.

So here, we had that this was-- oh, dear. This is a mistake. So this should have said y squared dM, and that's y squared delta dA. So I wrote the delta implicitly. I wrote it twice.

So what we meant to say is dI is y squared delta dA. And so in all these examples, the infinitesimal quantity that we're after is some straightforward coefficient times the infinitesimal area. And so once we know that, then we can just do a straightforward integral.

OK, and I'll leave it at that.

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